



Sharp Error Estimates for Spline Approximation

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Abstract

In this poster we provide a priori error estimates in standard Sobolev (semi-)norms for approximation in spline spaces of maximal smoothness on arbitrary grids. The error estimates are expressed in terms of a power of the maximal grid spacing, an appropriate derivative of the function to be approximated, and an explicit constant which is, in many cases, sharp. Some of these error estimates also hold in proper spline subspaces, which additionally enjoy inverse inequalities. Furthermore, we address spline approximation of eigenfunctions of certain differential operators, with a particular focus on the special case of periodic splines.

The results presented in this poster can be used to theoretically explain the benefits of spline approximation under k -refinement by isogeometric discretization methods. They also form a theoretical foundation for the outperformance of smooth spline discretizations of eigenvalue problems that has been numerically observed in the literature, and for optimality of geometric multigrid solvers in the isogeometric analysis context.

Introduction

Suppose $\tau := (\tau_0, \dots, \tau_N)$ is a knot vector such that

$$a := \tau_0 < \tau_1 < \dots < \tau_{N-1} < \tau_N := b,$$

and let $I_j := [\tau_j, \tau_{j+1})$, $j = 0, 1, \dots, N-2$, and $I_{N-1} := [\tau_{N-1}, \tau_N]$. For any $p \geq 0$, let $\mathcal{P}_p$ be the space of polynomials of degree at most p . Then, for $-1 \leq k \leq p-1$, we define the space $\mathcal{S}_{p,\tau}^k$ of splines of degree p and smoothness k by

$$\mathcal{S}_{p,\tau}^k := \{s \in C^k[a, b] : s|_{I_j} \in \mathcal{P}_p, j = 0, 1, \dots, N-1\}.$$

In the case of maximal smoothness we set

$$\mathcal{S}_{p,\tau} := \mathcal{S}_{p,\tau}^{p-1}.$$

Main Results

Define the projection $Q_p : H^1(a, b) \rightarrow \mathcal{S}_{p,\tau}$ by

$$\begin{aligned} (\partial Q_p u, \partial v) &= (\partial u, \partial v), \quad \forall v \in \mathcal{S}_{p,\tau}, \\ (Q_p u, 1) &= (u, 1). \end{aligned} \quad (1)$$

Theorem 1. For any knot vector τ , let h denote its maximum knot distance. Then, for any function $u \in H^r(a, b)$,

$$\begin{aligned} \|u - Q_p u\| &\leq \left(\frac{h}{\pi}\right)^r \|\partial^r u\|, \quad \forall p \geq r-1, \\ \|\partial(u - Q_p u)\| &\leq \left(\frac{h}{\pi}\right)^{r-1} \|\partial^r u\|, \quad \forall p \geq r-1. \end{aligned}$$

Here $\|\cdot\|$ denotes the L^2 norm. For $r = 1$ the above error estimates are satisfied for certain reduced spline spaces related to the optimal spline spaces in Figure 2.

Let

$$\begin{aligned} H_{\text{per}}^r &:= \{u \in H^r(a, b) : \partial^\alpha u(a) = \partial^\alpha u(b), \alpha = 0, 1, \dots, r-1\}, \\ \mathcal{S}_{p,\tau,\text{per}} &:= \{s \in \mathcal{S}_{p,\tau} : \partial^\alpha s(a) = \partial^\alpha s(b), \alpha = 0, 1, \dots, p-1\}, \end{aligned}$$

and define the projection $\tilde{Q}_p : H_{\text{per}}^1 \rightarrow \mathcal{S}_{p,\tau,\text{per}}$ analogously to (1).

Theorem 2. For any knot vector τ , let h denote its maximum knot distance. Then, for any function $u \in H_{\text{per}}^r$,

$$\begin{aligned} \|u - \tilde{Q}_p u\| &\leq \left(\frac{h}{\pi}\right)^r \|\partial^r u\|, \quad \forall p \geq r-1, \\ \|\partial(u - \tilde{Q}_p u)\| &\leq \left(\frac{h}{\pi}\right)^{r-1} \|\partial^r u\|, \quad \forall p \geq r-1. \end{aligned}$$

If τ is uniform and $\dim \mathcal{S}_{p,\tau,\text{per}} = 2m$, then the above result is optimal in the n -width sense.

