NTNU

Kunnskap for en bedre verden

Lipschitz metrics for nonlinear PDEs

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Joint work with J. A. Carrillo and H. Holden

The Hunter–Saxton equation

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2$$

has been introduced as model of the director field of a nematic liquid crystal by Hunter and Saxton in 1991.

- Weak solutions are not unique [Hunter, Zheng, 1995].
- Hunter and Zheng introduced conservative and dissipative solutions.
- The HS equation can be integrated in various ways.

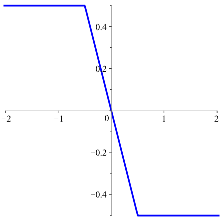
Wave breaking for the HS equation

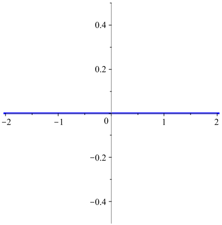
$$u_t(t,x) + uu_x(t,x) = \frac{1}{4}\int_{-\infty}^{x} u_x^2(t,z)dz - \frac{1}{4}\int_{x}^{\infty} u_x^2(t,z)dz$$

The HS equation enjoys wave breaking in finite time for a wide class of solutions, that means,

- the solution u remains bounded and continuous,
- the spatial derivative u_x becomes unbounded pointwise,
- the $L^2(\mathbb{R})$ norm of u_x remains bounded,
- $u_x^2 dx$ tends to a positive, finite Radon measure.

Non-uniqueness example

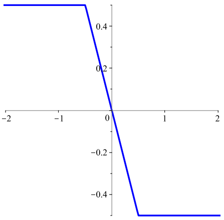


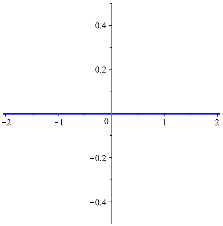


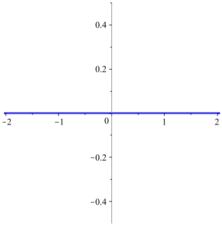
At collision time the solution vanishes but can be continued to a weak solution.

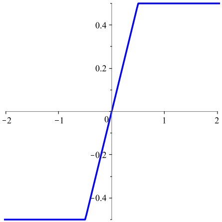
At collision time all the energy concentrates at the origin and can be described by a finite, positive Radon measure.

The solution $u(t, x)$

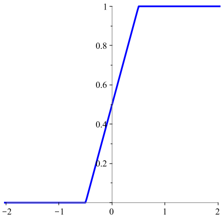


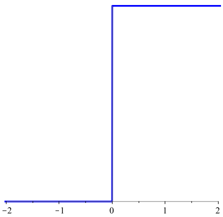


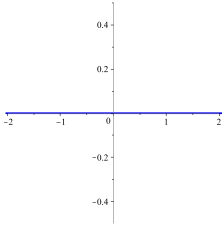


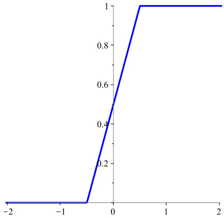


The energy $\mu(t,(-\infty,x))$

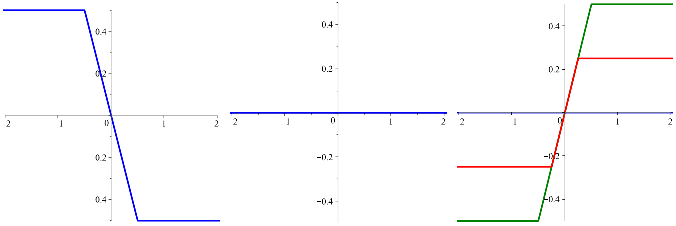








Non-uniqueness example



By manipulating the concentrated energy at breaking time various weak solutions can be constructed.

[Bressan, Constantin, 2005], [Dafermos, 2011], [Bressan, Holden, Raynaud, 2010], [Grunert, Nordli, 2016], . . .

