Travelling waves in non-local dispersive equations

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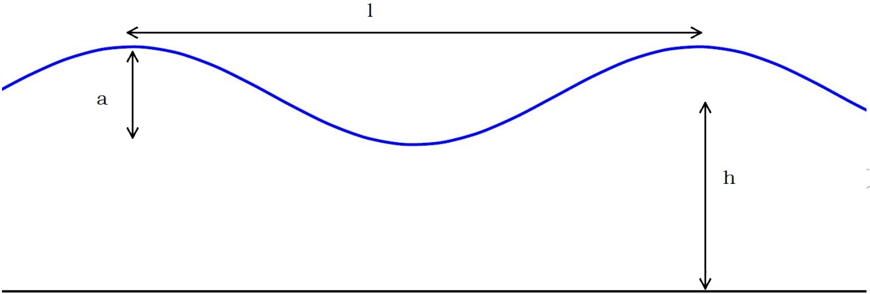

- *h* - depth of water at rest
- *a* - amplitude
- *l* - wavelength

In an incompressible and inviscid fluid with uniform and constant density ($=1$), the flow is governed by the Euler equations:

$$\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\nabla P + \vec{g},$$
$$\nabla \cdot \vec{v} = 0,$$

where $\vec{v}$ is the flow velocity vector, P is the mechanic pressure and $\vec{g}$ represents the body accelerations.

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— Often desirable to work with a simpler equation which approximates the Euler equations in a given setting.

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An Example

One of the most famous model equations is the Kortweg-deVries equation:

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Derived from the Euler equations under the assumption that

$$\frac{a}{h} \quad \text{and} \quad \left(\frac{h}{l}\right)^2$$

are small and equal.
That is, it is a model for the shallow-water/long-wave regime.

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More generally, we can consider equations of the form

$$u_t + n(u)_x + L(u)_x = 0, \tag{2}$$

where

- $u\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,
- n is a non-linear term, e.g. $n(u) = u^p$, $p > 1$,
- L is a Fourier multiplier operator: $\widehat{Lf}(\xi) = m(\xi)\widehat{f}(\xi)$.

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Can also consider

$$u_t + n(u)_x + L(u^2)_x = 0$$

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From (2) we get

$$-c\varphi + n(\varphi) + L(\varphi) = 0. \tag{3}$$

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We are interested in:

— Periodic waves,

— Solitary waves: $\lim_{x\to\pm\infty} \varphi(x) = 0$,

— Highest waves (peaked/cusped).

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Peaked/cusped waves cannot exist if L is differentiating:

$$(c - L)\,\varphi = \frac{1}{2}\varphi^2$$

if $c - m \neq 0$ and $L\colon\ C^{\alpha}(\mathbb{R}) \to C^{\alpha-s}(\mathbb{R})$, for some $s > 0$, then $\varphi \in C^{\infty}(\mathbb{R})$.

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For example, the KdV equation has no maximal height.

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Hence, if $\varphi < c$, and $L\colon C^\alpha(\mathbb{R}) \to C^{\alpha+r}(\mathbb{R})$, $r > 0$, then

$$\varphi = c - \sqrt{c^2 - 2L(\varphi)} \quad \Rightarrow \quad \varphi \in C^\infty.$$

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Clearly the same argument also works for

$$-c\varphi + \frac{1}{2}\varphi^2 + L(\varphi^2) = a \quad (DP)$$

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Examples:

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We will focus particularly on the Degasperis-Procesi equation:

$$-c\varphi + \frac{1}{2}\varphi^2 + \frac{3}{2}L(\varphi^2) = a, \quad m(\xi) = \frac{1}{1+\xi^2}. \tag{4}$$

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(a) An analysis of solutions that achieve the maximal height c,

(b) construction of a global curve of smooth periodic solutions $\varphi < c$ which approaches a wave of maximal height in the limit.

For $P \in (0, \infty]$, we consider non-constant P-periodic solutions φ that are:

— even,

— non-decreasing on $(-P/2, 0)$.

