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Solitary waves for dispersive equations of positive order with inhomogeneous nonlinearities

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Objective

Prove existence of solitary waves, $w(x,t) = u(x - \nu t)$ solving the equation

$$w_t + (Lw + n(w))_x = 0.$$

Inserting for w and integrating, u equivalently solves

$$(\star) \qquad -\nu u + Lu + n(u) = 0.$$

Nonlinearity: n

Has the decomposition $n = n_p + n_r$.

- For $p \in \mathbb{R}^+$, the homogenous term n_p takes either of the two forms
  (1) $x \mapsto c|x|^{1+p}$ and $c \neq 0$,
  (2) $x \mapsto cx|x|^p$ and $c > 0$.
- The remainder term n_r is locally Lipschitz and satisfies for some $r > p$, $n_r(x) = \mathcal{O}(|x|^{1+r})$ as $x \to 0$.

Dispersion: L

Has a real valued even symbol m,

$$\widehat{Lu}(\xi) = m(\xi)\hat{u}(\xi).$$

- Growth of $m - m(0)$:
$$\begin{cases} m(\xi) - m(0) \simeq |\xi|^{s'}, & \text{for } |\xi| \leqslant 1, \\ m(\xi) - m(0) \simeq |\xi|^{s}, & \text{for } |\xi| \geqslant 1, \end{cases}$$
  where $s' > \frac{p}{2}$ and $s > \frac{p}{2+p}$.
- Regularity of m:
  The function $\xi \mapsto m(\xi)/\langle \xi \rangle^s$ is uniformly continuous.

Examples

- Capillary Whitham equation (surface tension $T > 0$):
$$m(\xi) = \sqrt{\frac{(1+T\xi^2)\tanh\xi}{\xi}}, \quad \begin{matrix} n_p(x) = -x^2, \\ n_r(x) = 0, \end{matrix} \quad \begin{cases} s' = 2, & s = 1/2, \\ p = 1, & r = \infty. \end{cases}$$
- KdV equation with polynomial nonlinearities:
$$\begin{matrix} m(\xi) = \xi^2 \text{ (after time reversal)}, \\ n_p(x) = x^2, \quad n_r(x) = x^3 P(x), \end{matrix} \quad \begin{cases} s' = 2, & s = 2, \\ p = 1, & r \geqslant p + 1. \end{cases}$$

Main Theorem

For every sufficiently small $\mu > 0$, there exists a solution $u \in H^1$ of $(\star)$ with speed ν satisfying

(i) $\|u'\|_{L^2}^2 \lesssim \|u\|_{L^2}^2 = 2\mu$,

(ii) $m(0) - \nu \simeq \mu^\beta$, where $\beta = \frac{s'p}{2s'-p}$,

for implicit constants independent of μ.

Variational approach

- Define the functionals
$$\mathcal{L}(u) = \tfrac{1}{2}\int m|\hat{u}|^2 d\xi, \quad \mathcal{Q}(u) = \tfrac{1}{2}\int u^2,$$
$$\mathcal{N}(u) = \int N(u), \qquad \mathcal{E} = \mathcal{L} - \mathcal{N},$$
  where $N(x) = \int_0^x n(t)dt$, and define the constraint minimization problem
$$(\star\star) \qquad I_\mu := \inf_{\mathcal{Q}(u)=\mu} \mathcal{E}(u).$$
- By Lagrange's multiplier principle, minimizers of $(\star\star)$ solves
$$\nu u = \mathcal{E}'(u) = L(u) - n(u)$$
  for some $\nu \in \mathbb{R}$, and is consequently also a solution of $(\star)$.
- Goal: Construct a minimizer from a minimizing sequence (u_i) of $(\star\star)$.

Remark

An important observation is that it suffices to consider n being globally Lipschitz; the validity of the main theorem for this special case implies the theorem in general.

Concentration-compactness

A minimizing sequence (u_i), can

(1) Concentrate; the L^2–mass of u_i is uniformly localized (up to translation). Convergence follows.

(2) Vanishing; the L^2–mass of u_i spreads out as $i \to \infty$.

(3) Dichotomy; u_i splits in two parts of fixed mass separating as $i \to \infty$.

Excluding vanishing

The Gagliardo–Nirenberg inequality,

$$\|u\|_{L^{2+p}}^{2+p} \lesssim \|u\|_{\dot{H}^{s/2}}^{p/s} \|u\|_{L^2}^{(2+p)-p/s},$$

together with an ansatz-estimate of I_μ implies a lower and upper bound of

$$\|u\|_{L^{p+2}} \quad \text{and} \quad \|u\|_{\dot{H}^{s/2}},$$

respectively, for *near* minimizers u. This further implies some accumulation of mass (up to translation) of minimizing sequences.

Excluding dichotomy

For sufficiently small $\mu > 0$, it can be shown that

$$I_\mu < I_{\mu-\lambda} + I_\lambda$$

for $0 < \lambda < \mu$. If a minimizing sequence (u_i) splits in two parts (u_i^1) and (u_i^2) of mass $\mu - \lambda > 0$ and $\lambda > 0$, respectively, we obtain the contradiction

$$\begin{aligned} I_\mu &= \liminf \mathcal{E}(u_i^1) + \mathcal{E}(u_i^2) \\ &\geqslant I_{\mu-\lambda} + I_\lambda. \end{aligned}$$

Regularity of solutions

By rearranging $(\star)$, a solution u satisfies

$$u = (L - \nu)^{-1} n(u).$$

As $(L-\nu)^{-1}$ is a smoothing operator, the regularity of n gives a lower bound for the regularity of u; the Lipschitz continuity of n suffices to obtain $u \in H^1$. The method used to prove the main theorem also embeds solutions in $H^{s/2}$; a better regularity estimate for $s > 2$.

References

- M. N. Arnesen, "Existence of Solitary-Wave Solutions to Nonlocal Equations", Preprint, 2015.
- M. Ehrnström, M. D. Groves, and E. Wahlén, "On the existence and stability of solitary-wave solutions to a class of evolution equations of Whitham type", *Nonlinearity*, vol. 25, no. 10, pp. 2903–2936, 2012.
- R. L. Frank and E. Lenzmann, "Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$", *Acta Math.*, vol. 210, no. 2, pp. 261–318, 2013.
- M. Johnson, "Stability of small periodic waves in fractional kdv-type equations", *SIAM Journal on Mathematical Analysis*, vol. 45, no. 5, pp. 3168–3193, 2013. eprint: `https://doi.org/10.1137/120894397`.