Solitary waves to Degasperis-Procesi equation: exponential decay and symmetry

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Norwegian meeting on PDEs 2019

The Degaspries-Procesi (DP) equation

$$u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0$$

has nonlocal formulation

$$\partial_t u + u\partial_x u + \partial_x L(\frac{3}{2}u^2) = 0, \quad L := (1-\partial_x^2)^{-1}$$

The solitary wave $u(t,x) := \phi(x-ct)$ with speed c satisfies the steady equation

$$\frac{\phi}{3}(2c-\phi) = K * \phi^2 + a.$$

where $K(x) = e^{-|x|}$ denotes the kernel for L and a denotes integral constant.

1 Exponential decay of solitary wave

Preliminary estimate for the decay:

- The constant a is trivial actually for solitary waves.
- Positivity and strict upper bound for ϕ:

$$0 < \phi \le \sup_x \phi(x) < 2c$$

An improved convolution estimate based on [Bona-Li, 1997, JMPA]: For $0 < l < m$ and any $\sigma > 0$, the following inequality holds

$$\int_{\mathbb{R}} \frac{e^{l|x|}}{(1+\sigma e^{|x|})^m e^{m|x-y|}} dx \le B \frac{e^{l|y|}}{(1+\sigma e^{|y|})^m}, \quad y \in \mathbb{R},$$

where $B = (\min\{l, m-l\})^{-1}$.

The decay of solitary waves:
Theorem 1.1 *The map $x \mapsto e^{|x|}\phi(x) \in L^\infty(\mathbb{R},\mathbb{R})$.*

Hints for proof:

- Prove decay estimate in L^q: for any $\alpha \in (0,1)$ and $q > 1$, it is true that

$$e^{\alpha|\cdot|}\phi(\cdot) \in L^q(\mathbb{R})$$

The key estimate is

$$\int_{|y|\ge R_\delta} |\phi^2(y)|^q \left[\int_{|x|\ge R_\delta} \frac{e^{lq|x|}}{(1+\epsilon e^{|x|})^{\alpha q} e^{\alpha q|x-y|}} dx\right] dy \le \int_{|y|\ge R_\delta} |\phi^2(y)|^q \frac{B e^{lq|y|}}{(1+\epsilon e^{|y|})^{\alpha q} e^{\alpha q|y|}} dy$$

- Improve the decay in L^q to L^∞ and from $\alpha < 1$ to $\alpha = 1$.

$$e^{\alpha|x|}\phi(x) \lesssim \frac{3}{2c-M}\left[\left(e^{\alpha|\cdot|}K(\cdot)\right) * \left(e^{\alpha|\cdot|}\phi^2(\cdot)\right)\right](x) \in L^\infty(\mathbb{R})$$

2 Symmetry and one-side of monotonicity of solitary waves

Definition 1 (Super-solution & sub-solution) *A solution ϕ to the steady Degasperis-Procesi equation is called a* supersolution *if*

$$\frac{\phi}{3}(2c-\phi) \ge K * \phi^2$$

and a subsolution *if the inequality above is replaced by $\le$.*

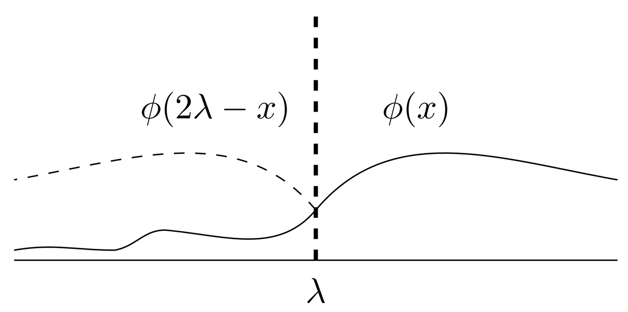
Lemma 2.1 (Touching lemma) *Let ϕ_1 and ϕ_2 be a super– and a subsolution of the steady Degasperis-Procesi equation on a subset $[\lambda,\infty) \subset \mathbb{R}$, respectively, satisfying $\phi_1 \ge \phi_2$ on $[\lambda,\infty)$ and $\phi_1^2 - \phi_2^2$ being odd with respect to λ, that is $(\phi_1^2-\phi_2^2)(x) = -(\phi_1^2-\phi_2^2)(2\lambda - x)$. Then either*

- *$\phi_1 = \phi_2$ in $[\lambda,\infty)$, or*
- *$\phi_1 > \phi_2$ with $\phi_1 + \phi_2 < 2c$ in (λ,∞) .*

