

Connections between L^1 -solutions of Hamilton-Jacobi-Bellman equations and L^∞ -solutions of convection-diffusion equations

Jørgen Endal

URL: <http://folk.ntnu.no/jorgeen>

Department of mathematical sciences
NTNU, Norway

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In collaboration with
N. Alibaud and E. R. Jakobsen

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Main results

- “ L^1 -stability”/Contractions for HJB

- “ L^1 -stability”/Contractions for HJB
- “ L^∞ -stability”/Weighted L^1 -contraction for CDE

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N. ALIBAUD, JE, E. R. JAKOBSEN. **Optimal and dual stability results for L^1 viscosity and L^∞ entropy solutions.** arXiv, 2018.

How are HJB and CDE connected?

$$\text{(HJ)} \quad \begin{cases} \partial_t \psi + H(\partial_x \psi) = 0 \\ \psi(\cdot, 0) = \psi_0 \end{cases} \quad \begin{array}{l} (x, t) \in \mathbb{R} \times (0, \infty) \\ x \in \mathbb{R} \end{array}$$

$$\text{(SCL)} \quad \begin{cases} \partial_t u + \partial_x(H(u)) = 0 \\ u(\cdot, 0) = u_0 \end{cases} \quad \begin{array}{l} (x, t) \in \mathbb{R} \times (0, \infty) \\ x \in \mathbb{R} \end{array}$$

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If u is the entropy solution of (SCL), then $\psi := \int^x u$ is the viscosity solution of (HJ) with $\psi_0 := \int^x u_0$.

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So, there is a connection, and in particular, information about u will give information about ψ .

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We will study the following Cauchy problems in $\mathbb{R}^N \times (0, \infty)$:

$$\text{(HJB)} \quad \begin{cases} \partial_t \psi = \sup_{\xi \in \mathcal{E}} \{ b(\xi) \cdot D\psi + \text{tr}(a(\xi) D^2 \psi) \} \\ \psi(\cdot, 0) = \psi_0 \end{cases}$$

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Why? And how are they related?

Note that if $\text{div}(A(u)Du) = \Delta\varphi(u)$, then we replace $\text{tr}(a(\xi)D^2\psi)$ by $a(\xi)\Delta\psi$.

How are HJB and CDE connected?

The Kato inequality for (CDE): For all $0 \leq \phi \in C_c^\infty$ and all $T \geq 0$,

$$\begin{aligned} \int_{\mathbb{R}^d} |u - v|(x, T) \phi(x, T) \, dx &\leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \phi(x, 0) \, dx \\ &+ \iint_{\mathbb{R}^d \times (0, T)} \left(|u - v| \partial_t \phi(x, t) \right. \\ &\left. + q(u, v) \cdot D\phi(x, t) + \text{tr}(r(u, v) D^2 \phi(x, t)) \right) \, dx \, dt, \end{aligned}$$

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$$q_i(u, v) := \text{sign}(u - v) \int_v^u F'_i(\xi) d\xi, \quad r_{ij}(u, v) := \text{sign}(u - v) \int_v^u A_{ij}(\xi) d\xi.$$

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By approximation, we can take $\phi(x, t) = \psi(x, T - t)$ in the above.

BUT if u_0, v_0 are only bounded, then ψ needs to be integrable.

The Cauchy problems

We consider the following Cauchy problem in $\mathbb{R}^N \times (0, \infty)$:

$$(HJB) \quad \begin{cases} \partial_t \psi = \sup_{\xi \in \mathcal{E}} \{ b(\xi) \cdot D\psi + \text{tr}(a(\xi)D^2\psi) \}, \\ \psi(\cdot, 0) = \psi_0, \end{cases}$$

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where $\psi_0 \in C_b(\mathbb{R}^N) \cap "L^1(\mathbb{R}^N)"$ and

$$(H1) \quad \begin{cases} \mathcal{E} \text{ is a nonempty set,} \\ b : \mathcal{E} \rightarrow \mathbb{R}^d \text{ bounded function,} \\ a = \sigma^a (\sigma^a)^T \text{ for some bounded } \sigma^a : \mathcal{E} \rightarrow \mathbb{R}^{d \times K}, \end{cases}$$

with K being a fixed integer.

