Spatial dynamics methods for solitary waves on a ferrofluid jet

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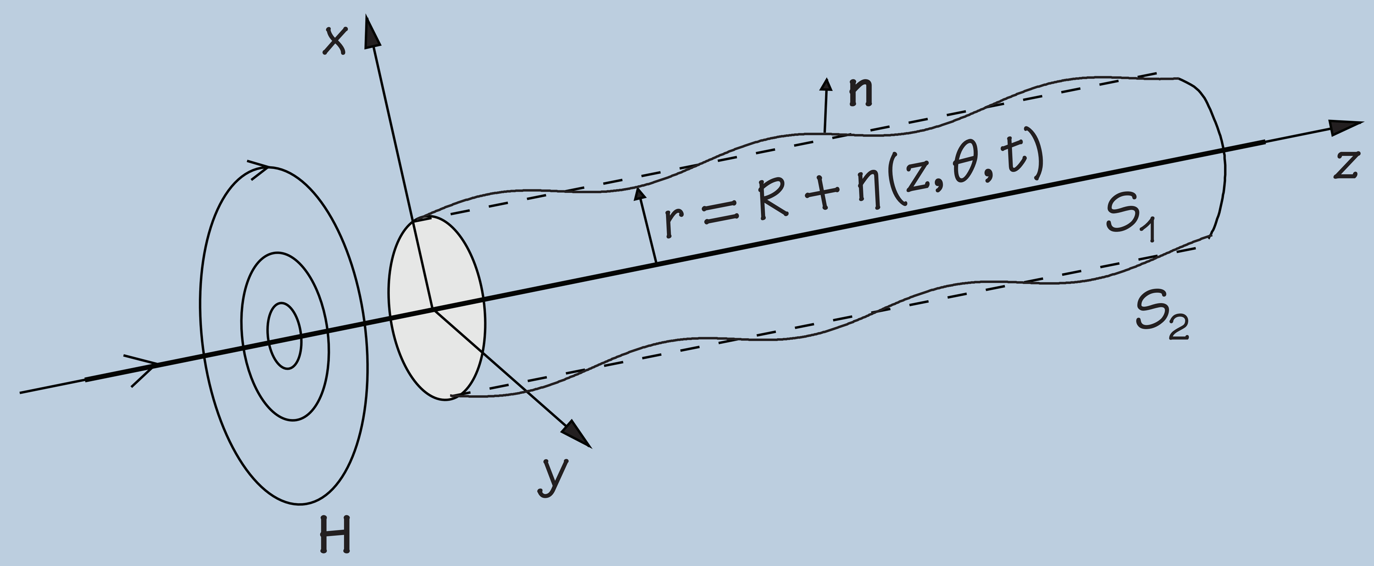
Introduction

We consider axisymmetric solitary waves propagating on the surface of an cylindrical ferrofluid jet surrounding a stationary metal rod. The ferrofluid, which is governed by a general (nonlinear) magnetisation law, is subject to an azimuthal magnetic field generated by an electric current flowing along the rod. Such waves have previously been studied in for example [1],[2].



Governing equations for traveling waves

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \qquad 0 < r < 1 + \eta(z,t), \tag{0.1}$$

$$\phi_r = 0, \qquad r = 0 \tag{0.2}$$

$$\eta_z + \phi_r - \phi_z\eta_z = 0, \qquad r = 1 + \eta(z,t) \tag{0.3}$$

$$-\phi_z + \frac{1}{2}(\phi_r^2 + \phi_z^2) - \alpha\frac{T'(\eta)}{1+\eta} + \beta\Bigg(\frac{1}{(1+\eta)(1+\eta_z^2)^{1/2}} - \frac{\eta_{zz}}{(1+\eta_z^2)^{3/2}} - 1\Bigg) = 0, \qquad r = 1 + \eta(z,t), \tag{0.4}$$

where ϕ is the velocity potential, η is the wave profile,

$$T(\eta) = \int_0^\eta \left(\nu\left(\frac{1}{1+s}\right) - \nu(1)\right)(1+s)\,\mathrm{d}s,$$

where $\nu'(s) = m_1(s)$ and m_1 is the magnetization. In addition $\alpha = \frac{\mu_0 J^2 \chi}{4\pi^2 R^2 c^2}$, $\beta = \frac{\sigma}{c^2 R}$ are dimensionless parameters. *Solitary waves* are nontrivial solutions of (0.1)–(0.4) with $\eta(z), \phi(r,z) \to 0$ as $z \to \pm\infty$.