Conservative solutions of the HS equation

$$u_t(t,x) + uu_x(t,x) = \frac{1}{4}\int_{-\infty}^{x} u_x^2(t,z)dz - \frac{1}{4}\int_{x}^{\infty} u_x^2(t,z)dz$$

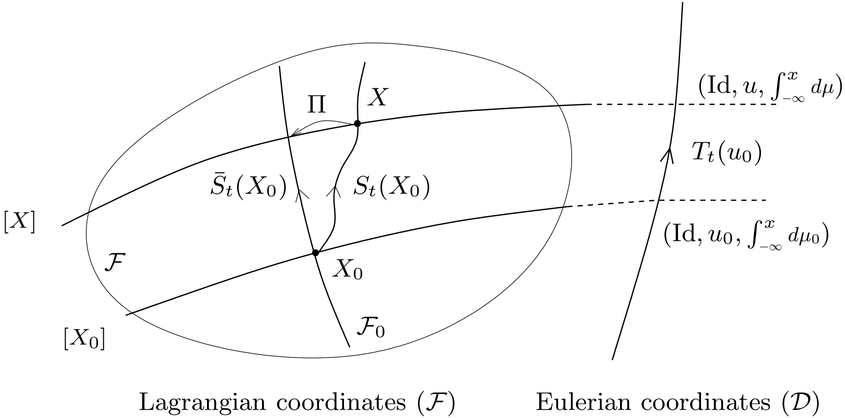
At any time t, the solution $(u(t,\cdot),\mu(t,\cdot)) \in \mathcal{D}$, i.e.

- $u(t,\cdot) \in L^\infty(\mathbb{R})$
- $u_x(t,\cdot) \in L^2(\mathbb{R})$
- $\mu(t,\cdot)$ a finite positive Radon measure such that

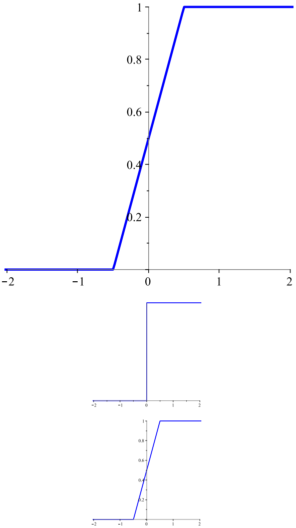
$$\mu_{ac} = u_x^2 dx$$

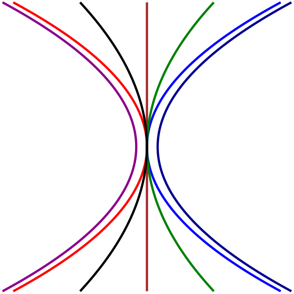
The total energy, given by $\mu(t,\mathbb{R})$, is independent of time.

Conservative solutions can be computed via a generalized method of characteristics.



Motivation $\mu(t,(-\infty,x)) \rightleftarrows y(t,\xi)\ldots$ characteristic





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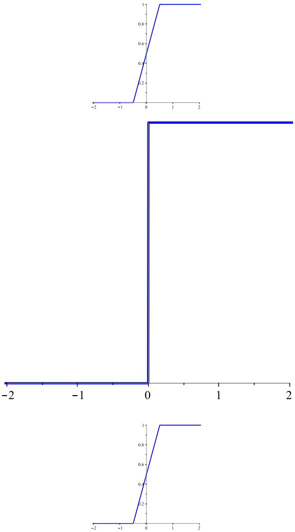
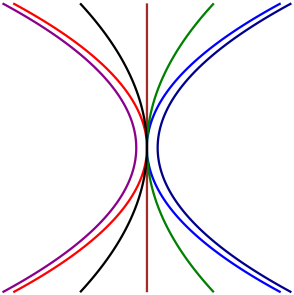
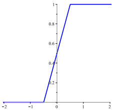
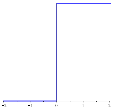
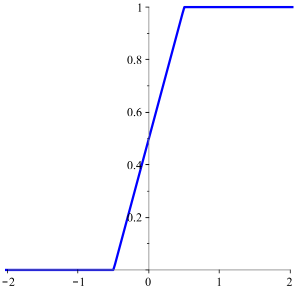


