A Physical Discussion of the Riemann Problem

Shallow Water Equations

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Abstract

This poster will briefly discuss the scenarios in which the Riemann problem for the shallow water system arises in a physically reasonable sense. Similar work has been carried out by Peregrine [5] and Leveque in [4]. They considered two colliding bores where one overtakes the other, creating a new shock with a left and right state defining a Riemann problem. We would like to build on that work by finding other regions for which the Riemann problem can originate and more importantly when it should not be observed in nature.

Shallow water equations

The study of the Riemann problem is important when trying to understand the behaviour of conservation laws. For example, it can used as a numerical method by decomposing general initial values into piecewise constants and then solve a series of Riemann problems [3]. The shallow water equations are given by

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{1}{2}gh^2 \end{pmatrix}_x = 0,$$

which describes conservation of mass and momentum, $q = hu$. The Riemann initial data in one space dimension is defined as follows,

$$\begin{pmatrix} h \\ hu \end{pmatrix}(x,0) = \mathbf{u}(x,0) = \begin{cases} \mathbf{u}_L & \text{for } x < 0, \\ \mathbf{u}_R & \text{for } x > 0. \end{cases}$$

These equations are discussed in detail by for example Whitham [1]. Imposing the entropy conditions and the Rankine-Hugoniot condition one can find a unique solution [2] for any right state, (h_R, q_R) and a given left state, (h_L, q_L). Drawn in phase space with momentum q on the vertical axis and height h on the horizontal axis is shown below.

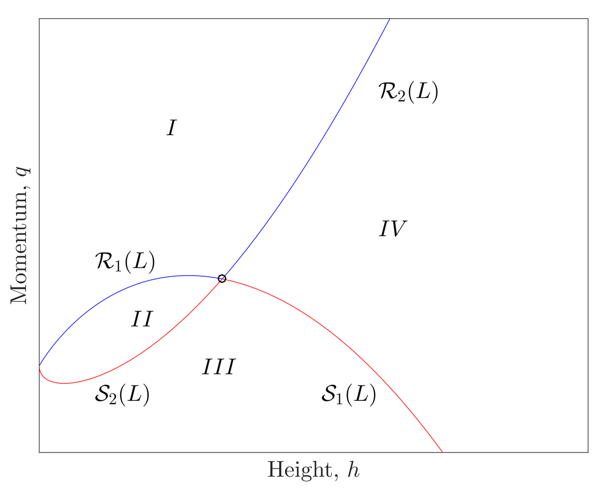

Figure 1: Phase space for a particular left state (h_L, q_L)

For any right state and a particular left state, the solution is given by a combination of rarefaction waves described by the curves

- $\mathcal{R}_1(L): \quad u = u_L - 2\sqrt{gh} + 2\sqrt{gh_L}, \quad u > u_L,$
- $\mathcal{R}_2(L): \quad u = u_L + 2\sqrt{gh} - 2\sqrt{gh_L}, \quad u > u_L,$

and the shock curves

- $\mathcal{S}_1(L): \quad u = u_L - (h - h_L)\sqrt{\frac{g}{2}(\frac{1}{h} + \frac{1}{h_L})}, \quad u < u_L,$
- $\mathcal{S}_2(L): \quad u = u_L + (h - h_L)\sqrt{\frac{g}{2}(\frac{1}{h} + \frac{1}{h_L})}, \quad u < u_L.$

Note that we have divided the phase space in Figure 1 into four regions. In addition, the entropy solution for each region is found by two elementary waves going through some middle state. For instance, we connect $\mathcal{S}_1(L)$ with $\mathcal{S}_2(M)$ to any right state in region III as drawn in $(h, u)-$coordinates below.

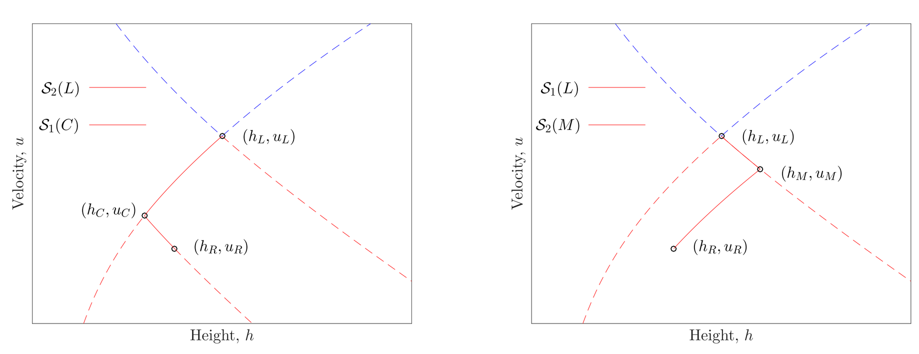

Figure 2: Collision of two bores (left) and solution of the initial value problem (on the right)

The solution is given in the figure on the right and satisfies the Rankine-Hugoniot and entropy condition [2]. On the other hand, the figure on the left represents two colliding bores connected by a center state (h_C, u_C) and creates the initial value problem.

