Statistical solutions of hyperbolic conservation laws

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6 June 2019

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Section 1

Introduction

Conservation laws

Hyperbolic system of conservation law

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

Conserved variables $u = u(x,t) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^N$

Initial data $u_0 : \mathbb{R}^d \to \mathbb{R}^N$

Flux function $f : \mathbb{R}^N \to \mathbb{R}^{N\times d}$

Conservation laws

Hyperbolic system of conservation law

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

Example (Euler equations for compressible, isentropic gases)

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho v \end{pmatrix} + \nabla \cdot \begin{pmatrix} \rho v \\ \rho v \otimes v + pI \end{pmatrix} = 0.$$

Here, $\rho =$ mass density, $v =$ velocity, $p =$ pressure, for instance

$$p(\rho) = \kappa \rho^{\gamma}.$$

Conservation laws

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Example (Euler equations for compressible, polytropic ideal gases)

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} + \nabla \cdot \begin{pmatrix} \rho v \\ \rho v \otimes v + pI \\ (E+p)v \end{pmatrix} = 0.$$

The density ρ, velocity field v, pressure p and total energy E are related by the equation of state

$$E = \frac{p}{\gamma - 1} + \frac{\rho |v|^2}{2}.$$

Weak (entropy) solutions

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

Definition

A **weak solution** satisfies (1) in the sense of distributions:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}_+} u\partial_t\varphi + f(u)\cdot\nabla\varphi \ dxdt + \int_{\mathbb{R}^d} u_0(x)\varphi(x,0)\ dx = 0 \qquad \forall\ \varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+).$$

- Weak solutions are generally non-unique
- Entropy conditions (hopefully!) single out the “physical” solution

Weak (entropy) solutions

$$
\begin{aligned}
\partial_t u + \nabla \cdot f(u) &= 0 \\
u(x,0) &= u_0(x)
\end{aligned}
\tag{1}
$$

Definition

An **entropy solution** satisfies for all entropy pairs (η, q)

$$\partial_t \eta(u) + \nabla \cdot q(u) \leqslant 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+)$$

$(\eta : \mathbb{R}^N \to \mathbb{R}$ is convex, $q'(u) = \eta'(u) f'(u))$

- Weak solutions are generally non-unique
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Well-posedness of conservation laws

$$
\begin{aligned}
\partial_t u + \nabla \cdot f(u) &= 0 \\
u(x,0) &= u_0(x)
\end{aligned}
\tag{1}
$$

Theorem (P. Lax 1957, J. Glimm 1965, N. H. Risebro 1993, A. Bressan et al. 2000)

For systems of equations in **one dimension** $d = 1$*, there exists a unique entropy solution of* (1) *whenever the initial data is* **sufficiently small** *(i.e., sufficiently close to a constant solution).*

Well-posedness of conservation laws

Theorem (C. De Lellis, L. Székelyhidi Jr. 2009)

The multi-D incompressible Euler equations

$$\begin{aligned} \partial_t v + \nabla \cdot (v \otimes v) + \nabla p &= 0 \\ \nabla \cdot v &= 0 \end{aligned}$$

are ill-posed in the space of continuous solutions. (There exists **"wild"** initial data with infinitely many "entropy solutions".)

Well-posedness of conservation laws

Theorem (C. De Lellis, L. Székelyhidi Jr. 2010)

The multi-D isentropic Euler equations

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho v) &= 0 \\ \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) + \nabla p(\rho) &= 0 \end{aligned} \tag{2}$$

are ill-posed in the sense of entropy solutions. (There exists **"wild"** initial data with infinitely many entropy solutions.)

Theorem (E. Chiodaroli, C. De Lellis, O. Kreml 2013–)

There exists Lipschitz continuous initial data for which (2) has infinitely many entropy solutions.

Well-posedness of conservation laws

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Question

- How should we think of these infinitely many solutions?

