The Hunter-Saxton Equations with Noise

a study in stochastic nonlocal wave equations

NTNU Kunnskap for en bedre verden

Helge Holden[1], Kenneth Karlsen[2], and Peter H.C. Pang[1]

1. Institutt for matematiske fag, NTNU, and 2. Matematisk institutt, Universitetet i Oslo

contact: peter.pang@ntnu.no

Introduction

We are interested in the stochastic Hunter-Saxton equations of the form:

$$0 = \mathrm{d}q + \left(u\partial_x q + \frac{1}{2}q^2\right)\mathrm{d}t + \partial_x(\sigma q)\circ \mathrm{d}W$$
$$q = \partial_x u,$$

on $[0,T]\times\mathbb{R}$, where W is a standard 1-dimensional Brownian motion on a stochastic basis $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$, and $\sigma:\mathbb{R}\to\mathbb{R}$ is a $W^{2,\infty}(\mathbb{R})$ function. These are sometimes considered the prototypal non-local wave equations exhibiting wave-breaking phenomena (explained below) [5].

This particular noise can be derived from a variational principle, by perturbing the Hamiltonian in the deterministic dynamics.

Background

In the deterministic ($\sigma=0$) setting, the archetypical example is the "box" initial condition:

$$q(0) = V_0\mathbb{1}_{[0,1]}(x) \qquad V_0\in\mathbb{R},$$

for which (i) wave-breaking occurs, and (ii) an explicit formula for the solution exists. These boxes can be patched together to form step function approximants to more general initial conditions.

We can write down characteristics (the Lagrangian variable) $X(t,x)$ satisfying

$$X(t,x) = x + \int_0^t u(t,X(s,x))\,\mathrm{d}s.$$

Solving for the the box, one arrives at

$$q(t,x) = \frac{2V_0}{2+V_0 t}\mathbb{1}_{\{2+V_0t>0;\,X(t,0)<x<X(t,1)\}}. \tag{1}$$

Therefore in the case $V_0<0$, $\|q\|_{L^\infty}\to\infty$ as $t\to -2/V_0$ even as $\|u\|_{L^\infty}$ remains bounded. This phenomenon is called *wave-breaking*. How solutions are continued beyond wave-breaking is the cardinal element in the theory of Hunter-Saxton equations.

