Singular solutions for a system of conservation laws with vanishing buoyancy

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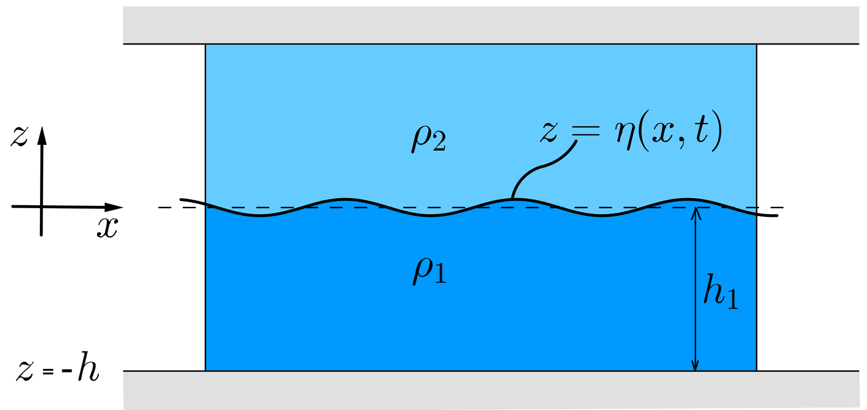
Introduction

Global warming and climate change are now understood to be partly caused by larger concentrations of CO_2 in the atmosphere and stabilizing the level of CO_2 in the atmosphere has been the focus of a large body of research. One potential method of reducing the rise of atmospheric CO_2 levels is to capture it in fossil-burning processes and sequester it elsewhere. Potential storage sites include depleted petroleum and natural gas reservoirs, saline aquifers, unminable coal beds and the world's oceans. While the oceans are the largest potential reservoir for dissolved CO_2, it appears to be preferable from ecological and climate considerations to store CO_2 in undissolved form.

It is well established in [3] that at predominant oceanic temperatures, CO_2 condenses to the liquid phase at depths of about 400m. Due to the relatively higher compressibility of liquid CO_2 than seawater, liquid CO_2 is denser than seawater at about 3000m depth. Storage of liquid CO_2 in the deep ocean is thus at least theoretically possible at depths exceeding 3000m.

The changes in the CO_2 density imply that at a certain depth it will coincide with the density of the ambient seawater. Moreover, unexpected large changes in the temperature of the ambient seawater may render a previously stable configuration unstable by making the CO_2 buoyant or neutrally buoyant. In the present work we focus on the borderline case of vanishing buoyancy which leads to a two-fluid system with fluids of equal density.



For a large underwater pool of CO_2, long waves will be the dominant wave phenomenon so we restrict our considerations to a shallow-water-like system of equations of the form (see [1])

$$\eta_t + u_x + (\eta u)_x = 0,$$
$$u_t + g\frac{\rho_1-\rho_2}{\rho_1}\eta_x + uu_x = 0.$$

In the neutrally buoyant case, the densities ρ_2 and ρ_1 will be equal and the system reduces to a triangular system of conservation laws of the form

$$u_t + uu_x = 0, \tag{1}$$
$$\eta_t + u_x + (\eta u)_x = 0. \tag{2}$$

This system is derived as a model for internal waves at the interface of a two-fluid system where a finite uniform layer fluid of density ρ_1 and approximate depth h_1 is located below an upper layer of density ρ_2 and very large depth as shown in the figure above.

Objective and Approach

- Investigate wave propagation at the interface of a two-fluid system.
- Derive a shallow-water-like system of equations to analyse the interface wave.
- Solve the Riemann problem for the derived system and show that a unique solution can be constructed in all cases.
- Present a δ-shock solution as a combination of a Dirac-δ distribution and a shock wave.
- Verify the solution in the context of the weak asymptotic method

The Riemann problem

The system (1), (2) is of the general form

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0,$$

where

$$\mathbf{U} = \begin{pmatrix} u \\ \eta \end{pmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} u^2/2 \\ (\eta+1)u \end{pmatrix}.$$

The flux Jacobian of $\mathbf{F}(\mathbf{U})$ is given by

$$J = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{pmatrix} u & 0 \\ \eta+1 & u \end{pmatrix},$$

Repeated eigenvalue: $\lambda_{1,2} = u$.