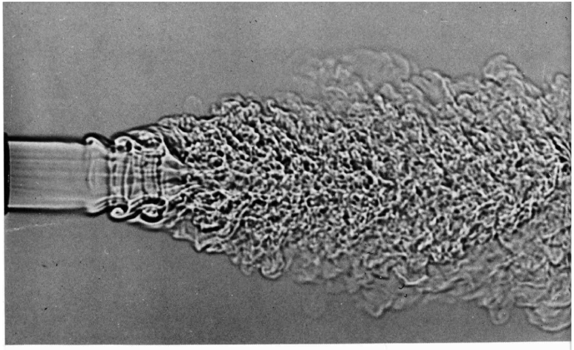
Section 2

Turbulence theory

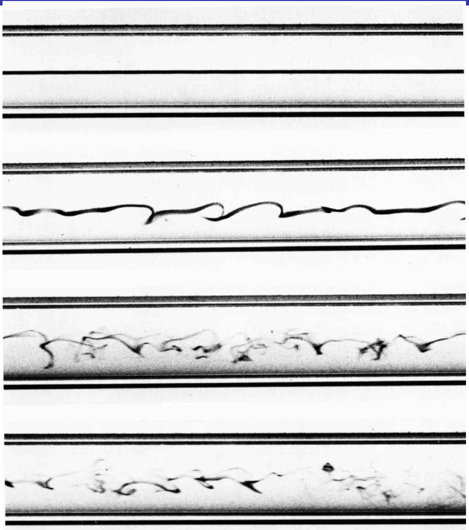
Turbulence

Quasi definition

Turbulence is a sudden chaotic, unpredictable behavior of fluids at a multitude of spatial scales.



Turbulence, real and simulated



Turbulence, real and simulated

Figure: The Navier–Stokes equations in real life

Turbulence, real and simulated

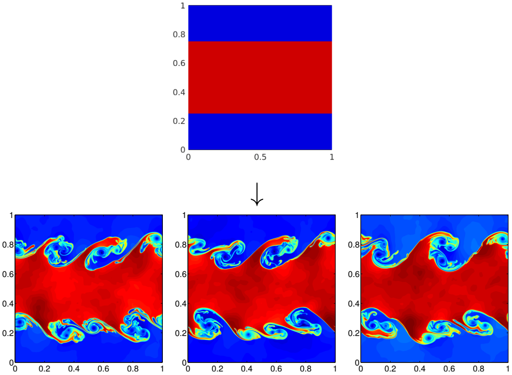


Figure: Numerical simulation of the compressible Euler equations

Approximate solutions and compactness

- A *viscous regularization* or *numerical method* for (1) might look like
$$\partial_t u^\varepsilon + \nabla \cdot f(u^\varepsilon) = \varepsilon Q^\varepsilon \qquad \text{where } Q^\varepsilon \text{ is (numerical) diffusion.}$$
- The diffusion provides (ε-dependent) **regularity** of u^ε, *but not enough for compactness in the limit* $\varepsilon \to 0$.

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Observations from turbulence theory

1. Turbulent flows are only predictable in a **statistical** sense (e.g., over long times or over many realizations)
2. **Ensembles** of turbulent flows might have higher regularity than individual realizations of the flow (*anomalous dissipation*)

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Questions

1. How do we represent an *uncertain solution*? What equations does it satisfy?
2. Can we utilize the *anomalous dissipation* to get compactness of **ensembles** of approximate solutions?

Section 3

Measure-valued and statistical solutions

Young measures

Definition

A **Young measure** is a map $\nu : x \mapsto \nu_x \in \mathcal{P}(\mathbb{R}^N)$. We denote

$$\langle \nu_x, f \rangle = \int_{\mathbb{R}^N} f(u)\, d\nu_x(u).$$

(Here, $\mathcal{P}(X) = \{\text{probability measures on } X\}$.)

Example

- $\nu_x = \delta_{u(x)}$ for some function $u = u(x)$ *("atomic measure")*
- $\nu_x = \frac{1}{M}\sum_{i=1}^{M} \delta_{u^i(x)}$ for functions $u^1, \dots, u^M$ *("empirical measure")*

Measure-valued solutions

$$\begin{aligned} \partial_t u(x,t) + \nabla \cdot f(u(x,t)) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

- The Young measure $\nu = \nu_{x,t}$ should satisfy (1) in an *averaged sense.*
- Consider $u = u(x,t)$ as a free variable and integrate over $u \in \mathbb{R}^N$ w.r.t. $\nu_{x,t}$ to get:

R. J. DiPerna. "Measure-valued solutions to conservation laws". In: Arch. Rational Mech. Anal. 88 (3 1985), 223–270.

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Definition

ν is a measure-valued (MV) solution of (1) if

$$\partial_t \langle \nu_{x,t}, u \rangle + \nabla \cdot \langle \nu_{x,t}, f(u) \rangle = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+). \tag{3}$$