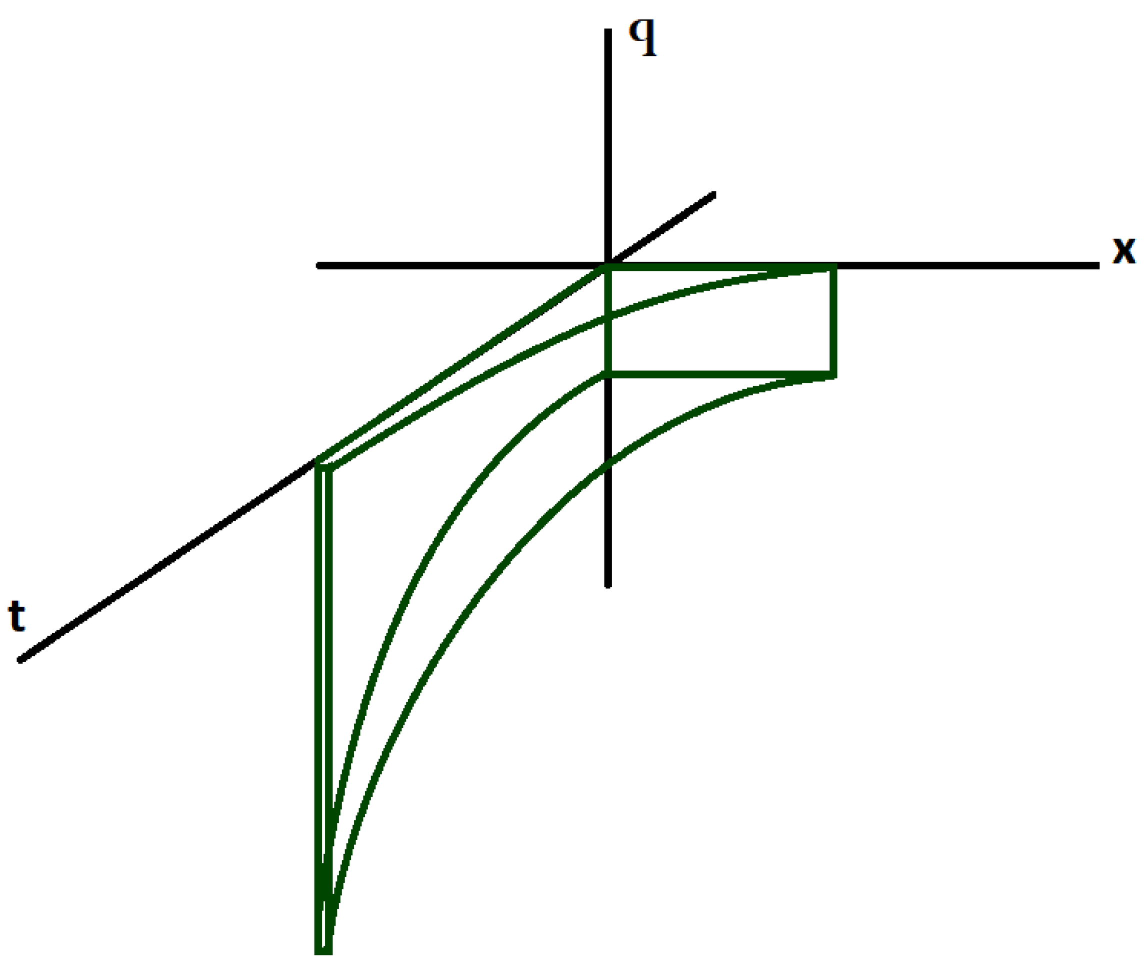

Figure 1: The evolution of the box up to near wave-breaking in the deterministic setting.

In the stochastic case one can do something similar — the characteristics become [1]:

$$\mathrm{d}X(t,x) = u(t,X(t,x))\,\mathrm{d}t + \sigma(X(t,x))\circ\mathrm{d}W.$$

The stochastic Hunter Saxton equation in the Lagrangian variable becomes:

$$\mathrm{d}q(t,X(t,x)) = \frac{-1}{2}q^2(t,X(t,x)) - \sigma'(X(t,x))q(t,X(t,x))\circ\mathrm{d}W.$$

The process $q(t,X(t,x))$ can, perhaps surprisingly, also be solved for explicitly:

$$q(t,X(t,x)) = \frac{Z(t,x)}{1/q(0,x)+\int_0^t Z(s,x)/2\,\mathrm{d}s}, \qquad Z(t,x) = \exp\left(\int_0^t \sigma'(X(s,x))\circ\mathrm{d}W\right). \tag{2}$$

The Case $\sigma''\equiv 0$

The constant σ case yields deterministic dynamics as the "pathwise" transformation $x\mapsto x+\sigma W(t)$ will immediately show [4]. This fails to perturb the difference between two characteristics.

The $\sigma''\equiv 0$ case exhibits truly random dynamics. It is a special case of the equation about which much can be explicitly derived. Fixing our attention on the the "box"-type scenario, we see first of all that over $x\in(X(t,0),X(t,1))$, the box remains constant (in space), and outside this random interval, $q(t)$ is everywhere nought (it remains a box). It can also be shown that:

Wave-breaking occurs when, and only when, characteristics meet.

Though very similar to the deterministic dynamics, the height and width of the box are random as it evolves. The wave-breaking time is also random.

By scaling distributions that Yor et al. [6] derived, it is possible to show that wave-breaking occurs at a stopping time t^* satisfying:

$$\mathbb{P}(\{t^*\geq t\}) = \mathbb{P}\big(\{A^{(0)}(\frac{(\sigma')^2 t}{4}) \leq \frac{-(\sigma')^2}{2V_0}\}\big),$$

where

$$\mathbb{P}(A^{(0)}(t)\in\mathrm{d}r) = \left[\int_{\mathbb{R}}\exp\left(-\frac{1+e^{2x}}{2r}\right)\vartheta(e^x/r,t)\,\mathrm{d}x\right]\frac{\mathrm{d}r}{r},$$
$$\vartheta(y,t) = \frac{y}{\sqrt{2\pi^3 t}}e^{\pi^2/(2t)}\int_0^\infty e^{-\xi^2/(2t)}e^{-y\cosh(\xi)}\sinh(\xi)\sin\left(\frac{\pi\xi}{t}\right)\mathrm{d}\xi.$$

Solutions Post Wave-Breaking

Non-uniqueness arises from the uncountably many ways solutions can be continued past wave-breaking.