Corresponding right eigenvectors: $r_1 = (0,1)^T$.

For an arbitrary constant left state (u_L, η_L) and a right state (u_R, η_R), the Rankine-Hugoniot conditions for (1), (2) are respectively

$$-c[u] + [u^2/2] = 0, \tag{3}$$
$$-c[\eta] + [(\eta+1)u] = 0, \tag{4}$$

where $[u] = u_R - u_L$ and $[\eta] = \eta_R - \eta_L$.
The shock speed in (3) is well known and has the form

$$c = (u_L + u_R)/2 \equiv \bar{u}, \tag{5}$$

and satisfies the Lax entropy condition

$$\lambda_i(u_R) \le c \le \lambda_i(u_L), \quad i = 1,2.$$

From (4) we obtain the condition: $\eta_R = -(\eta_L + 2)$
For constant states u_L, u_R, η_L and η_R, let the initial data for (1), (2):

$$u(\xi,0) = \begin{cases} u_L, & \text{if } \xi < 0, \\ u_R, & \text{if } \xi > 0, \end{cases} \qquad \eta(\xi,0) = \begin{cases} \eta_L, & \text{if } \xi < 0, \\ \eta_R, & \text{if } \xi > 0, \end{cases} \tag{6}$$

Theorem 1 *Let the constant states u_L, u_R, η_L and η_R be given such the (6) represents Riemann initial data for the system (1), (2).*

(a) If $u_L > u_R$, then u has a single shock whereas η has a single jump

$$\eta(x,t) = \begin{cases} \eta_L, & \text{if } x < \bar{u}t, \\ \eta_R, & \text{if } x > \bar{u}t, \end{cases}$$

together with a propagating Dirac mass whose strength is given by

$$[w] = (t/2)\big((u_L - u_R)(\eta_L + \eta_R + 2)\big). \tag{7}$$

(b) If $u_L < u_R$, then the weak solution of u is a rarefaction whereas η has two jump discontinuities given by

$$\eta(x,t) = \begin{cases} \eta_L, & \text{if } \quad x < u_L t, \\ -1, & \text{if } u_L y \le x \le u_R t, \\ \eta_R, & \text{if } \quad u_R t < x. \end{cases} \tag{8}$$

Proof: Define a function $w(x,t)$ by $w(x,t) = \int_{-\kappa}^{x} \eta(s,t)ds$ Transform (1), (2) into:

$$u_t + uu_x = 0, \tag{9}$$
$$w_t + uw_x = -u. \tag{10}$$

Riemann initial data for w:

$$w(\xi,0) \equiv w_0(\xi) = \begin{cases} \eta_L\kappa + \eta_L\xi, & \text{if } \xi \le 0, \\ \eta_L\kappa + \eta_R\xi, & \text{if } \xi \ge 0. \end{cases} \tag{11}$$

Case I: $u_L > u_R$
The solution of (9) is given by a shock wave travelling at a speed given in (5). The characteristics for (10) are

$$x(t) = \begin{cases} u_L t + \xi, & \text{if } \xi < (\bar{u} - u_L)t, \\ u_R t + \xi, & \text{if } \xi > (\bar{u} - u_R)t, \end{cases}$$

The solution for $w(x,t)$:

$$w(x,t) = \begin{cases} \eta_L\kappa + \eta_L(x - u_L t) - u_L t, & \text{if } x \le \bar{u}t, \\ \eta_R\kappa + \eta_R(x - u_R t) - u_R t, & \text{if } x > \bar{u}t. \end{cases}$$

Since the initial assumption is that $u_L > u_R$, the characteristics emanating from $\xi < 0$ will propagate values of $w(x,t)$ which are different from those propagated by the characteristics originating from $\xi > 0$. The characteristics together with the initial Riemann data (11) give

$$w_L = \eta_L\kappa + \big(\eta_l(\bar{u} - u_L) - u_L\big)t,$$
$$w_R = \eta_L\kappa + \big(\eta_R(\bar{u} - u_R) - u_R\big)t.$$