Here,

$$\langle \nu_{x,t}, u \rangle = \int_{\mathbb{R}^N} u \, d\nu_{x,t}(u), \qquad \langle \nu_{x,t}, f(u) \rangle = \int_{\mathbb{R}^N} f(u) \, d\nu_{x,t}(u).$$

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Deficiencies of MV solutions

Definition (Measure-valued solution)

$$\partial_t \langle \nu_{x,t}, u \rangle + \nabla \cdot \langle \nu_{x,t}, f(u) \rangle = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+) \tag{3}$$

However, measure-valued solutions are **generically non-unique:** Enforcing a condition on only two moments ($\langle \nu_{x,t}, u \rangle$ and $\langle \nu_{x,t}, f(u) \rangle$) does not uniquely determine the measure $\nu_{x,t}$.

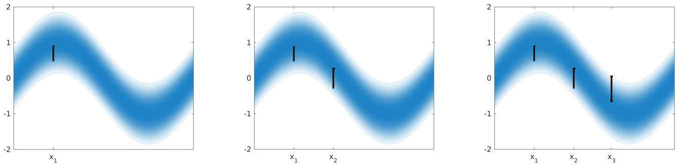
We add instead information about spatial correlations

Correlation measures

Definition (USF, S. Lanthaler, S. Mishra 2017)

A **correlation measure** is a hierarchy of Young measures $(\nu^1, \nu^2, \dots)$ where

- $\nu^1_x(A)$ = probability that $u(x) \in A$
- $\nu^2_{x,y}(A \times B)$ = probability that $u(x) \in A$ **and** $u(y) \in B$
- $\nu^3_{x_1,x_2,x_3}(A_1 \times A_2 \times A_k) = \dots$



(a) $\nu^1_{x_1}(A)$ (b) $\nu^2_{x_1,x_2}(A \times B)$ (c) $\nu^3_{x_1,x_2,x_3}(A \times B \times C)$

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- $\nu^3_{x_1,x_2,x_3}(A_1 \times A_2 \times A_k) = \dots$

- Each ν^k is a map $(x_1, \dots, x_k) \mapsto \nu^k_{x_1,\dots,x_k} \in \mathcal{P}((\mathbb{R}^N)^k)$
- The hierarchy $(\nu^1, \nu^2, \dots)$ must satisfy conditions on
  1. measurability
  2. consistency
  3. symmetry
  4. integrability (i.e. $\int_{\mathbb{R}^d} \langle \nu^1_x, |u|^p \rangle \, dx < \infty$)
  5. *diagonal continuity* (ν must satisfy the "Lebesgue Differentiation Theorem")

Correlation measures

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- $\nu^3_{x_1,x_2,x_3}(A_1 \times A_2 \times A_k) = \dots$

Example (Empirical measure)

For some $u_1, \dots, u_M : \mathbb{R}^d \to \mathbb{R}^N$, let $\nu^1_x = \frac{1}{M}\sum_{i=1}^M \delta_{u^i(x)}$,

$$\nu^k_{x_1,\dots,x_k} = \frac{1}{M}\sum_{i=1}^M \delta_{u^i(x_1)} \otimes \cdots \otimes \delta_{u^i(x_k)}$$

for all $k \in \mathbb{N}$, $x_j \in \mathbb{R}^d$.

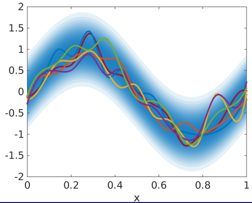
Equivalence between correlation measures and $\mathcal{P}(L^p)$

Theorem (USF, S. Lanthaler, S. Mishra 2017)

- *Fix $p \in [1, \infty)$. For every p-integrable correlation measure $\nu = (\nu^1, \nu^2, \dots)$ there exists a unique probability measure $\mu \in \mathcal{P}(L^p(\mathbb{R}^d, \mathbb{R}^N))$ satisfying the* **duality formula**

$$\int_{(\mathbb{R}^d)^k} \int_{(\mathbb{R}^N)^k} g(x, u)\ d\nu_x^k(u) dx = \int_{L^p} \int_{(\mathbb{R}^d)^k} g(x, u(x))\ dx d\mu(u) \qquad \forall\ g \in \mathcal{C}^k.$$

- *Conversely, for every probability measure $\mu \in \mathcal{P}(L^p(\mathbb{R}^d, \mathbb{R}^N))$ there exists a unique correlation measure ν satisfying the above.*