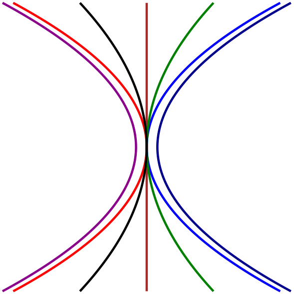


Motivation $\mu(t,(-\infty,x)) \rightleftarrows y(t,\xi) \ldots.$ characteristic

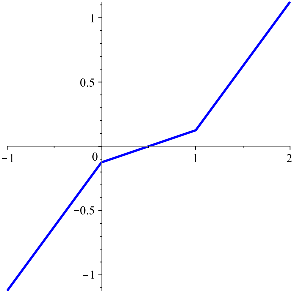


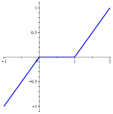


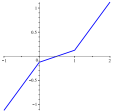


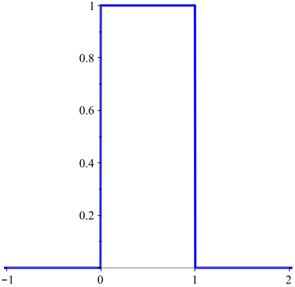


Motivation $\quad y(t,\xi) \rightleftarrows H_\xi(t,\xi) \ldots$ energy distribution









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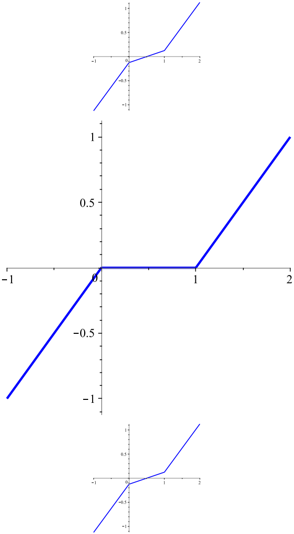

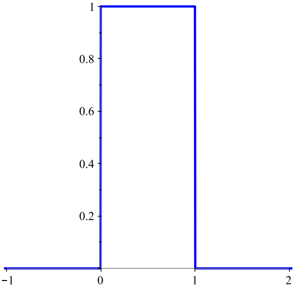

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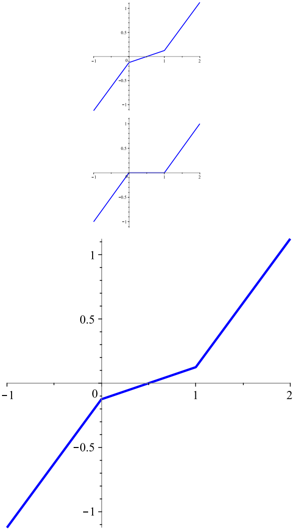

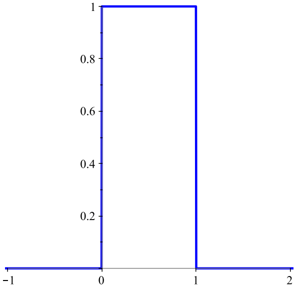

Motivation

Advantage:

- Linear system of ODEs
- The spreading of the concentrated energy is linked to the characteristics separating.

Disadvantage:

- One solution in Eulerian coordinates corresponds to an equivalence class of solutions in Lagrangian coordinates.
- Difficult to compute the distance between equivalence classes [Bressan, Holden, Raynaud, 2010].

Conservative solutions of the HS equation

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At any time t, the solution $(u(t,\cdot),\mu(t,\cdot)) \in \mathcal{D}$, i.e.