Two extreme ways exist to continue solutions beyond wave-breaking in the deterministic setting:

(i) *Conservative Solutions.* One can think of all energy $\|q(t)\|^2_{L^2}$ as passing into a defect-measure at the moment of wave-breaking [2]. If this energy is released immediately and totally back into the solution, the solution can be continued by $q(t^*+t) = -q(t^*-t)$ for $t<t^*$, and also thereafter as $-q$ is positive and hence can be continued indefinitely. This is equivalent to continuing q by retaining the explicit formula (1) without the factor $\mathbb{1}_{\{2+V_0t>0\}}$. The energy $\|q(t)\|_{L^2}$ is conserved in this scenario.

(ii) *Dissipative Solutions.* The defect measure can also hold up all energy eternally. Then the solution can be continued as $q(t^*+t)=0$ for $t>t^*$, and all energy is dissipated at wave-breaking. The quantities $|q(t)|_{H^{-1}_{\mathrm{loc}}}$ remain locally Lipschitz in time.

This non-uniqueness persists in the stochastic setting:

(i) *"Conservative" Solutions.* Solutions can be continued by retaining (2). But away from $\sigma=0$, there is only a bound and no conservation of $\|q(t)\|_{L^2}$ either before or after wave-breaking. By setting $\sigma=0$, deterministic conservative solutions are recovered.

(ii) *Dissipative Solutions.* Because it can be shown that $\mathbb{P}$ almost surely, $u(t,x)$ has no jumps due to

$$\lim_{t\to t^*} q(t,X(t,x))\exp\left(\int_0^t q(s,X(s,x))\,\mathrm{d}s + \int_0^t \sigma(X(s,x))\circ\mathrm{d}W\right) = 0,$$

it turns out that it is possible to continue solutions beyond wave-breaking by setting it to zero. The quantities $|q(t)|_{H^{-1}_{\mathrm{loc}}}$ remain continuous.

The Case $\sigma''\not\equiv 0$

Boxes are not preserved by the flow, and even with "box"-type initial conditions, wave-breaking can occur *at a point*, that is, it cannot be ruled out that at a particular point x,

$$\frac{\partial X(t,x)}{\partial x} = 0.$$

The wave-breaking dynamics becomes substantially more complicated. However, it remains the case that when this happens – when characteristics meet – wave-breaking must also occur, and vice versa.

Conclusions

- As in the deterministic problem, solutions can be approximated by step-functions, and analysis can be focused (to some extent) on negative "box"-type initial conditions.
- In the special case $\sigma''\equiv 0$, wave-breaking dynamics take on some properties of their deterministic counterpart. And the distributions of a number of associated quantities can be derived very explicitly.
- In the general case $\sigma\in W^{2,\infty}(\mathbb{R})$, wave-breaking dynamics becomes complicated, but can still be analysed by considering stochastic characteristics.

Forthcoming Research

We intend to extend this analysis to another notable nonlocal wave model exhibiting wave-breaking behaviour in the deterministic setting, the (stochastic) Camassa-Holm equations (as given in [3]):

$$0 = \mathrm{d}m + [\partial_x(um) + m\partial_x u]\,\mathrm{d}t + [\partial_x(\sigma m) + m\partial_x\sigma]\circ\mathrm{d}W,$$
$$m = u - \partial_x^2 u.$$

Acknowledgements

This research is supported by the Research Council of Norway under the project *Waves and Nonlinear Phenomena (WaNP)*.

References

[1] D. Alonso-Oran, A. Bethencourt de Léon, and S. Takao. The Burgers' equation with stochastic transport: shock formation, local and global existence of smooth solutions. *arXiv:1808.07821v1* 1 – 30.

[2] A. Bressan, H. Holden, and X. Raynaud. Lipschitz metric for the Hunter-Saxton equation. *J. Math. Pures Appl.* **94** (2010), 68 – 92.

[3] D. Holm, and T. Tyranowski. Variational principles for stochastic soliton dynamics. Proc. Royal Soc. A, **472 (2187)** (2016), 1 – 24.

[4] B. Gess, and M. Maurelli. Well-posedness by noise for scalar conservation laws. *Comm. P.D.E.*,**43(12)**(2018) 1702 – 1736.

[5] J.K. Hunter, and Y.-X. Zheng. On a nonlinear hyperbolic variational equation:I. Global existence of weak solutions. *Arch. Rat. Mech. Anal.***129** (1995) 305 – 353.

[6] H. Matsumoto, and M. Yor. Exponential functionals of Brownian motion, I: probability laws at fixed time. *Probability Surveys*, **2** (2005), 312 – 347.