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- *Conversely, for every probability measure $\mu \in \mathcal{P}(L^p(\mathbb{R}^d, \mathbb{R}^N))$ there exists a unique correlation measure ν satisfying the above.*

Example

The empirical correlation measure $\nu^k_{x_1,\dots,x_k} = \frac{1}{M}\sum_{i=1}^M \delta_{u^i(x_1)} \otimes \cdots \otimes \delta_{u^i(x_k)}$ corresponds to

$$\mu = \frac{1}{M}\sum_{i=1}^{M} \delta_{u^i} \qquad\qquad (\delta_u = \text{Dirac measure on } u \in L^p).$$

Statistical solutions

- Each correlation measure $\nu = (\nu^1, \nu^2, \dots)$ can be viewed as a probability measure $\mu \in \mathcal{P}(L^p(\mathbb{R}^d, \mathbb{R}^N))$
- Corr. meas./prob. meas. are uniquely determined by the correlation functions

$$\langle \nu^k_{x_1,\dots,x_k}, u_1 \cdots u_k \rangle = \int_{L^p} u(x_1) \cdots u(x_k) \; d\mu(u)$$

What evolution equations do correlation functions satisfy?

Evolution equation for correlation functions

Correlation function: $\langle \nu^k_{x_1,\dots,x_k}, u_1 \cdots u_k \rangle = \int_{L^1} u(x_1)\cdots u(x_k)\ d\mu(u)$

k = 1:

Deterministic solution:

$$\partial_t u(x) + \partial_x f(u(x)) = 0$$

Statistical solution:

$$\partial_t \langle \nu^1_x, u\rangle + \partial_x \langle \nu^1_x, f(u)\rangle = 0$$

Evolution equation for correlation functions

Correlation function: $\langle \nu^k_{x_1,\dots,x_k}, u_1 \cdots u_k \rangle = \int_{L^1} u(x_1)\cdots u(x_k)\; d\mu(u)$

k = 2:

Deterministic solution:

$$\partial_t\big(u(x_1)u(x_2)\big) + \partial_{x_1}\big(f(u(x_1))u(x_2)\big) + \partial_{x_2}\big(u(x_1)f(u(x_2))\big) = 0$$

Statistical solution:

$$\partial_t\langle \nu^2_{x_1,x_2}, u_1u_2\rangle + \partial_{x_1}\langle \nu^2_{x_1,x_2}, f(u_1)u_2\rangle + \partial_{x_2}\langle \nu^2_{x_1,x_2}, u_1 f(u_2)\rangle = 0$$

Evolution equation for correlation functions

Correlation function: $\langle \nu^k_{x_1,\dots,x_k}, u_1 \cdots u_k \rangle = \int_{L^1} u(x_1) \cdots u(x_k)\, d\mu(u)$

General $k \in \mathbb{N}$:

Deterministic solution:

$$\partial_t \big(u(x_1)\cdots u(x_k)\big) + \sum_{i=1}^{k} \partial_{x_i} \big(u(x_1)\cdots f(u(x_i))\cdots u(x_k)\big) = 0$$

Statistical solution:

$$\partial_t \langle \nu^k_{t,x_1,\dots,x_k}, u_1 \cdots u_k \rangle + \sum_{i=1}^{k} \partial_{x_i} \langle \nu^k_{t,x_1,\dots,x_k}, u_1 \cdots f(u_i) \cdots u_k \rangle = 0.$$

Note: *These equations are in divergence form, so they can be interpreted weakly!*

Evolution equation for statistical solutions (multi-D systems)

Let $\mu_0 \in \mathcal{P}(L^1(\mathbb{R}^d, \mathbb{R}^N))$ be given initial data.

Definition (USF, S. Lanthaler, S. Mishra 2017)

A map $t \mapsto \mu_t \in \mathcal{P}(L^1(\mathbb{R}^d, \mathbb{R}^N))$ is a **statistical solution** of (1) if

- $\lim_{t\to 0} \mu_t = \mu_0$
- the corresponding correlation measure $(\nu_t^1, \nu_t^2, \dots)$ satisfies

$$\partial_t \langle \nu_{t,x_1,\dots,x_k}^k, u_1 \otimes \cdots \otimes u_k \rangle + \sum_{i=1}^{k} \nabla_{x_i} \cdot \langle \nu_{t,x_1,\dots,x_k}^k, u_1 \otimes \cdots \otimes f(u_i) \otimes \cdots \otimes u_k \rangle = 0 \qquad \forall\ k \in \mathbb{N}$$

in $\mathcal{D}'(\mathbb{R}^{dk} \times [0,\infty), \mathbb{R}^N)$, for all $k \in \mathbb{N}$.