Hamiltonian formulation

The governing equations follow from the variational principle $\delta\mathcal{L} = 0$, where

$$\mathcal{L} = \int_{\mathbb{R}} L(\eta, \phi, \eta_z, \phi_z)\,\mathrm{d}z.$$

By performing a Legendre transform we introduce new variables $\omega = \frac{\delta L}{\delta \eta_z}$, $\xi = \frac{\delta L}{\delta \phi_z}$ and obtain the Hamiltonian

$$H(\eta,\omega,\phi,\xi) = \int_0^1 \left\{\frac{1}{2}\left(\frac{\xi}{(1+\eta)^2} + 1\right)^2 (1+\eta)^2 r - \frac{1}{2}r\phi_r^2\right\}\mathrm{d}r + \alpha T(\eta) - (1+\eta)\sqrt{\beta^2 - W^2} + \frac{1}{2}\beta(1+\eta)^2.$$

From this we obtain Hamilton's equation

$$u_z = v_H(u) \tag{0.5}$$

where $u = (\eta, \omega, \phi, \xi)$ and v_H is the Hamiltonian vector field corresponding to the Hamiltonian H.

References

[1] Rannacher, D. & Engel, A. 2006 Cylindrical Korteweg–de Vries solitons on a ferrofluid surface. *New J. Phys.* **8**, 108.

[2] Blyth, M. & Parau, E. 2014 Solitary waves on a ferrofluid jet. *J. Fluid Mech.* **750**, 401–420.

[3] Groves, M. D. & Wahlén, E. 2007 Spatial dynamics methods for solitary gravity-capillary water waves with an arbitrary distribution of vorticity. *SIAM J. Math. Anal.* **39**, 932–964.

Spatial dynamics

Equation (0.5) is a dynamical system where the spatial variable z plays the role of time. The idea of formulating the problem in this way is called the spatial dynamics approach. We write (0.5) on the form

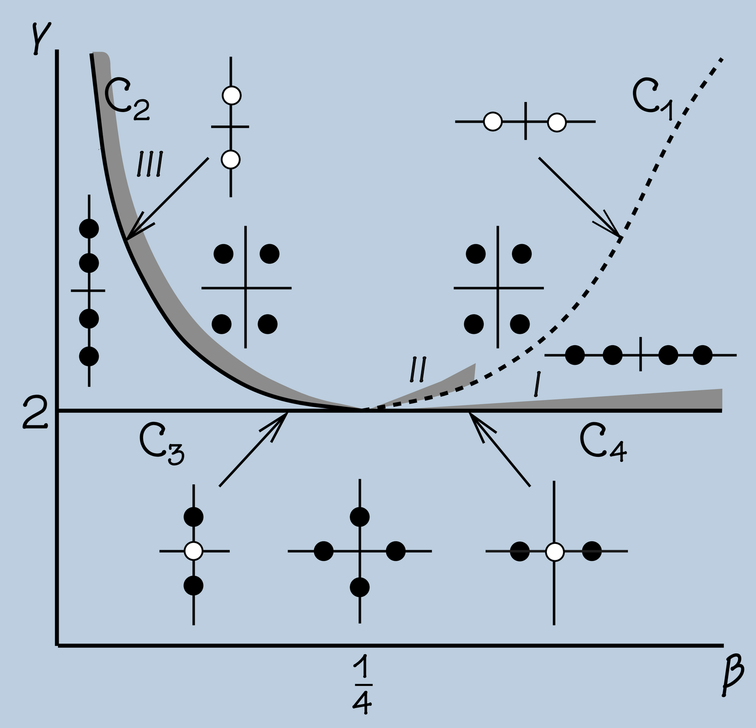
$$u_z = Lu + N(u), \tag{0.6}$$

where L is the linearization of $v_H(u)$ around the trivial solution $(0,0,0,0)$.

Bifurcations

In order to find solutions of (0.6) we apply the center manifold theorem. It is therefore necessary to know something about the spectrum of L, and in particular how it depends upon the dimensionless parameters. The bifurcation diagram to the right is then obtained, where $\gamma = \alpha - \beta$. The same diagram is obtained for surface waves, see for example [3] and the references therein.