Theorem 2.2 (Reflection) *There exists a $N > 0$ suffciently large such that*

$$\phi(x) < \phi_\lambda(x), \quad x < \lambda,$$

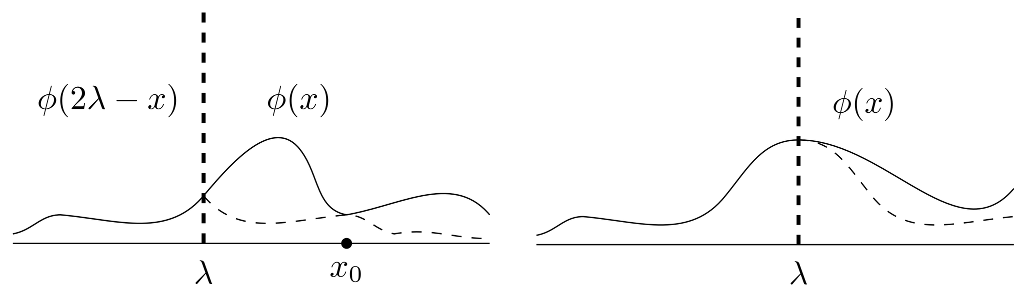
for any $\lambda \le -N$. In other words, $\Sigma_\lambda^- = \emptyset$ for any $\lambda \le -N$.



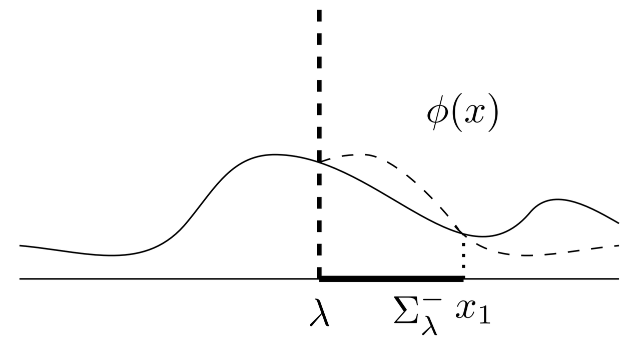
2.1 Symmetry of waves below the maximum height

Hints for proof:

- Make reflection of the solitary wave at $x = \lambda$ so that $\phi(x)$, $x < \lambda$, stays strictly below its reflection $\phi(2\lambda - x)$
- Move the line $x = \lambda$ to the right until it reaches a local maximum of ϕ or the reflection $\phi(2\lambda - x)$,$x < \lambda$, touches ϕ at some point $x = x_0$



- Exclude the possibility that the reflection $\phi(2\lambda - x)$,$x < \lambda$, touches ϕ at some point $x = x_0$
- Prove that if $x = \lambda$ reaches a local maximum of ϕ then it is a global maximum and the wave is symmetric



$$[2c-(\phi(x)+\phi_\lambda(x))](\phi_\lambda - \phi)(x) \le 3\int_{\Sigma_\lambda^-} \big(K(x-y) - K(2\lambda - x - y)\big)(\phi_\lambda^2(y) - \phi^2(y)) dy$$

2.2 Symmetry of waves of the maximum height

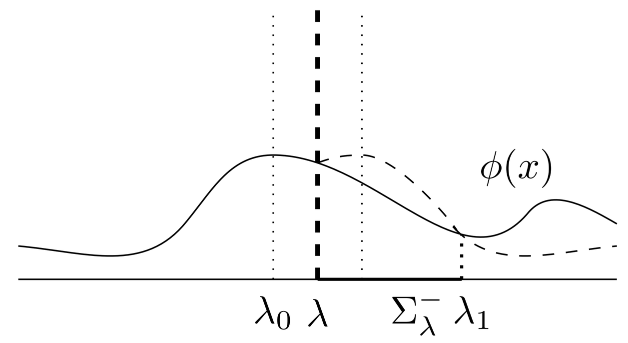
Key ingredient in proof: compare the contribution of

$$2c - (\phi(x) + \phi(2\lambda - x))$$

and

$$K(x-y) - K(2\lambda - x - y)$$

on Σ_z.



3 Symmetric waves are traveling waves

Symmetric solutions $u(t,x) = u(t, 2\lambda(t) - x)$ satisfies

$$u_t + \dot{\lambda} u_x = 0,$$
$$-\dot{\lambda} u_x + uu_x + 3L(uu_x) = 0.$$

We discover that these two equations determine the form and propagation speed of waves, respectively.