Section 4

Statistical solutions and turbulence

Inviscid limit of Navier–Stokes

Incompressible Navier–Stokes equation

$$\partial_t u + \nabla\cdot(u\otimes u) + \nabla p = \varepsilon\Delta u, \qquad \nabla\cdot u = 0. \tag{4a}$$

- Solutions satisfy

$$\frac{d}{dt}\int_{\mathbb{T}^3}\frac{|u(t)|^2}{2}\,dx + \varepsilon\int_{\mathbb{T}^3}|\nabla u|^2\,dx = 0. \tag{4b}$$

  What happens in the limit $\varepsilon\to 0$?

- The dissipation term

$$\mathcal{E} = \varepsilon\int_0^T\int_{\mathbb{T}^3}|\nabla u|^2\,dxdt$$

  might not vanish as $\varepsilon\to 0$ (anomalous dissipation)

Inviscid limit of Navier–Stokes

$$\frac{d}{dt}\int_{\mathbb{T}^3}\frac{|u(t)|^2}{2}\,dx+\varepsilon\int_{\mathbb{T}^3}|\nabla u|^2\,dx=0. \tag{4b}$$

Kolmogorov (1941) proved, under "reasonable assumptions", that

$$S_2(r)\simeq r^{1/3}\mathcal{E}^{1/3}$$

in the limit $\varepsilon\to 0$ for homogeneous, isotropic 3D turbulence. The **structure function** is defined as[4]

$$S_2(r):=\mathbb{E}\left[\int_{\mathbb{T}^3}\fint_{B_r(0)}|u(x+z)-u(x)|^2\,dzdx\right]^{1/2}$$

[4] $\mathbb{E}[\dots]$ is expected value w.r.t. a statistical solution.

Inviscid limit of Navier–Stokes

Theorem (USF, S. Mishra, F. Weber 2019)

Let μ^ε be statistical solutions of incompressible Navier–Stokes such that

$$\int_{\mathbb{T}^3} \langle \nu^{\varepsilon,1}_{t,x}, |u|^2 \rangle \, dx \leqslant C \qquad (L^2 \text{ bound})$$

$$\varepsilon S_2^\varepsilon(\varepsilon)^2 \lesssim \varepsilon \qquad \forall\, \varepsilon > 0 \qquad (\text{weak } H^1 \text{ bound})$$

$$S_2^\varepsilon(\lambda r) \lesssim \lambda^\alpha S_2^\varepsilon(r) \qquad \forall\, \lambda, r > 0 \qquad (\text{scaling law})$$

for some $\alpha > 0$ and $S_2^\varepsilon(r) := \left[\int_{\mathbb{T}^3} ⨍_{B_r(0)} \langle \nu^{\varepsilon,2}_{t,x,x+z}, |u_1 - u_2|^2 \rangle \, dzdx\right]^{1/2}$. Then $\exists$ a statistical solution μ of incompressible Euler, and[a]

$$\mu^\varepsilon \rightharpoonup \mu \qquad \textit{as } \varepsilon \to 0 \qquad \textit{(along a subsequence).}$$

[a] $\mu^\varepsilon \rightharpoonup \mu$ denotes weak ("narrow") convergence in the sense of measures

Inviscid limit of Navier–Stokes

Idea of proof.

1. The domain $\mathbb{T}^3$ is bounded
2. (weak H^1 bound)+(scaling law) yield bounds on oscillations
3. A "Kolmogorov compactness theorem"[a] yields compactness.

□

[a] U. S. Fjordholm, S. Mishra, K. Lye, and F. Weber. "Statistical solutions of hyperbolic systems of conservation laws: numerical approximation". In preparation. 2019.