The problem is often given as

$$\partial_t \psi = H(D\psi, D^2\psi) \quad \text{with} \quad H(p, X) = \sup_{\xi \in \mathcal{E}} \{b(\xi) \cdot p + \text{tr}(a(\xi)X)\}.$$

- It is a fully nonlinear equation in nondivergence form.
- The vector b and the matrix a may degenerate.
- Classical solutions may not exist, and a.e.-solutions may be nonunique.

- The works of Crandall, Lions, Evans, Ishii, Jensen,... suggest that viscosity solutions are indeed the right solution concept: existence, uniqueness and stability in C_b .
- Viscosity solutions are pointwise solutions, and the test function test the equation at local extremal points.

We also consider the following Cauchy problem in $\mathbb{R}^N \times (0, \infty)$:

$$(CDE) \quad \begin{cases} \partial_t u + \operatorname{div} F(u) = \operatorname{div} (A(u) Du), \\ u(\cdot, 0) = u_0, \end{cases}$$

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where $u_0 \in L^\infty(\mathbb{R}^N)$ and

$$(H2) \quad \begin{cases} F \in W_{\text{loc}}^{1,\infty}(\mathbb{R}, \mathbb{R}^d), \\ A = \sigma^A (\sigma^A)^T \text{ with } \sigma^A \in L_{\text{loc}}^\infty(\mathbb{R}, \mathbb{R}^{d \times K}). \end{cases}$$

The problem was given as

$$\partial_t u + \operatorname{div} F(u) = \operatorname{div} (A(u) Du).$$

- It is an equation in divergence form.
- The vector F and the matrix A may degenerate, and we get a mixture of hyperbolic and parabolic equations. Moreover, the diffusion is anisotropic.
- Classical solutions may not exist, and distributional solutions may be nonunique.
- The works of Kružkov, Carrillo, Chen, Perthame,... suggest that entropy solutions are indeed the right solution concept: existence, uniqueness and stability in L^1 .
- Entropy solutions are “signed” distributional solutions.

Previously known L^∞ -stability for (CDE)

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Note that $\mathbf{1}_{B(x_0, R+L_F t)}$ is a “supersolution” of

$$\begin{cases} \partial_t \psi = L_F |D\psi|, \\ \psi(\cdot, 0) = \mathbf{1}_{B(x_0, R)}. \end{cases}$$



S. N. KRUŽKOV. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81(123):228–255, 1970.

Finally, finite speed of propagation is encoded in the estimate.

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When $A(u) = \varphi'(u)l$ in (CDE), we have

$$\begin{aligned} & \int_{\mathbb{R}^N} |u(x, t) - v(x, t)| \mathbf{1}_{B(x_0, R)} \, dx \\ & \leq \int_{\mathbb{R}^N} |u_0(x) - v_0(x)| \Phi(\cdot, L_\varphi t) *_x \mathbf{1}_{B(x_0, R+1+L_F t)}(x) \, dx. \end{aligned}$$

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JE, E. R. JAKOBSEN. L^1 contraction for bounded (nonintegrable) solutions of degenerate parabolic equations. *SIAM J. Math. Anal.*, 46(6):3957–3982, 2014.

Note the finite infinite speed of propagation.

Previously known L^∞ -stability for (CDE)

When $A(u)$ “general” in (CDE), we have

$$\begin{aligned} & \int_{\mathbb{R}^N} |u(x, t) - v(x, t)| e^{-|x|} dx \\ & \leq e^{(L_F + L_A)t} \int_{\mathbb{R}^N} |u_0(x) - v_0(x)| e^{-|x|} dx. \end{aligned}$$

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G.-Q. CHEN, E. DiBENEDETTO. Stability of entropy solutions to the Cauchy problem for a class of nonlinear hyperbolic-parabolic equations. *SIAM J. Math. Anal.*, 33(4):751–762, 2001.



H. FRID. Decay of Almost Periodic Solutions of Anisotropic Degenerate Parabolic-Hyperbolic Equations. In *Non-linear partial differential equations, mathematical physics, and stochastic analysis*, EMS Ser. Congr. Rep., pages 183–205. Eur. Math. Soc., Zürich, 2018.

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Can we obtain

$$\|\psi(\cdot, t) - \hat{\psi}(\cdot, t)\|_{L^1} \leq \|\psi_0 - \hat{\psi}_0\|_{L^1}$$

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Not really studied, and only some results for (HJ).



C.-T. LIN, E. TADMOR. L^1 -stability and error estimates for approximate Hamilton-Jacobi solutions. *Numer. Math.*, 87(4):701–735, 2001.

