Convergence rates of finite volume methods for conservation laws

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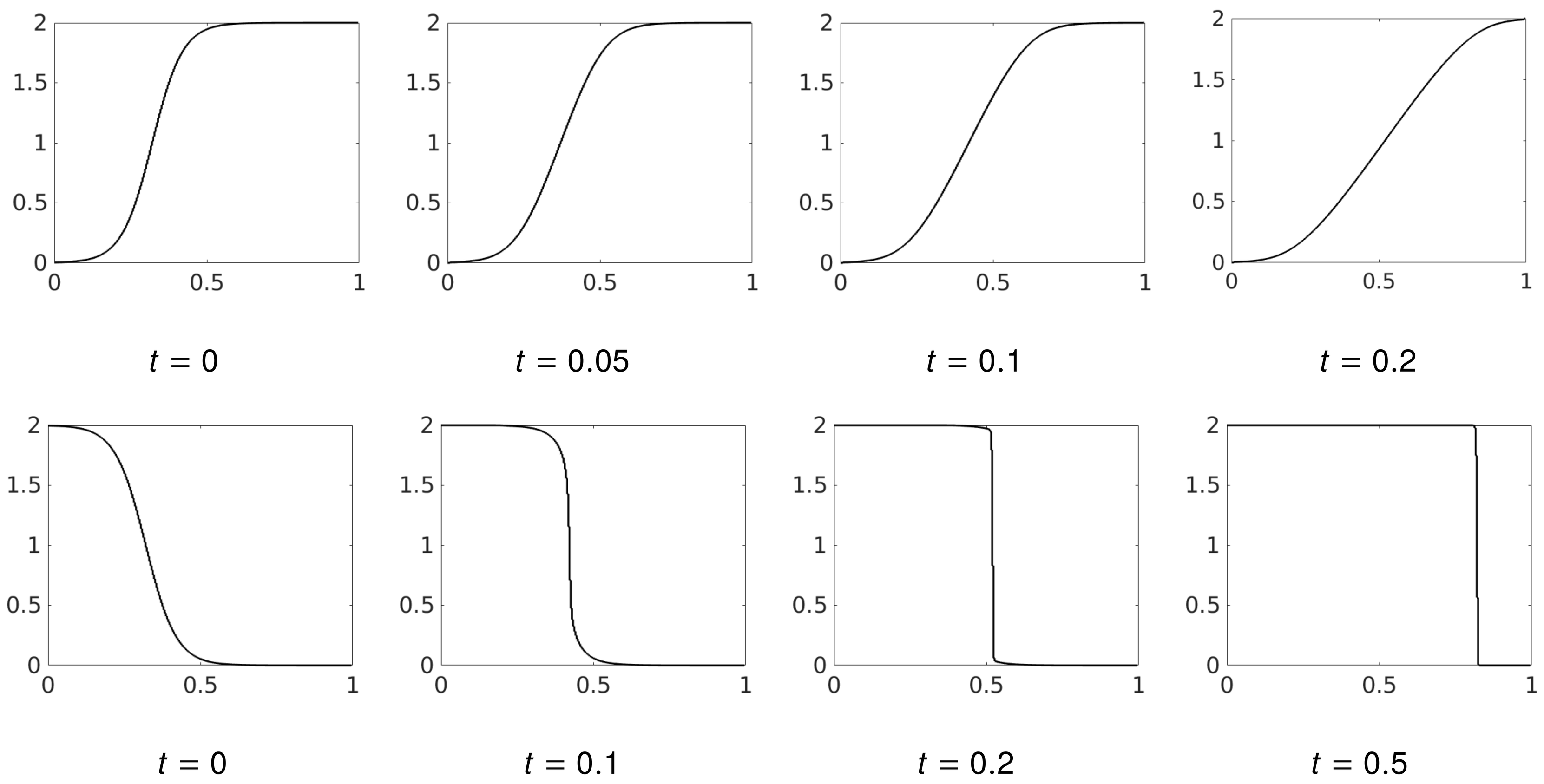
Equation and numerical methods

Conservation laws of the form

$$\begin{aligned} u_t + f(u)_x &= 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}_+, \\ u(x,0) &= u_0(x), \end{aligned} \tag{CL}$$

exhibit two main types of behaviour:



Finite volume methods (FVMs) aims at approximating

$$\frac{1}{\Delta x}\int_{C_i} u(x,t^n)\,dx, \qquad C_i = [x_{i-1/2}, x_{i+1/2}),$$

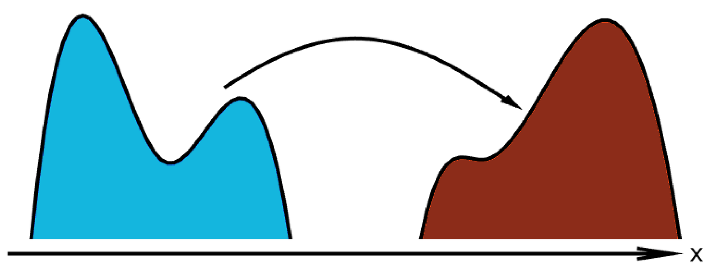
$$x_{i+1/2} - x_{i-1/2} = \Delta x, \qquad t^n = n\Delta t.$$