Kolmogorov n -Widths

For a subset A of $L^2(a, b)$ and an n -dimensional subspace $\mathcal{X}_n$ of $L^2(a, b)$, let

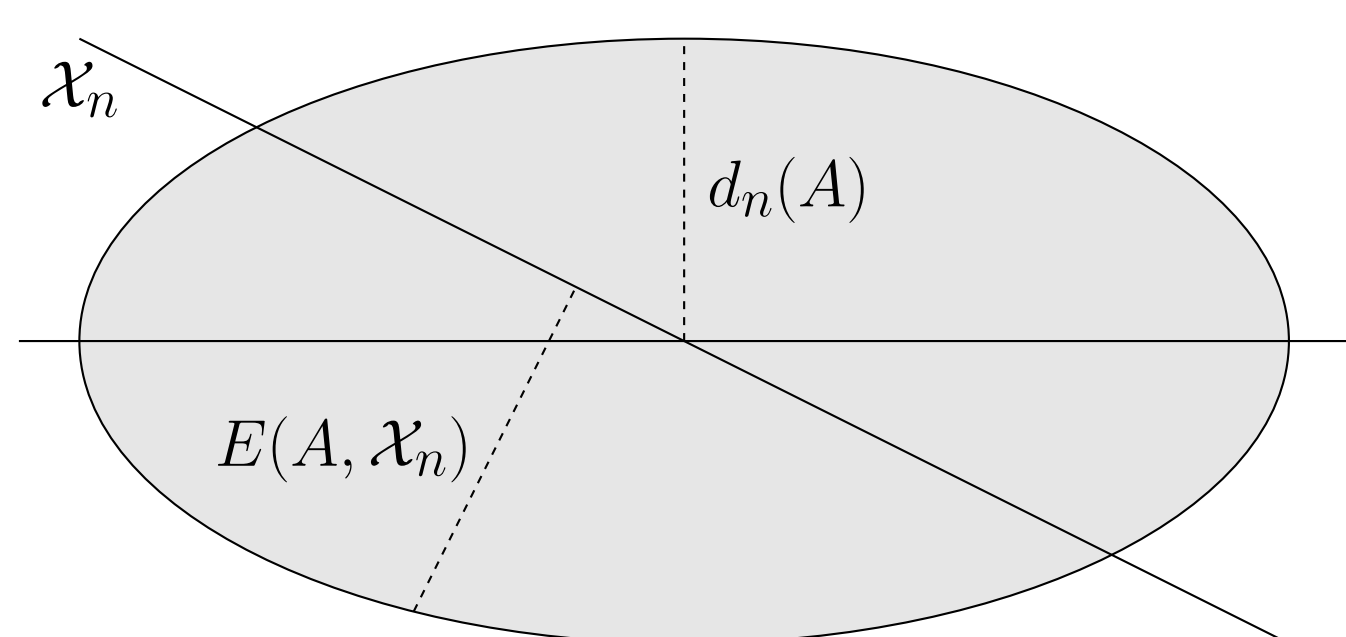
$$E(A, \mathcal{X}_n) := \sup_{u \in A} \inf_{v \in \mathcal{X}_n} \|u - v\|$$

be the distance to A from $\mathcal{X}_n$. Then the Kolmogorov n -width of A is defined by

$$d_n(A) := \inf_{\mathcal{X}_n} E(A, \mathcal{X}_n).$$

A subspace $\mathcal{X}_n$ is called an optimal subspace for A provided that

$$d_n(A) = E(A, \mathcal{X}_n).$$



Let

$$A^r := \{u \in H^r(a, b) : \|u^{(r)}\| \leq 1\}.$$

Then for any subspace $\mathcal{X}_n \subset L^2(a, b)$ we have the sharp error estimate

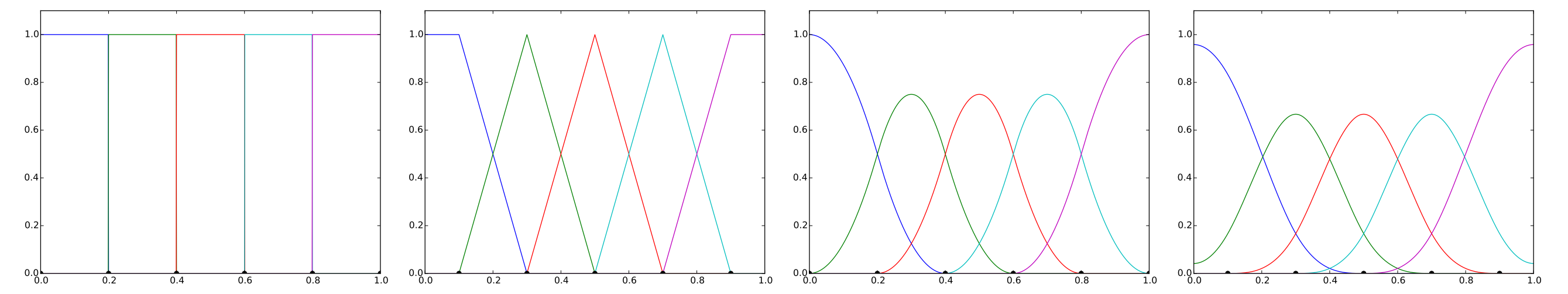
$$\inf_{v \in \mathcal{X}_n} \|u - v\| \leq E(A^r, \mathcal{X}_n) \|u^{(r)}\|.$$