Numerical approximation of hyperbolic conservation law

Theorem (USF, K. Lye, S. Mishra, F. Weber 2019)

Let $\mu^{\Delta x}$ be numerically computed approximate statistical solutions of a hyperbolic conservation law (1) *such that, for some $p \geqslant 1$,*

$$\Delta x^d \sum_{i\in\mathbb{Z}^d} \left|u_i^{\Delta x}(t)\right|^p \leqslant C \qquad (L^p \text{ bound})$$

$$\Delta x^d \int_0^T \sum_{m=1}^d \sum_{\mathbf{i}\in\mathbb{Z}^d} \left|u_{\mathbf{i}+\mathbf{e}_m}^{\Delta x}(t) - u_{\mathbf{i}}^{\Delta x}(t)\right|^s dt \leqslant C\Delta x \qquad \textit{for } s \geqslant p \qquad (\text{weak } W^{1,s} \text{ bound})$$

$$S_p^\varepsilon(\lambda r) \lesssim \lambda^\alpha S_p^\varepsilon(r) \qquad \forall\ \lambda, r > 0 \qquad (\text{scaling law})$$

for some $\alpha > 0$. Then $\exists$ a statistical solution μ of (1)*, and*

$$\mu^\varepsilon \rightharpoonup \mu \qquad \textit{as } \varepsilon \to 0 \qquad \textit{(along a subsequence).}$$

Section 5

Uniqueness, regularity and numerical approximation

1. Weak-strong uniqueness

Weak-strong uniqueness

"If there exists a classical solution w, then any other solution u coincides with w."

Technique: Uses the method of relative energy: Compute

$$\frac{d}{dt}\|u(t) - w(t)\|_{L^2}^2.$$

$\partial_t w$ exists strongly, and $\partial_t u$ exists weakly.

Result: Entropy condition on u + Gronwall estimate yields

$$\|u(t) - w(t)\|_{L^2}^2 \leqslant e^{Ct}\|u(0) - w(0)\|_{L^2}^2.$$

1. Weak-strong uniqueness

Theorem (USF, K. Lye, S. Mishra, F. Weber 2019)

Consider a hyperbolic system of conservation laws (1).

- *Let* ρ *be a* **strong statistical solution** *(concentrated on strong solutions of* (1)*)*
- *Let* μ *be a* **dissipative statistical solution** *(satisfies an additional entropy condition)*

Then

$$W_p(\mu_t, \rho_t) \leqslant e^{Ct} W_p(\mu_0, \rho_0) \qquad (W_p \text{ is Wasserstein distance}).$$

2. Energy conservation for incompressible Euler

Incompressible Euler equations

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0, \qquad \nabla \cdot u = 0 \tag{5}$$

Onsager's conjecture

Let u be an α-Hölder continuous solution of (5). Lars Onsager conjectured (1949) that if

- $\alpha > 1/3$ then u preserves the energy $\int |u|^2\,dx$,
- $\alpha \leqslant 1/3$ then u might dissipate energy.

Energy preservation: Proved by Eyink 1994; Constantin, E, Titi in 1994

Energy dissipation: Proved by Ph. Isett 2016

2. Energy conservation for incompressible Euler

Incompressible Euler equations

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0, \qquad \nabla \cdot u = 0 \tag{5}$$

Theorem (USF, E. Wiedemann 2017)

Let μ be a statistical solution of (5) satisfying

$$S_3(h) \lesssim C|h|^\alpha, \qquad S_3(h) := \left[\int_{L^2(\mathbb{T}^3)} \int_{\mathbb{T}^3} |u(x+h) - u(x)|^3 \, dx \, d\mu_t(u)\right]^{1/3}$$

for some $\alpha > 1/3$. Then μ preserves energy:

$$\int_{L^2(\mathbb{T}^3)} \int_{\mathbb{T}^3} \frac{|u|^2}{2} \, dx d\mu_t(u) = \int_{L^2(\mathbb{T}^3)} \int_{\mathbb{T}^3} \frac{|u|^2}{2} \, dx d\mu_0(u) \qquad \forall\ t > 0.$$

3. Numerical methods for stat. soln.

Approximate stat. soln. can be generated by numerical methods:

Monte Carlo algorithm

1. Generate i.i.d. random variables $u_0^1, u_0^2, \dots u_0^M$ according to $\mu_0 \in \mathcal{P}(L^1)$
2. Propagate with a numerical method: $u^i(t) = \mathcal{S}_t^{\Delta x} u_0^i$
3. Compose into the empirical measure

$$\mu_t^{\Delta x} = \frac{1}{M} \sum_{i=1}^{M} \delta_{u^i(t)}$$

Theorem (USF, K. Lye, S. Mishra 2018)

For scalar conservation laws the above MC and MLMC methods converge to a statistical solution as $\Delta x \to 0,\ M \to \infty$.

Proof: Standard Monte Carlo technique + convergence of $\mathcal{S}^{\Delta x}$.

Thank you for your attention!

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