Strong continuity of weak solutions to the compressible Euler equations

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Prologue

Weak continuity

$$\mathbf{U} \in C_{\rm weak}([0,T]; L^p(\Omega; R^d)),\ \ t \mapsto \int_{\mathbb{T}^d} \mathbf{U} \cdot \varphi \ {\rm d}x \in C[0,T]$$

$\varphi \in L^{p'}(\Omega; R^d)$

Strong continuity

$$\tau \in [0,T],\ \ \| \mathbf{U}(t,\cdot) - \mathbf{U}(\tau,\cdot) \|_{L^p(\Omega;R^d)} \text{ whenever } t \to \tau$$

Strong vs. weak

$$\text{strong } \Rightarrow \text{weak},\ \ \text{weak} \not\Rightarrow \text{strong}$$

Euler system for a barotropic inviscid fluid

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum equation

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \ p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1$$

Impermeability boundary conditions

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Initial conditions

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

First and Second law – energy

Energy

$$\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \begin{cases} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0 \\ \infty \text{ if } \varrho = 0, |\mathbf{m}| \neq 0 \end{cases} \quad \text{is convex l.s.c}$$

Energy balance (conservation)

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E}\frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p\frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t \mathcal{E} + \mathrm{div}_x(\mathcal{E}\mathbf{u}) + \mathrm{div}_x(p\mathbf{u}) \boxed{\leq} 0$$

$$E = \int_{\mathbb{T}^d} \mathcal{E}\ \mathrm{d}x,\ \ \partial_t E \leq 0,\ \ E(0+) = \int_{\mathbb{T}^d} \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right]\ \mathrm{d}x$$

Weak solutions

Field equations

$$\int_0^\infty \int_{\mathbb{T}^d} [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi] \ \mathrm{d}x\mathrm{d}t = - \int_{\mathbb{T}^d} \varrho_0 \varphi(0, \cdot) \ \mathrm{d}x, \ \varphi \in C^1_c([0,\infty) \times \overline{\Omega})$$

$$\int_0^\infty \int_{\mathbb{T}^d} \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

$$= - \int_{\mathbb{T}^d} \mathbf{m}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x, \ \boldsymbol{\varphi} \in C^1_c([0, T) \times \overline{\Omega}; R^N), \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Admissible weak solutions

$$\int_0^\infty \int_{\mathbb{T}^d} \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x \ \partial_t \psi \ \mathrm{d}t \geq \psi(0) \int_{\mathbb{T}^d} \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \ \mathrm{d}x$$

$$\psi \in C^1_c[0,\infty), \ \psi \geq 0$$

Riemann integrability

Class $\mathcal{R}$

The complement of the points of continuity of **U** is of zero Lebesgue measure in a domain Q

Riemann integrability

A function **U** is Riemann integrable in Q only if **U** belongs to the class $\mathcal{R}$

Oscillations

$$\operatorname{osc}[v](y) = \lim_{s \searrow 0} \left[\sup_{B((y),s)\cap\overline{Q}} v - \inf_{B((y),s)\cap\overline{Q}} v \right],$$

$$A_\eta = \left\{ (y) \in \overline{Q} \;\middle|\; \operatorname{osc}[v](y) \geq \eta \right\} \text{ is closed and of zero content}$$

$$A_\eta \subset \cup_{i \in \mathrm{fin}} Q_i, \; \sum_i |Q_i| < \delta \text{ for any } \delta > 0, \;\; Q_i \text{ - a box}$$

Main result

Theorem

Let $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$, and E_0 be given such that

$$\varrho_0 \in \mathcal{R},\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

$$0 \leq E \leq \overline{E},\ E \in \mathcal{R}.$$

Then there exists a positive constant E_∞ (large) such that the Euler problem admits infinitely many weak solutions with the energy profile

$$\int_{\mathbb{T}^d} \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (t, \cdot) \, \mathrm{d}x = E_\infty + E(t) \text{ for a.a. } t \in (0, T)$$

Strongly discontinuous solutions

Theorem

Let $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R},\ 0 \le \underline{\varrho} \le \varrho_0 \le \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times.
Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\rm weak}([0, T]; L^{\gamma}(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)] \ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i,\ i = 1, 2, \dots .$$

Convex integration ansatz

Helmholtz decomposition of the initial data

$$\mathbf{m}_0 = \mathbf{v}_0 + \nabla_x \Phi_0,\ \mathrm{div}_x \mathbf{v}_0 = 0,\ \Delta_x \Phi_0 = \mathrm{div}_x \mathbf{m}_0,\ (\nabla_x \Phi_0 - \mathbf{m}_0) \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\varrho(t,x) = \varrho_0 + h(t)\Delta_x \Phi_0,\ h(0) = 0,\ h'(0) = -1$$

$$\mathbf{m}(t,x) = \mathbf{v} - h'(t)\nabla_x \Phi_0,\ \mathrm{div}_x \mathbf{v} = 0,\ \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega},\ \mathbf{v}(0,\cdot) = \mathbf{v}_0$$

Balance of momentum

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} - h'(t)\nabla_x \Phi_0) \otimes (\mathbf{v} - h'(t)\nabla_x \Phi_0)}{\varrho} - \frac{1}{d} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} \mathbb{I} \right)$$
$$= 0$$

Energy

$$\frac{1}{2} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} = \Lambda(t) - \frac{d}{2} p(\varrho) + \frac{d}{2} h''(t) \Phi_0$$

