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Existence of traveling waves in nonlinear dispersive equations of Whitham-type

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Objective

- Study class of evolution equations
$$u_t + (Lu + n(u))_x = 0.$$
- Show existence of small-amplitude traveling waves
$$u = u(x - \nu t)$$
with speed ν satisfying
$$(\star) \qquad -\nu u + Lu + n(u) = 0$$
in fractional Sobolev spaces H^s.

Dispersion: L

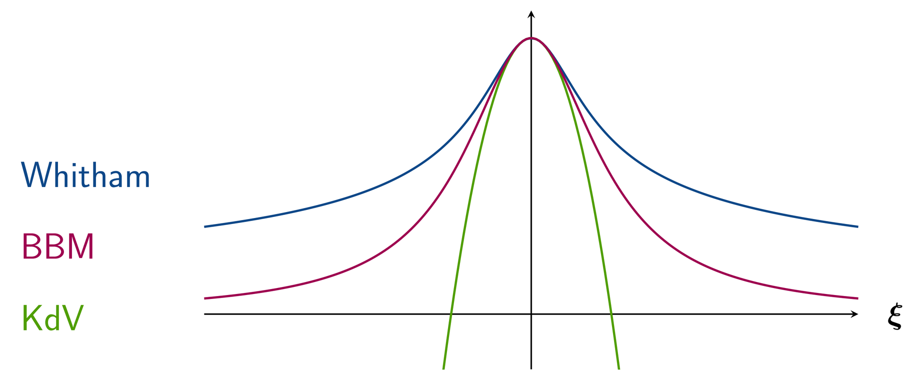
- Smoothing Fourier multiplier:
$$\widehat{Lu} = \mathfrak{m}\,\widehat{u}, \quad \mathfrak{m}(\xi) \lesssim \langle \xi \rangle^{\sigma}, \quad \sigma < 0.$$
- Even symbol $\mathfrak{m}$ with expansion
$$\mathfrak{m}(\xi) = \mathfrak{m}(0) - c_\lambda \xi^{2\lambda} + \mathcal{O}\left(\xi^{2\lambda+2}\right)$$
for some $\lambda = 1, 2, \ldots$ and $c_\lambda > 0$.

Nonlinearity: n

- Power-type, essentially of the form
$$n(x) \approx |x|^{1+q} + \mathcal{O}(|x|^{1+q'})$$
with $q \in (0, 4\lambda)$ and $q' > q$.
- Locally Lipschitz when $s \in (0, 1)$; or $n \in \mathrm{C}^{1+q}(\mathbb{R})$ if $s \in [1, 1+q)$.

Whitham equation

- Model for surface gravity water-waves with exact linear dispersion relation:
  - ⋆ $n(u) = u^2$
  - ⋆ $\mathfrak{m}(\xi) = \sqrt{\frac{\tanh \xi}{\xi}} = \underbrace{1 - \tfrac{1}{6}\xi^2}_{\text{KdV}} + \mathcal{O}(\xi^4)$
  $$\lesssim \langle \xi \rangle^{-1/2}$$



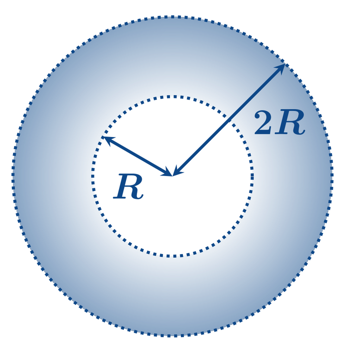
- Periodic waves in H^s for all $s > 0$, and solitary waves for all $s > \frac{1}{6}$.

Variational approach

- Fix a small parameter $\mu > 0$ and define functionals
$$\mathcal{L}(u) = -\tfrac{1}{2}\int uLu, \quad \mathcal{Q}(u) = \tfrac{1}{2}\int u^2,$$
$$\mathcal{N}(u) = -\int N(u), \qquad \mathcal{E} = \mathcal{L} + \mathcal{N},$$
where N is the primitive of n.
- Solutions of $(\star)$ $\leftrightarrow$ critical points of $\mathcal{E}$ constrained to $\mathcal{Q}(u) = \mu$ by Lagrange's multiplier principle.
- Find constrained local minimizers of $\mathcal{E}$.
  - ⋆ Problem: $\mathcal{N}$ unbnd. locally in H^s for low s.
  - ⋆ Solution: Cut off n at scale dep. on μ. *A priori* estimates of $\|u\|_\infty$ show that the cut-off is not seen for small μ.