Here $E(A^r, \mathcal{X}_n)$ is the least possible constant for the subspace $\mathcal{X}_n$. Moreover, if $\mathcal{X}_n$ is optimal for A^r , then

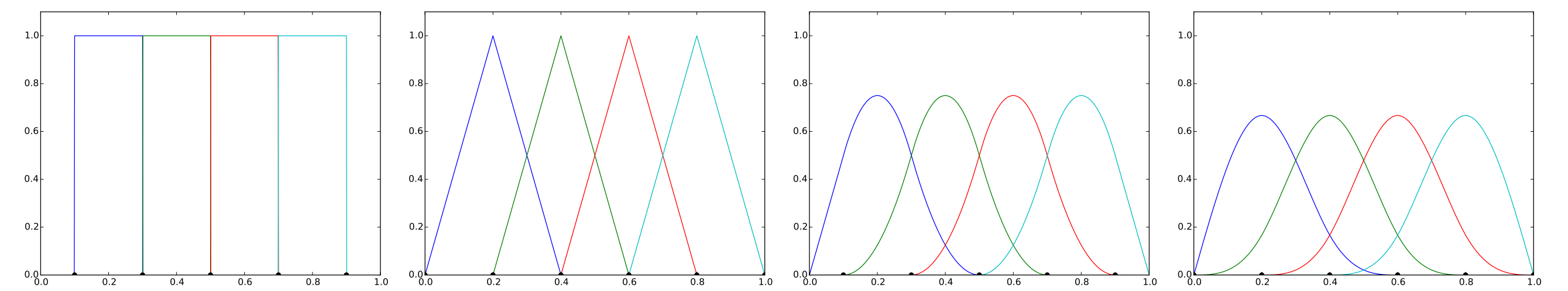
$$\inf_{v \in \mathcal{X}_n} \|u - v\| \leq d_n(A^r) \|u^{(r)}\|,$$

and $d_n(A^r)$ is the least possible constant over all n -dimensional subspaces of $L^2(a, b)$.

Optimal Spline Spaces



Basis functions for the optimal spline spaces for A^1 of degree 0 to 3 with knot-distance 0.2 ($n = 5$).



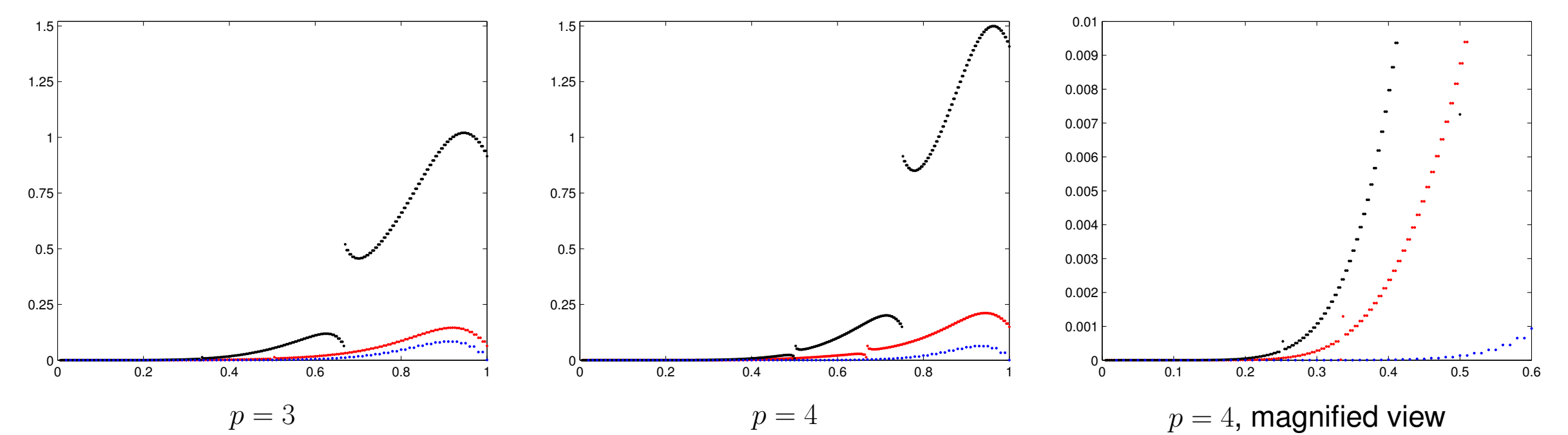
Basis functions for optimal spline spaces for A_0^1 of degree 0 to 3 with knot-distance 0.2 ($n = 4$).

