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A framework for non-local, non-linear initial value problems

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$$(1)\qquad \begin{cases} \partial_t u + \mathcal{L}_u u = 0 & \text{on } [0,\infty)\times\mathbb{R}^N \\ u(0) = u_0 & \text{on } \mathbb{R}^N \end{cases}$$

$$\big(\mathcal{L}_v u\big)(x) = \int_{\mathbb{R}^N} \big[u(x) - u(y)\big]\,\rho\big(v(x), v(y); x, y\big)\,dy$$

Main result

Theorem

Let ρ be a homogeneous jump kernel. For every initial condition $u_0 \in L^\infty(\mathbb{R}^N)\cap BV(\mathbb{R}^N)$, problem (1) has a unique very weak solution u such that

$$u \in L^\infty\big([0,\infty), L^\infty(\mathbb{R}^N)\cap BV(\mathbb{R}^N)\big)\cap W^{1,1}_{\text{loc}}\big([0,\infty), L^1(\mathbb{R}^N)\big).$$

This solution has the following properties

- *mass is conserved: $\int_{\mathbb{R}^N} u(t,x)\,dx = \int_{\mathbb{R}^N} u_0(x)\,dx$ for all $t \geq 0$;*
- *L^p-norms are non-increasing: $\|u(t)\|_p \leq \|u_0\|_p$ for all $p\in[1,\infty]$ and $t\geq 0$;*
- *if $u_0(x)\geq 0$ for almost every $x\in\mathbb{R}^N$ then $u(t,x)\geq 0$ for almost every $x\in\mathbb{R}^N$ and $t\geq 0$.*

Moreover, for two solutions u and $\widetilde{u}$ corresponding to initial conditions u_0 and $\widetilde{u}_0$, respectively, we have

$$\|u(t)-\widetilde{u}(t)\|_1 \leq \|u_0 - \widetilde{u}_0\|_1 \quad \text{for every } t\geq 0$$

and if $u_0(x)\geq \widetilde{u}_0(x)$ for almost every $x\in\mathbb{R}^N$ then $u(t,x)\geq\widetilde{u}(x,t)$ for almost every $x\in\mathbb{R}^N$ and $t\geq 0$.

Corollary

Let ρ be a homogeneous jump kernel. For every initial condition $u_0\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, problem (1) has a very weak solution u such that

$$u\in L^\infty\big([0,\infty), L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\big)\cap C\big([0,\infty), L^1(\mathbb{R}^N)\big).$$

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- *if $u_0(x)\geq 0$ for almost every $x\in\mathbb{R}^N$ then $u(t,x)\geq 0$ for almost every $x\in\mathbb{R}^N$ and $t\geq 0$.*

Jump kernel

Definition

We say a function $\rho : (\mathbb{R}\times\mathbb{R})\times(\mathbb{R}^N\times\mathbb{R}^N)\to\mathbb{R}$ is a homogeneous jump kernel if it satisfies conditions (A1)–(A6). For all $a,b,c,d\in\mathbb{R}$ and for almost every $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ we assume that

- (A1) ρ is a non-negative Borel function;
- (A2) ρ is symmetric, i.e. $\rho(a,b;x,y) = \rho(b,a;y,x)$;
- (A3) ρ is monotone in the following sense:
$$(a-b)\rho(a,b;x,y)\geq(c-d)\rho(c,d;x,y) \quad \text{whenever } a\geq c\geq d\geq b;$$
- (A4) ρ is homogeneous
$$\rho(a,b;x,y) = \rho\big(a,b;|x-y|\big);$$
- (A5) for every $R>0$ there exists a function $m_R:[0,\infty)\to[0,\infty)$ such that $\sup_{-R\leq a,b\leq R}\rho(a,b;x,y)\leq m_R\big(|x-y|\big)$ and
$$\int_{\mathbb{R}^N}(1\wedge|y|)m_R\big(|y|\big)\,dy = K_R < \infty;$$
- (A6) ρ is continuous with respect to the first two variables and it is locally Lipschitz-continuous outside diagonals, i.e. for every $\varepsilon>0$ and every $R>\varepsilon$ there exists a constant $C_{R,\varepsilon}$ such that
$$\big|\rho(a,b,x,y)-\rho(c,d,x,y)\big|\leq C_{R,\varepsilon}\big(|a-c|+|b-d|\big)\,m_R\big(|x-y|\big)$$
for every $a,b,c,d\in[-R,R]$ such that $|a-b|\geq\varepsilon$ and $|c-d|\geq\varepsilon$.

Decoupled jump kernels

$$(2)\qquad \rho(a,b;x,y) = F(a,b)\times\mu(|x-y|),$$

where $F\geq 0$ and μ is a density of a Lévy measure with low singularity, i.e.

$$\int_{\mathbb{R}^N}(1\wedge|y|)\mu(|y|)\,dy\leq\infty.$$

Fractional porous medium equation

Let $f\in C^1(\mathbb{R})$ be a non-decreasing function. If

$$F(a,b) = \frac{f(a)-f(b)}{a-b}$$

then ρ given by formula (2) is a homogeneous jump kernel. This example, for $\mu(|y|) = |y|^{-N-\alpha}$, $\alpha\in(0,1)$ and $f(u) = u|u|^{m-1}$, corresponds to the following operator

$$\mathcal{L}_u u = \Delta^{\alpha/2}\big(u|u|^{m-1}(x)\big).$$

Fractional p-Laplacian

Let $\Phi:\mathbb{R}\to\mathbb{R}$ be a continuous, non-decreasing function satisfying $\Phi(z)\geq 0$ for $z\geq 0$, $\Phi(-z) = -\Phi(z)$ such that $\Phi(z)$ is locally Lipschitz-continuous and $\lim_{z\to 0}\frac{\Phi(z)}{z}<\infty$. If

$$F(a,b) = \frac{\Phi(a-b)}{a-b}$$

then ρ given by formula (2) is a homogeneous jump kernel. The function $\Phi(z) = |z|^{p-2}z$ satisfies the hypothesis of this Proposition if and only if $p\geq 2$. Then we may take $s<\frac{1}{p}$ (so that $sp<1$) and we recover the non-local s-fractional p-Laplace operator.

Convex diffusion operator (new!)

Let $f:\mathbb{R}\to[0,\infty)$ be a convex function and

$$F(a,b) = f(a)+f(b).$$

Then ρ given by formula (2) is a homogeneous jump kernel.

Entangled jump kernels

Homogeneous jump kernels which cannot be decomposed as in formula (2) are also part of our framework. Let

$$\Psi(\mathfrak{a};z) = \Psi_1(\mathfrak{a})\mathbb{1}_{z<1}(z)+\Psi_2(\mathfrak{a})\mathbb{1}_{z\geq 1}(z)+\Theta(z),$$

where $\Psi_1:[0,\infty)\to\mathbb{R}$ is non-decreasing, $\Psi_2:[0,\infty)\to\mathbb{R}$ is non-increasing, both Ψ_1 and Ψ_2 are locally Lipschitz-continuous, $\Theta:[0,\infty)\to\mathbb{R}$ is measurable and

$$0<A_1\leq\Psi(\mathfrak{a};z)\leq A_2<1, \qquad A_1\leq\Theta(z)\leq A_2$$

Then

$$\rho(a,b;x,y) = |x-y|^{-N-\Psi(|a-b|;|x-y|)}$$

is a homogeneous jump kernel.