The optimal convergence rate of monotone schemes for conservation laws in the Wasserstein distance

Adrian Montgomery Ruf
joint work with Espen Sande and Susanne Solem

Introduction

In their 1994 paper, Nessyahu, Tadmor and Tassa [3] showed that a large class of monotone finite volume methods converge to the entropy solution of the hyperbolic conservation law

$$\begin{aligned} u_t + f(u)_x &= 0, \qquad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{1}$$

at a rate of $O(\Delta x)$ in the 1-Wasserstein distance W_1 under the assumption that f is strictly convex and the initial datum u_0 is compactly supported and Lip^+-bounded.
Recently, Fjordholm and Solem [1] showed a convergence rate of $O(\Delta x^2)$ in W_1 for initial data consisting of finitely many shocks. This raises the question whether the first-order rate in W_1 of [3] can be improved. In this paper we show that this is not possible. by constructing a suitable counterexample.

Heuristic argument

Our proof is based on the following heuristic argument. Monotone schemes provide approximations of the type shown in Figure 1 and the W_1-distance can be thought of as measuring the minimal amount of work needed to move mass from one place to another.

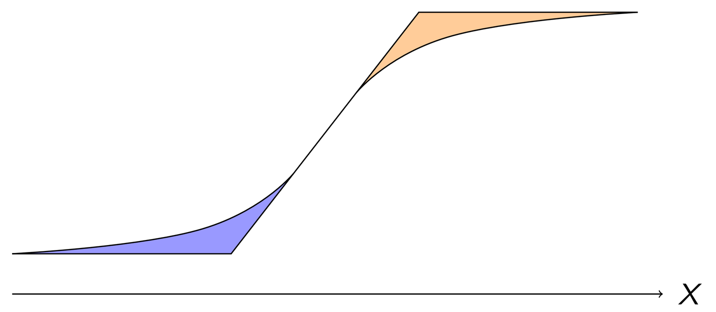

Figure 1: Exact and approximate solution of (1)

Since the surplus of mass on the left (blue area) is $O(\Delta x)$ and it needs to be moved a distance of $O(1)$ to the shortage of mass on the right (orange area) we expect the W_1-error to be no better than $O(\Delta x) \cdot O(1) = O(\Delta x)$ in this case.

Outline of the proof

- We first write the W_1 distance as the L^1 norm of $E(x,t) = \int_{-\infty}^{x} (u(y,t) - v(y,t))\,dy$ and show that E satisfies a certain transport equation.
- Next, we prove some elementary properties of the cell averaging operator $\mathcal{A}$ that is used in the Godunov scheme.
- We then show that the W_1 error is bounded from below by a sum over all preceding time steps of W_1 errors between the cell averaging operator of the numerical solution and the numerical solution itself, i.e.,

$$W_1(u(t), u_{\Delta x}(t)) \geq \sum_{n=0}^{N} W_1(\mathcal{A}\tilde{u}_{\Delta x}(t^n-), \tilde{u}_{\Delta x}(t^n-)).$$

- Lastly we use the fact that the Godunov scheme evolves exactly in time to show that these W_1 errors accumulate to $O(\Delta x)$, more precisely,

$$W_1(\mathcal{A}\tilde{u}_{\Delta x}(t^n-), \tilde{u}_{\Delta x}(t^n-)) \geq C \Delta t \Delta x.$$

UiO : University of Oslo

The **first-order convergence rate** of monotone schemes for conservation laws in the **Wasserstein distance**, proved in 1994, is **optimal.**

Take a picture to
download the full paper
or go to `arxiv.org/abs/1808.04661`

Numerical validation

We consider two numerical experiments using Burger's equation,

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

with two different initial data. The first initial datum (Figure 2 top, gray) is compactly supported Lip^+-bounded initial datum. The second initial datum (Figure 2 bottom, gray) is Lip^+-unbounded and convergence rate results in this case are unknown.