Also note that for particular solutions in region I, there are states connected by a middle state with zero height [2]. This is known as the cavitation state in gas dynamics [4] and is a well defined concept. However, separating two counter-propagating waves with a dry area in between does not seem reasonable in shallow water theory. It is therefore interesting to see if these types of solutions are excluded when imposing the condition of having a Riemann problem from colliding bores.

Development of the Riemann problem

Collision of $\mathcal{S}_1$ and $\mathcal{S}_2$ shocks

Lets now consider the origin of the Riemann problem as a combination of traveling bores. First, we will prove that certain right states in region III emanate from two counter-propagating bores, one on $\mathcal{S}_1$ and one on the $\mathcal{S}_2$ curve. Secondly, a short argument will be given for why it cannot happen in any other regions.

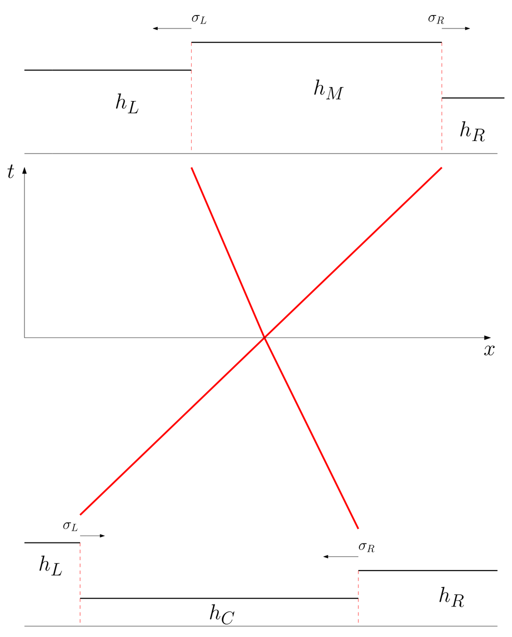

Figure 3: Two colliding bores forming the Riemann Problem

Figure 3 displays the special case in Figure 2 with two bores colliding at $t = 0$ which then forms the initial value problem. The general statement is formulated below.

Theorem 1: For a given left state and a right state in region III there exists a center state connecting them via a $\mathcal{S}_2 - \mathcal{S}_1$ collision creating the Riemann problem by two counter-propagating bores if the momentum of the center state is greater than the right state.

Proof. We need to prove there is a center state connecting two colliding shock waves satisfying the bore conditions. Guided by the discussion above, we should seek a point (h_C, u_C) on $\mathcal{S}_2(L)$ giving rise to a $1-$shock, $\mathcal{S}_1(C)$ through (h_R, u_R). The existence of this center state is clear as u is strictly increasing on $\mathcal{S}_2(L)$ for $h \in (0, h_L]$ and range from $(-\infty, u_L)$, while any $1-$shock going through the right state is strictly decreasing and unbounded above. Consequently, we have two curves that intersects which is described by the equations

$$\mathcal{S}_2(L): \quad u_C = u_L + (h_C - h_L)\sqrt{\frac{g}{2}(\frac{1}{h_C} + \frac{1}{h_L})} \tag{1}$$

and

$$\mathcal{S}_1(C): \quad u_R = u_C + (h_R - h_C)\sqrt{\frac{g}{2}(\frac{1}{h_R} + \frac{1}{h_C})}. \tag{2}$$

Adding (1) and (2) we may solve for h_C and then use (1) again to find u_C. Furthermore, since $(h_C, u_C) \in \mathcal{S}_2(L)$ we must have $h_C < h_L$ and $u_C < u_L$. By the bore properties discussed in [6] we see that the left shock is moving to the right. Indeed since $h_C/h_L < 1$, we have from the Rankine-Hugoniot condition that the shock speed of the left most bore must satisfy

$$\sigma_L = \frac{h_L u_L - h_C u_C}{h_L - h_C} = \frac{h_L(u_L - u_C\frac{h_C}{h_L})}{h_L - h_C} > 0.$$