Convergence to Eigenfunctions

Suppose τ is a uniform knot vector and let $n = \dim \mathcal{S}_{p,\tau,\text{per}}$. The error estimates in Theorem 2 can then be used to prove convergence, in p , of all the eigenvalues and eigenfunctions of the discrete eigenvalue problem: find $u_j^h \in \mathcal{S}_{p,\tau,\text{per}}$ and $\lambda_j^h \in \mathbb{R}$, $j = 0, 1, \dots, n-1$, such that

$$(\partial u_j^h, \partial v) = \lambda_j^h (u_j^h, v), \quad \forall v \in \mathcal{S}_{p,\tau,\text{per}}.$$

Similar results hold for the optimal spline spaces in Figures 2 and 3.

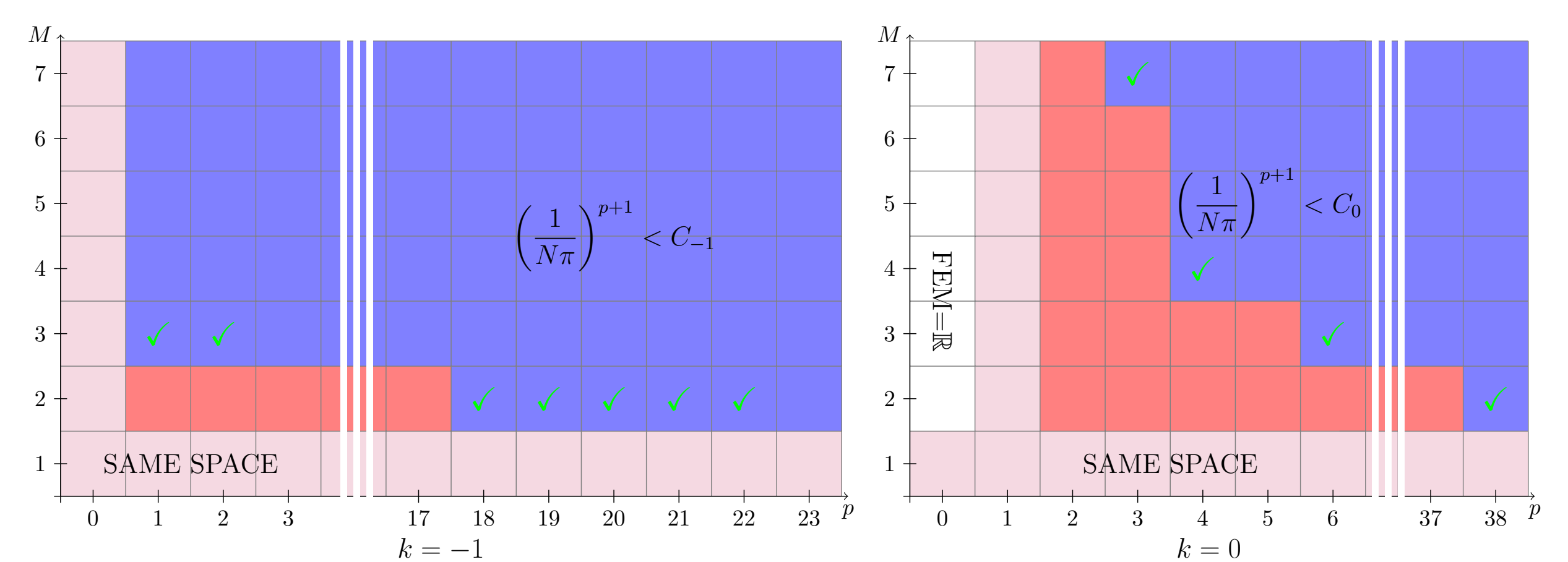


Discretization of the periodic eigenvalue problem in the space $\mathcal{S}_{p,\tau}^k \cap H_{\text{per}}^1$ with fixed $n = 100$ and varying k ($k = 0$ in black, $k = 1$ in red, $k = p-1$ in blue): relative eigenvalue approximation error $\lambda_j^h / \lambda_j - 1$, $j = 2, \dots, n(p-k)$, where each λ_j^h denotes the approximated value of the j -th eigenvalue λ_j .

Comparing Error Estimates

Let τ be the uniform knot vector on N segments and let ξ be the uniform knot vector on M segments. Defining $C_k := E(A^{p+1}, \mathcal{S}_{p,\xi}^k)$ we compare a lower bound on C_k , $k = -1, 0$, with the upper bound on C_{p-1} in Theorem 1, under the constraint

$$\dim \mathcal{S}_{p,\tau} = \dim \mathcal{S}_{p,\xi}^k.$$



The blue areas indicate for which p and M we can conclude that approximation with maximally smooth splines is better than DG approximation ($k = -1$) and FEM approximation ($k = 0$) with the same number of degrees of freedom. The red areas indicate where no conclusion can be obtained from our estimates. The spaces coincide in the pink areas.

References

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