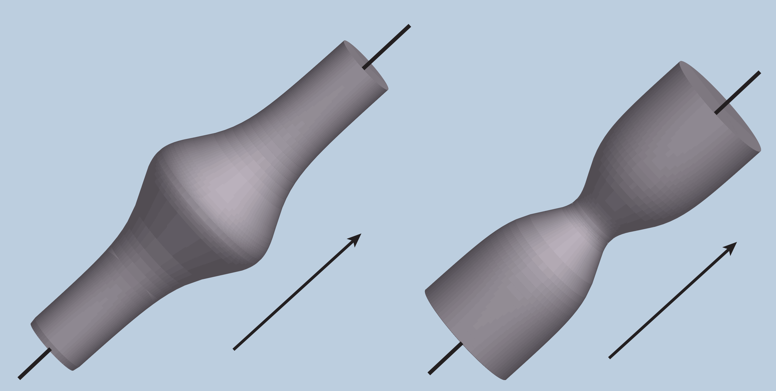
Diagram



Region I

Truncated reduced equation obtained from the center manifold reduction:

$$q'' - q + cq^2 = 0,\ c = \frac{1}{2}(\alpha m_1'(1) - 6)$$

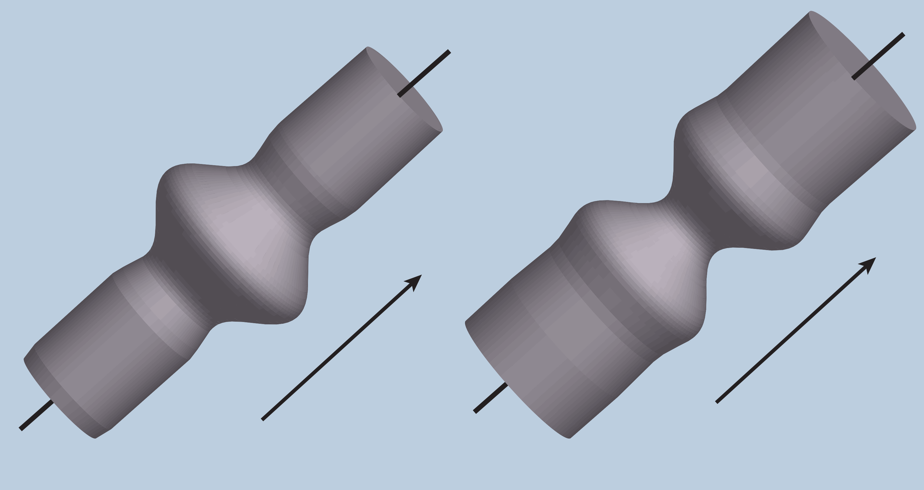


Solitary waves of elevation and depression in region I.

Region II

Truncated reduced equation obtained from the center manifold reduction:

$$q'''' + 2(1+\delta)q'' + q - 3cq^2 = 0,\ c = 48\sqrt{6}(3m_1'(1) - 8)$$

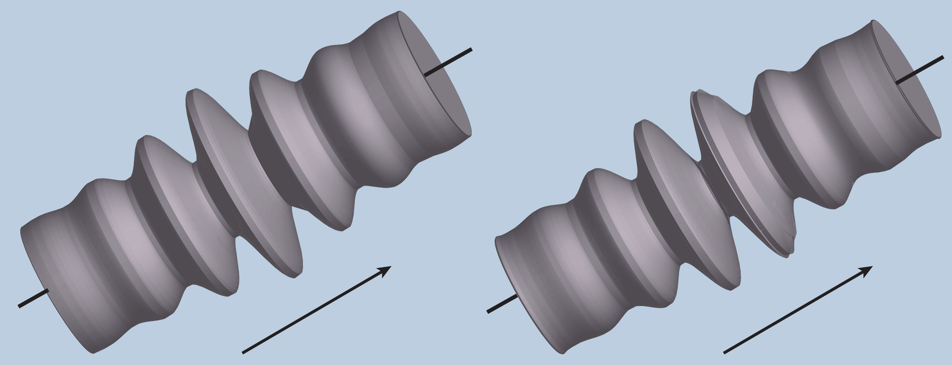


Primary solitary waves of elevation and depression in region II.

Region III

Truncated reduced equation obtained from the center manifold reduction:

$$q'' + c_1 q + c_2 q|q|^2 = 0,\ c_1 < 0\,, c_3 > 0$$



Solitary waves of elevation and depression in region III.