Metric

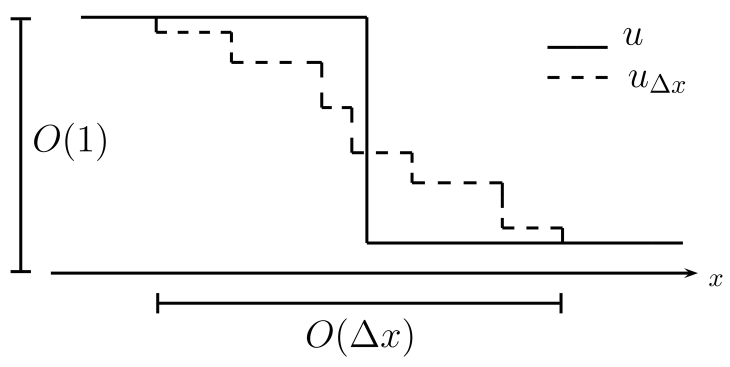
The Wasserstein distance

$$W_1(\mu,\nu) = \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \int_{\mathbb{R}} \varphi(x)\,d(\mu-\nu)(x)$$

measures the minimum amount of work (mass×distance) needed to move the probability density of μ (blue) onto the one of ν (red).

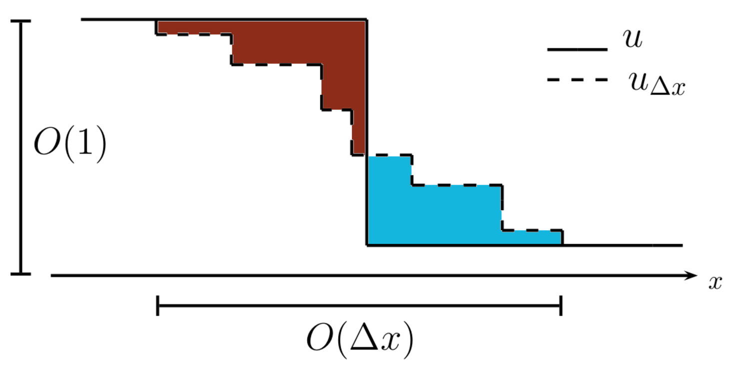


For u and the approximation $u_{\Delta x}$, W_1 measures the work needed to move the excess of mass (blue) to the shortage (red).

