Finite element systems for elasticity

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Outline

- Elasticity complexes
- New finite element discretization
- Finite element systems

Elasticity Strain Complex

- Continuous metrics:

$$\mathrm{H}^2(U, \mathbb{V}) \xrightarrow{\mathsf{def}} \mathrm{H}^1_{\mathsf{sven}}(U, \mathbb{S}) \xrightarrow{\mathsf{sven}} \mathrm{H}^0(U, \mathbb{R}). \tag{1}$$

  with:

$$\mathrm{H}^1_{\mathsf{sven}}(U, \mathbb{S}) = \{u \in \mathrm{H}^1(U, \mathbb{S}) \ : \ \mathsf{sven}\, u \in \mathrm{H}^0(U, \mathbb{R})\}. \tag{2}$$

- Exactness and rigid motions.
- Saint Venant compatibility and linearized curvature.
- Lower regularity and partitions of unity.

New finite element

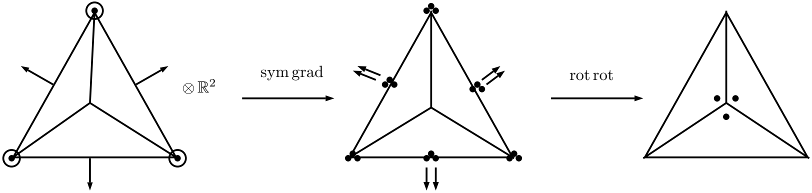


Figure: Strain complex with continuous metrics.

Spaces

- Vector valued Clough Tocher:

$$A^0(T) = \mathrm{C}^1\mathrm{P}^3(\mathcal{R}(T), \mathbb{V}), \tag{3}$$

- Continuous P^2 metrics with integrable sven $(\partial_\nu u\tau \cdot \tau)$:

$$A^1(T) = \mathrm{C}^0_{\mathsf{sven}}\mathrm{P}^2(\mathcal{R}(T), \mathbb{S}), \tag{4}$$

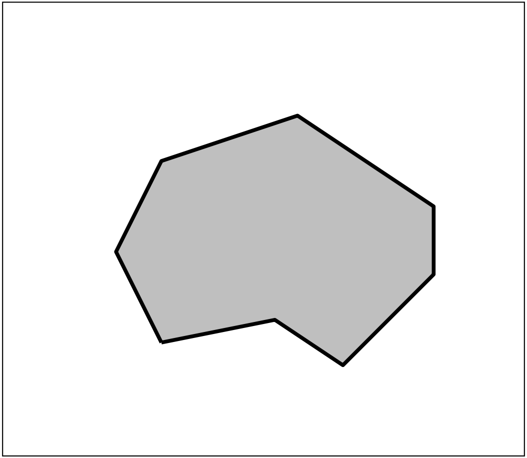
  DoFs: values at vertices (3×3),
  pairings with $M(E)$ for each edge E (3×3),
  integral against normal vector on edges (3×2).

- Piecewise constants:

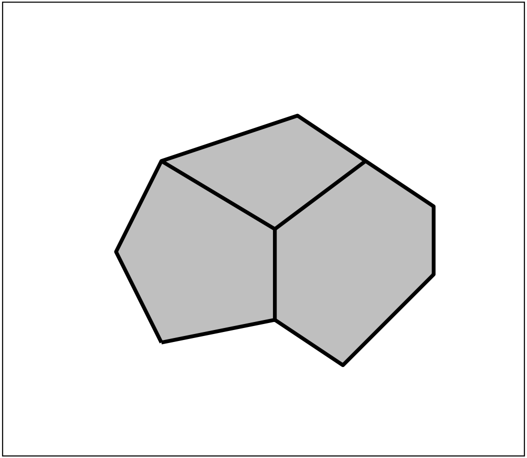
$$A^2(T) = \mathrm{P}^0(\mathcal{R}(T), \mathbb{R}), \tag{5}$$

  DoFs: integration against affine functions.

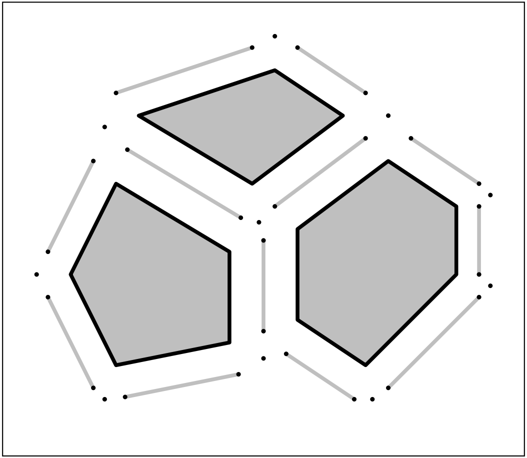
Cellular complexes



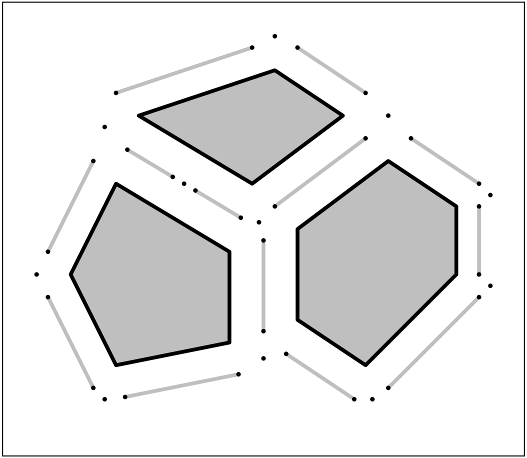
Cellular complexes



Cellular complexes



Cellular complexes



Finite element systems [C. 08, C.-Hu 18]