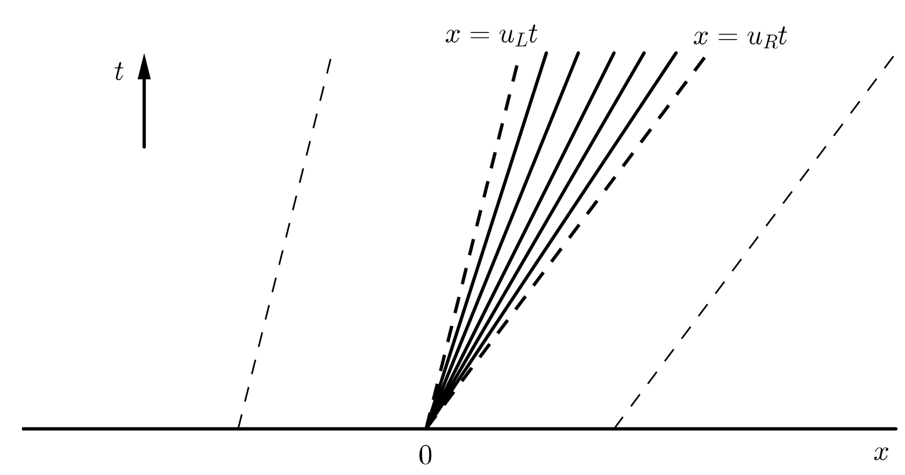
The strength of the jump:

$$[w] = w_R - w_L = \frac{t}{2}(u_L - u_R)\big(\eta_L + \eta_R + 2\big). \tag{12}$$

Case II: $u_L < u_R$
The solution of (9) is a rarefaction wave. The characteristic equations for (10):

$$x(t) = \begin{cases} u_R t + \xi, & \text{if } \quad \xi > 0, \\ \gamma t, & \text{if } \quad \xi = 0, \qquad u_L < \gamma < u_R \\ u_L t + \xi, & \text{if } \quad \xi < 0, \end{cases}$$

A graphical representation of the characteristics:



The solution of (10) is

$$w(x,t) = \begin{cases} \eta_L\kappa + \eta_L(x - u_L t) - u_L t, & \text{if } \quad x < u_L t, \\ \eta_L\kappa - x, & \text{if } \quad u_L t \le x \le u_R t, \\ \eta_l\kappa + \eta_R(x - u_R t) - u_R t, & \text{if } \quad u_R t < x. \end{cases}$$

A partial derivative with respect to x gives the solution of $\eta(x,t)$.

Weak asymptotic solution

Definition 1 *Let $f_\varepsilon(x,t) \in \mathscr{D}'(\mathbb{R})$ denote a collection of distributions which depend on $\varepsilon \in (0,1)$. If the estimate*

$$\langle f_\varepsilon(x,t), \varphi(x)\rangle = o(1), \quad \text{as} \quad \varepsilon \to 0, \tag{13}$$

holds uniformly for any test function $\varphi(x) \in \mathscr{D}(\mathbb{R})$, then we have $f_\varepsilon = o_{\mathscr{D}'}(1)$.

Definition 2 *The family of smooth, complex-valued (real-valued) distributions (u_ε) and (η_ε) is a weak asymptotic solution to the system (1), (2) if u, $\eta \in C(\mathbb{R}_+; \mathscr{D}'(\mathbb{R}))$ are real-valued distributions such that*

$$u_\varepsilon \rightharpoonup u, \quad \eta_\varepsilon \rightharpoonup \eta \quad \text{as} \quad \varepsilon \to 0,$$

holds for any fixed $t \in (0,\infty)$ in the sense of distributions in $\mathscr{D}'(\mathbb{R})$ and

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{1}{2}\frac{\partial u_\varepsilon^2}{\partial x} = o_{\mathscr{D}'}(1), \tag{14}$$
$$\frac{\partial \eta_\varepsilon}{\partial t} + \frac{\partial\big((\eta_\varepsilon + 1)u_\varepsilon\big)}{\partial x} = o_{\mathscr{D}'}(1). \tag{15}$$

In addition, the initial data must satisfy

$$u_\varepsilon(\xi,0) \to u(\xi,0) \quad \text{and} \quad \eta_\varepsilon(\xi,0) \to \eta(\xi,0),$$