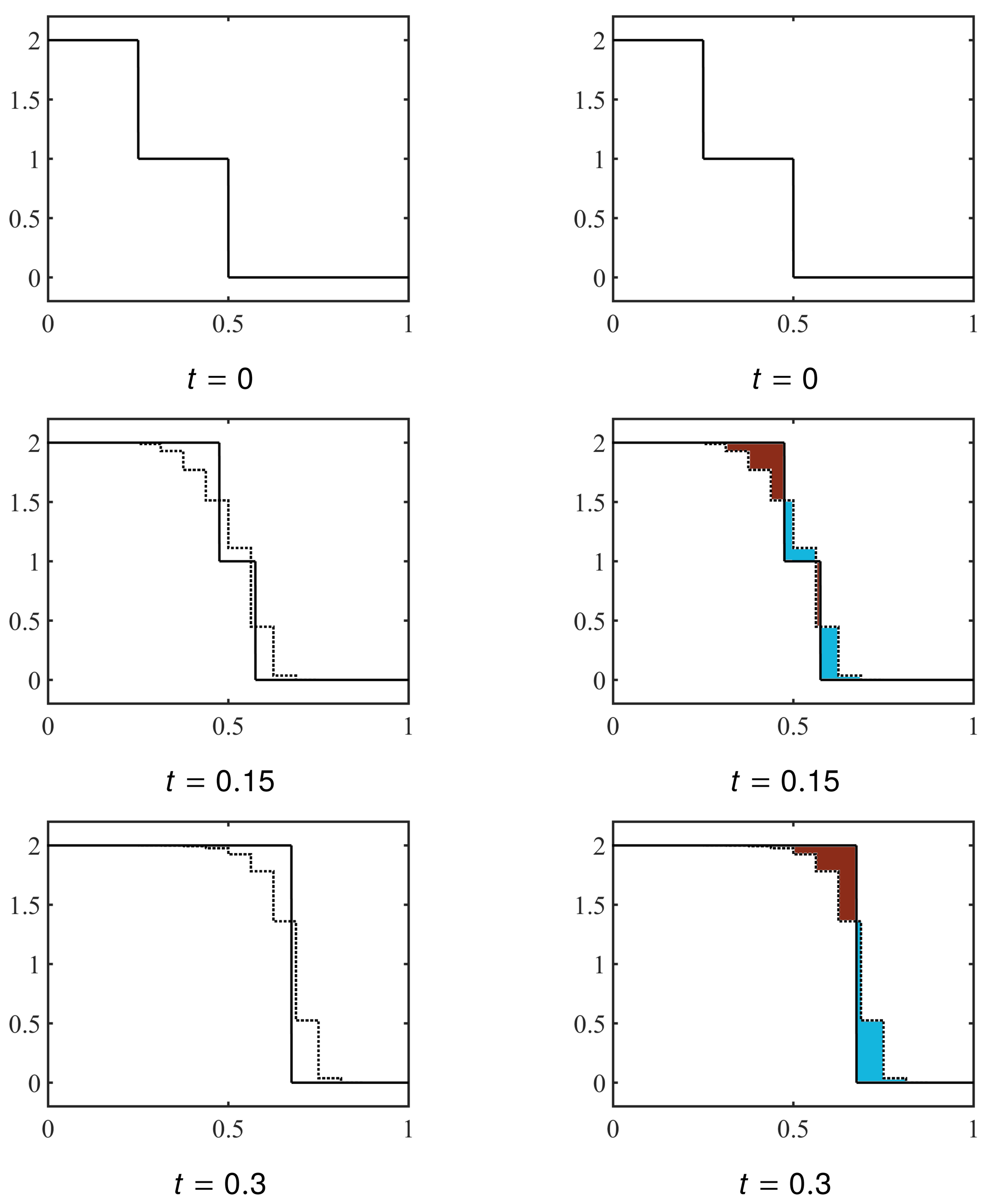


Area: $\|u - u_{\Delta x}\|_{L^1} = O(\Delta x)$

Work: $W_1(u, u_{\Delta x}) = O(\Delta x^2)$



Convergence rates



Monotone FVMs and shocks

$$u_0(x) = \begin{cases} u^{(0)}, & x < x^1, \\ u^{(k)}, & x^k \leqslant x < x^{k+1}, \\ u^{(K)}, & x^K \leqslant x. \end{cases}$$

Theorem ([1])
A large class of monotone finite volume schemes will converge to the exact solution of (CL) at a rate of Δx^2 in W_1.

Reconstruction based FVMs and shocks

Increase the formal order by reconstruction:

$$\mathcal{R}u_{\Delta x}(x,t^n) = u_i^n + \sigma_i^n(x - x_i), \qquad x \in C_i,$$

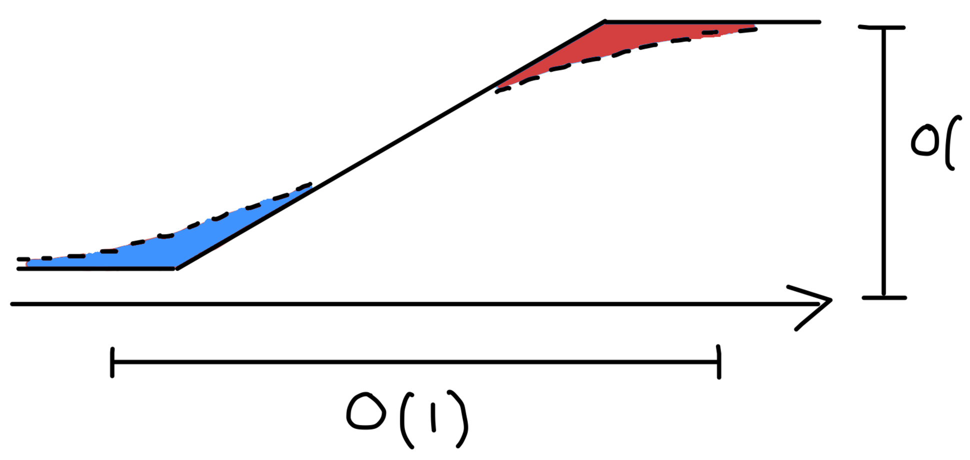
where σ_i^n is a (carefully chosen) slope.

Theorem ([4])
With a suitable minmod type slope σ_i^n,

$$W_1(u(t), u_{\Delta x}(t)) \leqslant C\Delta x^2.$$

Monotone FVMs and general initial data

Theorem ([2], [3])
The optimal convergence rate of monotone FVMs is $O(\Delta x)$ in W_1.



$$L^1 \sim O(\Delta x) \quad \rightarrow \quad W_1 \sim O(\Delta x)$$

References

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[2] H. Nessyahu, E. Tadmor, and T. Tassa. The convergence rate of Godunov type schemes. *SIAM J. Numer. Anal.*, 31(1):1–16, 1994.

[3] A. M. Ruf, E. Sande, and S. Solem. The optimal convergence rate of monotone schemes for conservation laws in the Wasserstein distance. *ArXiv e-prints*, arXiv:1808.04661, 2018.

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Comments

- The results depend on the flux being convex.
- Convergence rates of reconstruction based FVMs when u_0 is nondecreasing?
- Convergence rates for 1D systems or multi-dimensional scalar conservation laws?