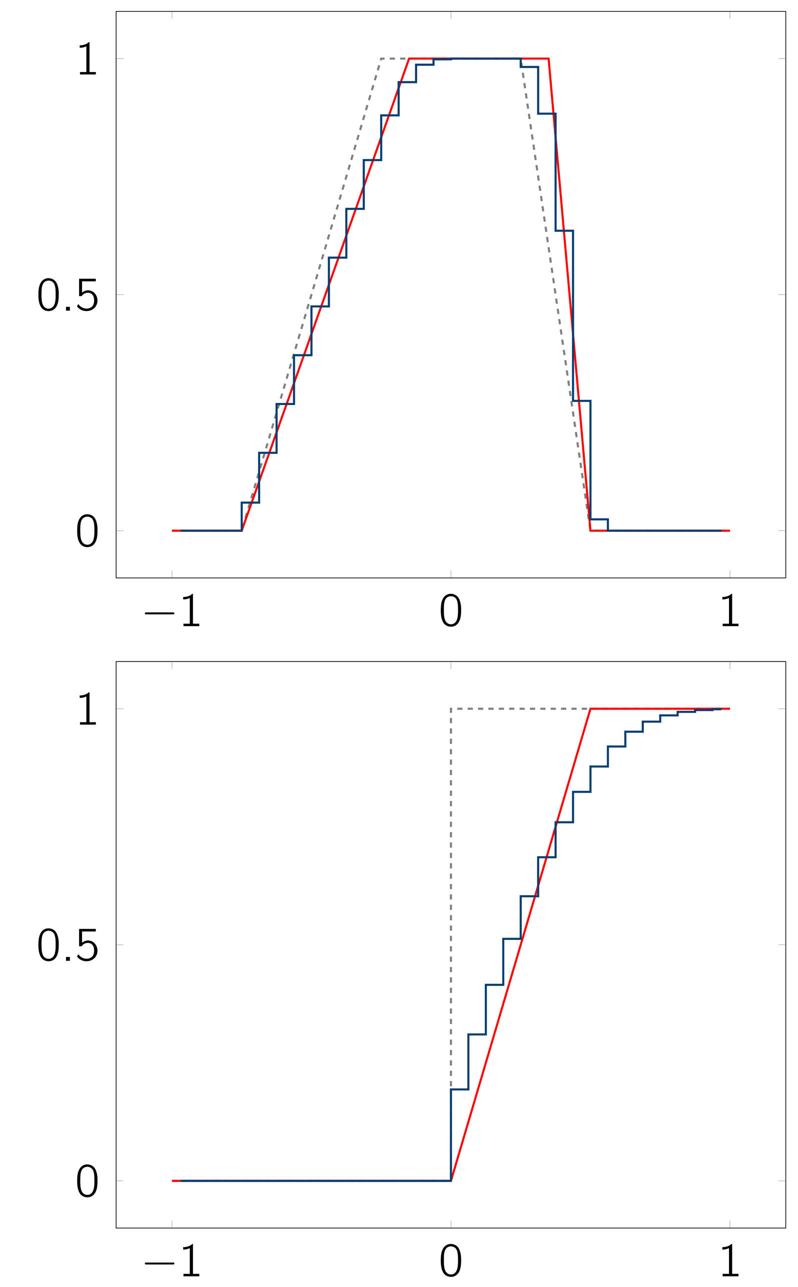

Figure 2: Exact solution, numerical approximation and initial datum for Experiment 1 and 2.

Table 1 (left) numerically illustrates the optimality result of this work. Table 1 (right) indicates that in the case of a single upward jump, i.e., Lip^+-unbounded initial datum, we can expect a convergence rate of $O(\Delta x |\log \Delta x|)$ not only in L^1 as shown by Harabetian [2], but also in W_1. This is consistent with the rate $O(\varepsilon |\log \varepsilon|)$ in W_1 proved in [4] for the viscous regularization of conservation laws with Lip^+-unbounded initial data.

n	L^1 OOC	W_1 OOC
32	0.822	1.196
64	0.896	1.123
128	0.861	1.075
256	0.884	1.046
512	0.900	1.029

n	L^1 OOC	W_1 OOC
32	0.598	0.764
64	0.641	0.759
128	0.675	0.761
256	0.708	0.769
512	0.739	0.782

Table 1: Observed order of convergence in L^1 and W_1 for Experiment 1 and 2.

References

[1] U. S. Fjordholm and S. Solem, *Second-order convergence of monotone schemes for conservation laws*, SIAM J. Numer. Anal., 54 (2016), pp. 1920–1945.

[2] E. Harabetian, *Rarefactions and large time behavior for parabolic equations and monotone schemes*, Comm. Math. Phys., 114 (1988), pp. 527–536.

[3] H. Nessyahu, E. Tadmor, and T. Tassa, *The convergence rate of Godunov type schemes*, SIAM J. Numer. Anal., 31 (1994), pp. 1–16.

[4] H. Nessyahu and T. Tassa, *Convergence rate of approximate solutions to conservation laws with initial rarefactions*, SIAM J. Numer. Anal., 31 (1994), pp. 628–654.