Now consider σ_R and note that the right state is below the $\mathcal{S}_2(L)$ curve, connected by $\mathcal{S}_1(C)$ for which u is decreasing. Hence, $h_C < h_R$ and $u_R < u_C$. With respect to the bore properties it is clear that the bore must move to the left which is only true for $q_R < q_C$. In this case we have

$$\sigma_R = \frac{q_R - q_C}{h_R - h_C} < 0.$$

As a result, we have two bores moving toward each other creating a Riemann problem in a head-on collision. □

Additionally, we will leave as a note that it is not possible to form a Riemann problem by a collision of two counter-propagating bores for a given left state if the right state is in region I, II or IV. This is a result of the entropy condition, as it turns out that any of these cases would give rise to energy creating bores.

Collision of two $\mathcal{S}_2$ shocks

There is still a number of cases to consider, but we will close this discussion with the case of two fast shocks forming a Riemann problem. A specific case is drawn in phase space below, using momentum coordinates to guide our intuition.

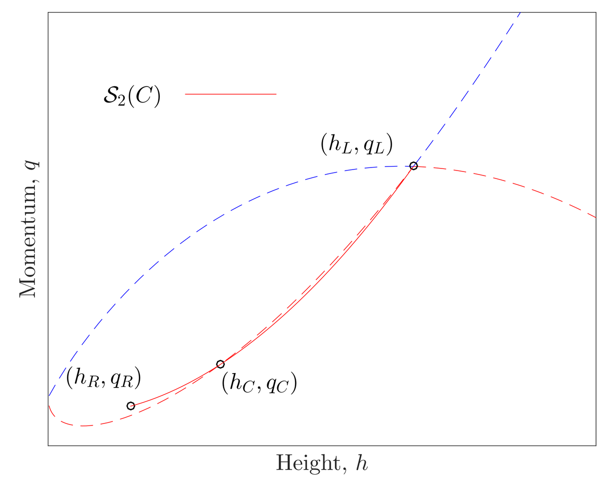

Figure 4: Development of the Riemann problem due to a collision of two $\mathcal{S}_2-$shocks

In Figure 4 we observe the relationship $h_R < h_C < h_L$ and that the line joining each state has a positive slope. This implies that both states moves in the positive direction. In addition, we need the left state to move faster, but that's clear due to $\mathcal{S}_2$ being convex i.e. the rate of change (shock speed σ) is positive. Although, we also need to choose an admissible connection which we will find in the next theorem.

Theorem 2: If $h_R < h_C < h_L$ and $u_R < u_C < u_L$ such that there is a fast shock curve, $\mathcal{S}_2(L)$ connecting left state and center state state and a $\mathcal{S}_2(C)$ curve connecting center state with right state then it creates a Riemann problem.

Proof. For a general case where this is true we would need both bores to move in a positive direction. We observe that the bore on the right moves with speed

$$\sigma_R = \frac{h_C u_C - h_R u_R}{h_C - h_R} = \frac{h_C(u_C - u_R\frac{h_R}{h_C})}{h_C - h_R} > 0.$$

Similarly for the bore on the left

$$\sigma_L = \frac{h_L u_L - h_C u_C}{h_L - h_C} = \frac{h_L(u_L - u_C\frac{h_C}{h_L})}{h_L - h_C} > 0.$$

In addition, we need the right side to be overtaken by the left side as indicated in Figure 5 below.

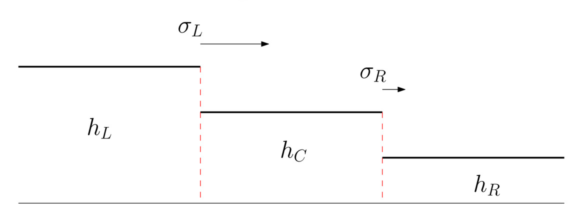

Figure 5: Two colliding bores forming the Riemann Problem

Clearly we need the left shock to be faster than the right shock. Considering the momentum coordinates we note that a fast shock is given by a convex function. Thus, the slope joining the left and center states respectively is greater than the line connecting the center state with the right whenever $h_R < h_L$. We may therefore conclude that such a scenario would create a Riemann problem. □

Conclusion

We have considered the Riemann problem associated to the shallow water equations and imposed a condition that such a problem should arise from the collision of two bores. Imposing this condition, we were able to recover solutions consisting of both rarefaction waves and shock waves, but excluded the possibility of cavitation. This is made clear in Theorem 1 when having a combination of fast and slow shocks, finding that admissible shock curves only live in region III.

References

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