Theorem 2 *Let the constant states u_L, u_R, η_L and η_R be given such that (6) represents Riemann initial data for the system (1), (2) and c is the admissible shock speed given in (5). Then there exist weak asymptotic solutions u_ε and η_ε such that the families (u_ε) and (η_ε) have distributional limits given by*

$$u(x,t) = u_L + (u_R - u_L)H(x - ct), \tag{16}$$
$$\eta(x,t) = \eta_L + (\eta_R - \eta_L)H(x - ct) + \alpha(t)\delta(x - ct), \tag{17}$$

where H is the Heaviside function, δ is the Dirac delta distribution, and

$$\alpha(t) = [w].$$

Proof: Define an approximate delta distribution

$$\delta_\varepsilon(x,t) = \frac{1}{2\varepsilon}\rho\left(\frac{x - ct - 3\varepsilon}{\varepsilon}\right) + \frac{1}{2\varepsilon}\rho\left(\frac{x - ct + 3\varepsilon}{\varepsilon}\right).$$

Define a regularized smooth function

$$H_\varepsilon(x,t) = \begin{cases} 1, & \text{if } \quad x \le ct - 10\varepsilon, \\ \frac{1}{2}, & \text{if } \quad ct - 5\varepsilon < x < ct + 5\varepsilon, \\ 0, & \text{if } \quad x \ge ct + 10\varepsilon, \end{cases}$$

which continuous smoothly in $(-10\varepsilon, -5\varepsilon)$ and $(5\varepsilon, 10\varepsilon)$. Then:

$$\delta_\varepsilon(x - ct) \rightharpoonup \delta(x - ct) \quad \text{as} \quad \varepsilon \to 0 \tag{18}$$
$$H_\varepsilon(x - ct)\delta_\varepsilon(x - ct) \rightharpoonup \frac{1}{2}\delta(x - ct) \tag{19}$$
$$H_\varepsilon(x,t)\frac{\partial H_\varepsilon(x,t)}{\partial x} = \frac{1}{2}\delta_\varepsilon(x,t) + o_{\mathscr{D}'}(1) \tag{20}$$

We start with the singular ansatz:

$$u_\varepsilon(x,t) = u_L + (u_R - u_L)H_\varepsilon(x - ct), \tag{21}$$
$$\eta_\varepsilon(x,t) = \eta_L + (\eta_R - \eta_L)H_\varepsilon(x - ct) + \alpha(t)\delta_\varepsilon(x - ct). \tag{22}$$

Equation (2) becomes:

$$\begin{aligned}(\eta_R - \eta_L)\partial_t H_\varepsilon + \alpha'(t)\delta_\varepsilon - c\alpha(t)\delta' + (u_R - u_L)\partial_x H_\varepsilon + u_L(\eta_R - \eta_L)\partial_x H_\varepsilon \\ + u_L\alpha(t)\delta' + \eta_L(u_R - u_L)\partial_x H_\varepsilon + (u_R - u_L)(\eta_R - \eta_L)\partial_x H_\varepsilon^2 \\ + \frac{\alpha(t)}{2}(u_R - u_L)\delta' = o_{\mathscr{D}'}(1).\end{aligned}$$

By using (18)–(20), it follows from Definition 2 that

$$\alpha'(t) = \frac{1}{2}(u_L - u_R)\big(\eta_L + \eta_R + 2\big),$$

Conclusion

- A hyperbolic system arising in the study of long waves in two-fluid systems has been studied. The system is not strictly hyperbolic and hence, the standard theory of hyperbolic conservation laws cannot be used to find admissible weak solutions.
- However, the structure of the system makes it possible to reformulate (2) in terms of the primitive w of the unknown η.
- An exact weak solution to the Riemann problem associated to the original system (1), (2) has been found by solving a transport equation to obtain unique solutions.
- The unique solutions to the hyperbolic system is given by a singular solution featuring a Dirac delta distribution whose strength is $\alpha(t)$ travelling with the Delta shock.
- The solution is redefined in terms of the theory of weak asymptotic [2] solutions leading to a Delta shock with a strength $\alpha(t)$, where the derivative $\alpha'(t)$ represents the Rankine–Hugoniot deficit

References

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