What is possible? Initial guess

Consider the eikonal equation

$$\begin{cases} \partial_t \psi = C(|\partial_{x_1} \psi| + |\partial_{x_2} \psi| + \cdots + |\partial_{x_N} \psi|), \\ \psi(\cdot, 0) = \psi_0. \end{cases}$$

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Control theory gives the following representation formula:

$$\psi(x, t) = \sup_{x+Ct[-1,1]^N} \psi_0 = \sup_{\bar{Q}_{Ct}(x)} \psi_0.$$

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Moreover,

$$\int_{\mathbb{R}^N} \sup_{\bar{Q}_r(x)} \psi(\cdot, t) dx = \int_{\mathbb{R}^N} \sup_{\bar{Q}_{r+Ct}(x)} \psi_0(x) dx \leq \tilde{C}(t) \int_{\mathbb{R}^N} \sup_{\bar{Q}_r(x)} \psi_0 dx.$$

We consider the normed space

$$L_{\text{int}}^\infty(\mathbb{R}^N) := \{\psi \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) : \|\psi\|_{L_{\text{int}}^\infty(\mathbb{R}^N)} < \infty\}$$

where

$$\|\psi\|_{L_{\text{int}}^\infty(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \text{ess sup} \frac{|\psi|}{\bar{Q}_1(x)} dx.$$

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Theorem

- L_{int}^∞ is a Banach space.
- The space L_{int}^∞ is continuously embedded into $L^1 \cap L^\infty$.
- $\int_{\mathbb{R}^N} \text{ess sup}_{\overline{Q_{r+\varepsilon}(x)}} |\psi| dx \leq C_{r,\varepsilon} \int_{\mathbb{R}^N} \text{ess sup}_{\overline{Q_r(x)}} |\psi| dx.$

The same for second order equations?

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It seems that L_{int}^∞ is a good space for (HJB).

Largest subspace of L^1 stable by the equation (HJB)

Consider a space E such that

$$\begin{cases} E \text{ is a vector subspace of } C_b \cap L^1, \\ E \text{ is a normed space,} \\ E \text{ is continuously embedded into } L^1, \end{cases}$$

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and the C_b -semigroup $G(t)$ associated with (HJB) such that

$G(t)$ maps E into itself and $G(t) : E \rightarrow E$ is continuous.

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Theorem (Best possible E , [Alibaud & JE & Jakobsen, 2018])

The space $C_b \cap L_{\text{int}}^\infty$ satisfies the above properties. Moreover, any other E satisfying the above properties is continuously embedded into $C_b \cap L_{\text{int}}^\infty$.

Theorem (L_{int}^{∞} -stability, [Alibaud & JE & Jakobsen, 2018])

Assume (H1). Then

$$\|\psi - \hat{\psi}\|_{L_{\text{int}}^{\infty}} \leq (1 + t|H|_{\text{conv}})^N (1 + \omega_N(t|H|_{\text{diff}})) \|\psi_0 - \hat{\psi}_0\|_{L_{\text{int}}^{\infty}}.$$

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- Any other viscosity solution $\hat{\psi}$ of (HJB) will satisfy $\psi \leq \hat{\psi}$. Hence, it includes ALL previous results of this type.
- To make the right-hand side finite, we could require $u_0 - v_0 \in L^1$ or $\psi_0 \in L_{\text{int}}^\infty$.

A duality result

For the respective unique solutions u, ψ of (CDE),(HJB) define

$$S(t) : u_0 \in L^\infty(\mathbb{R}^d) \mapsto u(\cdot, t) \in L^\infty(\mathbb{R}^d) \quad \forall t \geq 0,$$

$$G_{m,M}(t) : \psi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d) \mapsto \psi(\cdot, t) \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d) \quad \forall t \geq 0.$$

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Theorem (Semigroup duality [Alibaud & JE & Jakobsen, 2018])

Assume (H2), $m < M$, and consider the above semigroups. Then $\{G_{m,M}(t)\}_{t \geq 0}$ is the smallest strongly continuous semigroup on $C_b \cap L_{\text{int}}^\infty(\mathbb{R}^d)$ satisfying, for all $t \geq 0$,

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Given $S(t)$, then the above inequality characterizes $G_{m,M}(t)$.

A duality result, open problem

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Is $S(t)$ the ONLY such semigroup satisfying such an inequality? If no, which ones do?

Thank you for your attention!