- Fix a cellular complex $\mathcal{T}$.
- A finite element system A is
  $A^k(T)$ for $k \in \mathbb{N}$ and $T \in \mathcal{T}$ of all dimensions.
  – differentials: $\mathrm{d} : A^k(T) \to A^{k+1}(T)$.
  – restrictions: $T' \subseteq T$ gives $\mathrm{r} : A^k(T) \to A^k(T')$.
  commutation relations. de Rham map.
- Associated global space:

  $$A^k(\mathcal{T}) = \{u \in \bigoplus_{T \in \mathcal{T}} A^k(T) \ : \ T' \subseteq T \Rightarrow u_T|_{T'} = u_{T'}\}.$$

  Encodes continuity.
- – FES is a contravariant functor
  from a cellular complex to differential complexes.
  – Global space is the inverse limit.

Induced operators

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\Gamma(T,\mathbb{V}) \arrow[r, "\mathrm{def}"] \arrow[d, "\begin{matrix} u \\ \downarrow \\ (u\cdot\tau,\, u\cdot\nu,\, \partial_\nu u\cdot\tau,\, \partial_\nu u\cdot\nu) \end{matrix}"] \arrow[dd, bend right=60, swap, "\begin{matrix} u \\ \downarrow \\ (u,\, \operatorname{grad} u) \end{matrix}"] & \Gamma(T,\mathbb{S}) \arrow[r, "\mathrm{sven}"] \arrow[d, "\begin{matrix} u \\ \downarrow \\ (u\tau\cdot\tau,\, u\tau\cdot\nu,\, u\nu\cdot\nu,\, \partial_\nu u\tau\cdot\tau) \end{matrix}"] & \Gamma(T,\mathbb{R}) \\
\Gamma(E,\mathbb{R}^2\times\mathbb{R}^2) \arrow[r, swap, "(u,v,u',v')\to(\partial_\tau u,\, \frac{1}{2}(u'+\partial_\tau v),\, v',\, \partial_\tau u')"] \arrow[d, "\begin{matrix} \downarrow \\ (u\tau+v\nu,\, \partial_\tau u\tau\tau^{\mathrm{T}}+\partial_\tau v\nu\tau^{\mathrm{T}}+u'\tau\nu^{\mathrm{T}}+v'\nu\nu^{\mathrm{T}}) \end{matrix}"] & \Gamma(E,\mathbb{R}^3\times\mathbb{R}) \arrow[d, "\begin{matrix} (u,v,w,k) \\ \downarrow \\ u\tau\tau^{\mathrm{T}}+v(\tau\nu^{\mathrm{T}}+\nu\tau^{\mathrm{T}})+w\nu\nu^{\mathrm{T}} \end{matrix}"] & \\
\Gamma(V,\mathbb{V}\times\mathbb{M}) \arrow[r, swap, "(u,v)\to\mathrm{sym}(v)"] & \Gamma(V,\mathbb{S}) &
\end{tikzcd}
$$

Cochains with coefficients

- For each $T \in \mathcal{T}$, a vectorspace $L(T)$. A discrete vectorbundle.
- When T' is a codim 1 face of T, an isomorphism $\mathrm{t}_{TT'} : L(T') \to L(T)$. A discrete connection.
- Flatness:
$$\mathrm{t}_{TT'_0}\mathrm{t}_{T'_0T''} = \mathrm{t}_{TT'_1}\mathrm{t}_{T'_1T''}. \tag{6}$$
- Cochains $\mathcal{C}^k(\mathcal{T}, L)$: $(u(T))_{T\in\mathcal{T}^k}$ such that $u(T) \in L(T)$.
- The differential $\delta^k_{\mathrm{t}} : \mathcal{C}^k(\mathcal{T}, L) \to \mathcal{C}^{k+1}(\mathcal{T}, L)$ is defined by:
$$(\delta^k_{\mathrm{t}} u)(T) = \sum_{T' \trianglelefteq T} \mathrm{o}(T, T')\mathrm{t}_{TT'}u(T'). \tag{7}$$
- Flatness gives $\delta^{k+1}_{\mathrm{t}} \circ \delta^k_{\mathrm{t}} = 0$.

FES and cochains

- $\mathrm{e} : A^k(T) \to L(T)$. Stokes: For $u \in A^{k-1}(T)$:

$$\mathrm{e}_T \mathrm{d}_T u = \sum_{T' \in \partial T} \mathrm{o}(T, T') \mathrm{t}_{TT'} \mathrm{e}_{T'} \mathrm{r}_{T'T} u. \tag{8}$$

- Commutes with differentials:

$$\mathrm{e} : A^\bullet(\mathcal{T}') \to \mathcal{C}^\bullet(\mathcal{T}', L). \tag{9}$$

- de Rham theorem:
  induces isomorphisms on cohomology groups.