Periodic waves

- Minimization of $\mathcal{E}_P$ constrained to $\mathcal{Q}_P(u) = \mu$ over a ball of radius R in P-periodic Sobolev space H^s_P.
- Coercivity-issue: Minimizers may approach boundary of the ball.
- Consider instead min. of penalized functional
$$\mathcal{E}_P(u) + \varrho(\|u\|^2_{\mathrm{H}^s_P})$$
over enlarged $2R$-ball.
  - ⋆ $\varrho(t) = 0$ for $t < R^2$; blows up as $t \uparrow (2R)^2$.
  - ⋆ Existence of minimizer u_P by standard arguments (Weierstrass theorem).



- *A priori* estimates $\Rightarrow$ penalizer inactive:
  - ⋆ Upper bound on infimum of penalized $\mathcal{E}_P$.
  - ⋆ Lower bound on wave speed ν_P.
  - ⋆ $\|u_P\|^2_{\mathrm{H}^s_P} \lesssim \mu$ for small μ.
- Small-amplitude: $\|u_P\|_\infty \lesssim \|u_P\|_{\mathrm{H}^s_P}$.

Theorem

If $s \in \left(\max\left\{\frac{1}{2} - |\sigma|, 0\right\}, 1+q\right)$ and μ is sufficiently small, there exists $P_\mu > 0$ such that $(\star)$ admits a nontrivial solution $u_P \in \mathrm{H}^s_P \cap \mathrm{L}^\infty$ for all $P \geqslant P_\mu$, satisfying
$$\|u_P\|^2_\infty \lesssim \|u_P\|^2_{\mathrm{H}^s_P} \lesssim \mu$$
uniformly in $P \geqslant P_\mu$, with supercritical speed
$$\mathfrak{m}(0) + c\left(\tfrac{2\mu}{P}\right)^{q/2} < \nu_P < C.$$

Periodic $\xrightarrow[P \to \infty]{}$ solitary

- Translate and truncate $\{u_P\}_P$ so that its restriction $\widetilde{u}_P \in \mathrm{H}^s(\mathbb{R})$ to $(-\frac{P}{2}, \frac{P}{2})$ stays sufficiently far away from $\pm\frac{P}{2}$.
- As $P \to \infty$, all of
$$\mathcal{E}_P(u_P) - \mathcal{E}(\widetilde{u}_P), \quad \|\mathcal{E}'(\widetilde{u}_P)\|_{\mathrm{H}^s(|x| > \frac{P}{2})},$$
$$\|\mathcal{E}'_P(u_P) - \mathcal{E}'(\widetilde{u}_P)\|_{\mathrm{H}^s(|x| < \frac{P}{2})}$$
go to 0 (and similarly for $\mathcal{Q}$, $\mathcal{Q}_P$).
- $\{u_k\}_k$ with $u_k = \widetilde{u}_{P_k}$ and $P_k \to \infty$ is a special minimizing sequence for the solitary problem with $\sup\|u_k\|^2_{\mathrm{H}^s} \lesssim \mu$.

Solitary waves

- Concentration-compactness principle for minimizing sequence $\{u_k\}$ bounded away from boundary of the H^s-ball,
$$\mathcal{E}(u_k) \to I_\mu := \inf \mathcal{E}, \quad \begin{array}{l} \mathcal{Q}(u_k) = \mu, \\ \sup\|u_k\|_{\mathrm{H}^s} < R. \end{array}$$
- $\{u_k\}_k$ can either
  - ⋆ *vanish* (wave dissolves into ripples); ruled out with estimates on $\mathcal{N}$;
  - ⋆ *dichotomize* (wave splits into two parts); ruled out by strict sub-additivity
  $$I_{\mu_1+\mu_2} < I_{\mu_1} + I_{\mu_2}$$
  and frequency decomp. and scaling arguments for "near-minimizers";
  - ⋆ *concentrate* (behavior—up to translations—as for periodic problem); and since $\{u_k\}$ stays inside H^s-ball, it also concentrates in frequency. By Kolmogorov–Riesz' comp. theorem + interpolation, $\{u_k\}$ converges to a minimizer.
- *A priori* estimates similar as before.

Theorem

Let $s > \frac{1}{2} - |\sigma|$, $s < 1 + q$ and $2s > \frac{q}{2+q}$. For μ sufficiently small, there exists a nontrivial solitary wave $u \in \mathrm{H}^s(\mathbb{R}) \cap \mathrm{L}^\infty$ with
$$\|u\|^2_\infty \lesssim \|u\|^2_{\mathrm{H}^s(\mathbb{R})} \lesssim \mu$$
and supercritical speed $\nu > \mathfrak{m}(0)$.

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