Subsolutions

Energy profile

$$e = e(t,x) = \frac{E(t)}{|\Omega|} + \Lambda_0(t) - \frac{d}{2} p(\varrho) + \frac{d}{2} h''(t)\Phi_0, \ \ e \in \mathcal{R}([0,T] \times \overline{\Omega}).$$

Field equations

$$\mathrm{div}_x \mathbf{v} = 0, \ \ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \ \ \mathbf{v}(0,\cdot) = \mathbf{v}_0, \ \ \mathbb{U}(t,x) \in R^{d\times d}_{\mathrm{sym},0}$$

Convex constraint

$$\frac{d}{2} \sup_{[0,T]\times\overline{\Omega}} \lambda_{\max} \left[\frac{(\mathbf{v} - h'(t)\nabla_x \Phi_0) \otimes (\mathbf{v} - h'(t)\nabla_x \Phi_0)}{\varrho} - \mathbb{U} \right] < \inf_{[0,T]\times\overline{\Omega}} e$$

Algebraic inequality

$$\frac{1}{2} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} \leq \frac{d}{2} \lambda_{\max} \left[\frac{(\mathbf{v} - h'(t)\nabla_x \Phi_0) \otimes (\mathbf{v} - h'(t)\nabla_x \Phi_0)}{\varrho} - \mathbb{U} \right]$$

Critical points

Convex functional

$$I[\mathbf{v}] = \int_0^T \int_{\mathbb{T}^d} \left(\frac{1}{2} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} - e \right) \, \mathrm{d}x\mathrm{d}t \text{ for } \mathbf{v} \in X.$$

Zero points

$$I[\mathbf{v}] = 0 \ \Rightarrow \ \mathbf{v} \text{ is a weak solution of the problem}$$

Points of continuity

$$\mathbf{v} \text{ – a point of continuity of } I \text{ on } X \ \Rightarrow I[\mathbf{v}] = 0$$

Oscillatory Lemma (De Lellis, Székelyhidi)

Oscillatory Lemma, basic form

Let $Q = (0,1) \times (0,1)^d$, $d = 2, 3$. Suppose that $\mathbf{v} \in R^d$, $\mathbb{U} \in R^{d \times d}_{0,\mathrm{sym}}$, $e \leq \overline{e}$ are given constant quantities such that

$$\frac{d}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] < e.$$

Then there is a constant $c = c(d, \overline{e})$ and sequences of vector functions $\{\mathbf{w}_n\}_{n=1}^{\infty}$, $\{\mathbb{V}_n\}_{n=1}^{\infty}$,

$$\mathbf{w}_n \in C^\infty_c(Q; R^d), \ \mathbb{V}_n \in C^\infty_c(Q; R^{d \times d}_{0,\mathrm{sym}})$$

satisfying

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{V}_n = 0, \ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } Q,$$

$$\frac{d}{2} \lambda_{\max} \left[(\mathbf{v} + \mathbf{w}_n) \otimes (\mathbf{v} + \mathbf{w}_n) - (\mathbb{U} + \mathbb{V}_n) \right] < e \text{ in } Q \text{ for all } n = 1, 2, \dots,$$

$$\mathbf{w}_n \to 0 \text{ in } C_{\mathrm{weak}}([0,1]; L^2((0,1)^d; R^d)) \text{ as } n \to \infty,$$

$$\liminf_{n \to \infty} \int_Q |\mathbf{w}_n|^2 \mathrm{d}x\mathrm{d}t \geq c(d, \overline{e}) \int_Q \left(e - \frac{1}{2} |\mathbf{v}|^2 \right)^2 \mathrm{d}x\mathrm{d}t$$

Oscillatory Lemma

Oscillatory lemma

$$\mathbf{v} \in \mathcal{R}(\overline{Q}; R^d),\ \mathbb{U} \in \mathcal{R}(\overline{Q}; R^{d\times d}_{0,\mathrm{sym}}),\ e \in \mathcal{R}(\overline{Q}),\ r \in \mathcal{R}(\overline{Q}),\ Q = (0, T) \times \Omega$$

$$0 < \underline{r} \leq r(t, x) \leq \overline{r},\ e(t, x) \leq \overline{e} \text{ for all } (t, x) \in \overline{Q},$$

$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max} \left[\frac{\mathbf{v} \otimes \mathbf{v}}{r} - \mathbb{U} \right] < \inf_{\overline{Q}} e.$$

Then there is a constant $c = c(d, \overline{e})$ and sequences $\{\mathbf{w}_n\}_{n=1}^{\infty}$, $\{\mathbb{V}_n\}_{n=1}^{\infty}$,

$$\mathbf{w}_n \in C^{\infty}_c(Q; R^d),\ \mathbb{V}_n \in C^{\infty}_c(Q; R^{d\times d}_{0,\mathrm{sym}})$$

satisfying

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{V}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } Q,$$

$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max} \left[\frac{(\mathbf{v} + \mathbf{w}_n) \otimes (\mathbf{v} + \mathbf{w}_n)}{r} - (\mathbb{U} + \mathbb{V}_n) \right] < \inf_{\overline{Q}} e,$$

$$\mathbf{w}_n \to 0 \text{ in } C_{\mathrm{weak}}([0, T]; \Omega; R^d)) \text{ as } n \to \infty,$$

$$\liminf_{n\to\infty} \int_Q \frac{|\mathbf{w}_n|^2}{r} \mathrm{d}x\mathrm{d}t \geq c(d, \overline{e}) \int_Q \left(e - \frac{1}{2} \frac{|\mathbf{v}|^2}{r} \right)^2 \mathrm{d}x\mathrm{d}t$$