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$$L(f)(x) = K * f(x) = \int_{\mathbb{R}} K(x-y)f(y)\,\mathrm{d}y, \quad K = \mathscr{F}^{-1}(m)$$

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If m and φ is even and $\varphi(0) = c$, then

$$\begin{aligned}(c - \varphi(x))^2 =& L(\varphi)(0) - L(\varphi)(x) \\ =& \frac{1}{2} \int_{\mathbb{R}} (2K(y) - K(x-y) - K(x+y)) \, \varphi(y) \, \mathrm{d}y\end{aligned}$$

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If $|y| > |x|$, then

$$2K(y) - K(y-x) - K(y+x) = -\frac{x^2}{2}(K''(\xi_1) + K''(\xi_2)).$$

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If

$$m^{(j)}(\xi) \lesssim (1+|\xi|)^{-r-j}, \quad j = 0, 1(\ldots),$$

then for $|x| < 1$

$$K(x) \simeq \begin{cases} |x|^{r-1}, & \text{if } 0 < r < 1, \\ \ln\left(\frac{1}{|x|}\right), & \text{if } r = 1, \\ 1, & \text{if } r > 1. \end{cases}$$

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Hence if $\varphi(0) = c$, we expect

$$\begin{aligned} c_1|x|^r \leq |c - \varphi(x)| \leq c_2|x|^r, &\quad \text{if} \quad 0 < r < 1, \\ c_1|x| \leq |c - \varphi(x)| \leq c_2|x|, &\quad \text{if} \quad r > 1. \end{aligned}$$

Regularity at the crest

Theorem 1 (M. N. A.)

Let $\varphi \leq c$ be a P-periodic solution to (4) *which is even, non-constant and non-decreasing on $(-P/2, 0)$ with $\varphi(0) = c$. Then*

(i) *φ is smooth on $(-P/2, 0)$.*

(ii) *$\varphi \in C^{0,1}(\mathbb{R})$, i.e. φ is Lipschitz.*

(iii) *φ is exactly Lipschitz at $x = 0$; that is, there exist constants $0 < c_1 < c_2$ such that*

$$c_1|x| \leq |c - \varphi(x)| \leq c_2|x|, \quad |x| \ll 1.$$

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In particular, the DP equation does not have cuspon solutions.

Global bifurcation

For $\alpha > \min\{r, 1\}$ and $P > 0$, let

$$U := \{(\varphi, c) \in C^{\alpha}_{even}(\mathbb{S}_P) \times (0, \infty) : \varphi < c\}.$$

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The waves of maximal height lie on the boundary of U.

— Create a global curve of (smooth) solutions in U that approaches the boundary of U,

— Show that the curve approaches a wave of maximal height in the limit.

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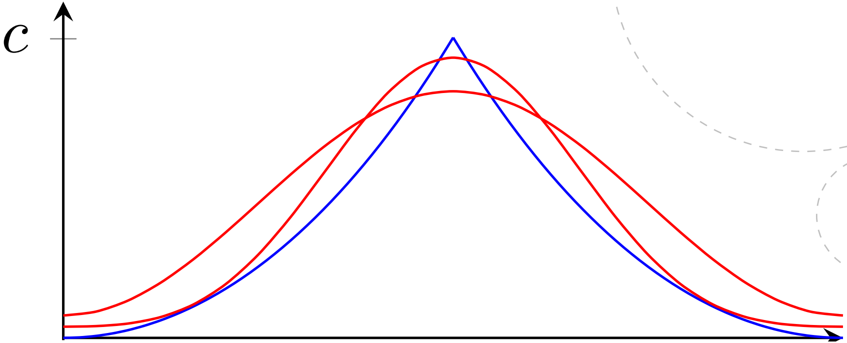

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Theorem 2

For some $c > 0$, there exists curves $s \to (\varphi(s), c(s))$, $s \geq 0$ of nontrivial solutions, bifurcating from a constant solution, such that at least one of the three alternatives hold:

(i) $\|(\varphi(s), c(s))\|_{C^\alpha(\mathbb{S}_P)\times\mathbb{R}} \to \infty$ *as* $s \to \infty$.

(ii) $(\varphi(s), c(s))$ *approaches the boundary of U as* $s \to \infty$.

(iii) *The function* $s \mapsto (\varphi(s), c(s))$ *is (finitely) periodic.*

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Use Arzela-Ascoli's theorem to show that any sequence of solutions

$$\{(\varphi_n, c_n)\}_n, \quad \{c_n\}_n \text{ bounded}$$

has a subsequence that converges uniformly to a solution.

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To obtain periodic peakons/cuspons Theorem 2, we need to show that:

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If the above holds, then (i) and/or (ii) must occur, but our previous analysis shows that both (i) and (ii) occur if and only if

$$\lim_{s \to \infty} c(s) - \varphi(s)(0) = 0,$$

which is exactly what we need.

Some results

Whitham equation: P-periodic $C^{1/2}$ cusped solutions for all periods $P > 0$. [Ehrnström, Wáhlen 2016]

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Degasperis-Procesi: P-periodic peakons for all sufficiently small $P > 0$. [M. N. A. 2019]

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Thank you for your attention!

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