- $u(t,\cdot) \in L^\infty(\mathbb{R})$
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- $\mu(t,\cdot)$ a finite positive Radon measure such that

$$\mu_{ac} = u_x^2 dx$$

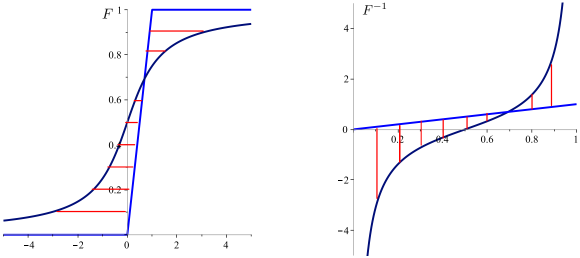
The total energy, given by $\mu(t,\mathbb{R})$, is independent of time.

Main idea

- Any measure $\mu(\cdot)$ with $\mu(\mathbb{R}) = 1$ can be seen as a curve in the space of probability measures.
- Given two probability measures $\mu_i(x)$, $i = 1, 2$, their distance can be measured by Wasserstein metrics, i.e.

$$d_p(\mu_1, \mu_2) = \|F_1^{-1} - F_2^{-1}\|_{L^p([0,1])} \qquad p \in [1, \infty],$$

where $F_i(x) = \mu_i((-\infty, x))$.

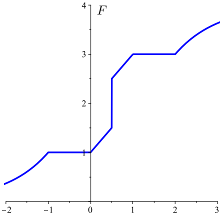


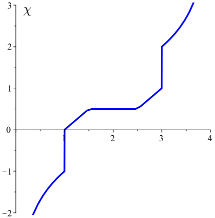
New system

- Any measure $\mu(t,\cdot)$ with $\mu(t,\mathbb{R}) = 1$ can be seen as a curve in the space of probability measures.

Let $\chi(t,\eta) = \sup\{x \mid F(t,x) < \eta\}$, where $F(t,x) = \mu(t,(-\infty,x))$, then

- $\chi(t,\cdot) : [0,1] \mapsto \mathbb{R}$ is non-decreasing.





- $(\chi(t,\eta), u(t,\chi(t,\eta)))$ describes the solution and has compact support.

New system

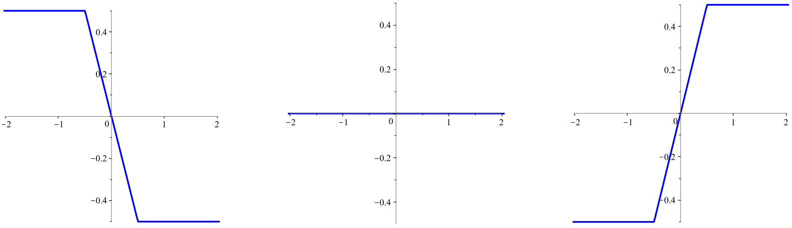
$$\chi_t(t,\eta) = \mathcal{U}(t,\eta)$$
$$\mathcal{U}_t(t,\eta) = \frac{1}{2}\eta - \frac{1}{4}$$

Thus $\chi(t,\eta)$ can be seen as characteristic, which takes care of the rarefaction behavior.

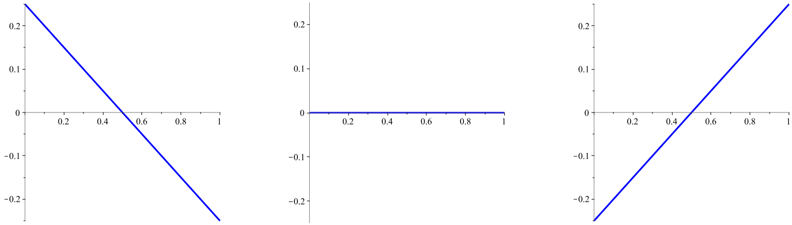
- Jumps in $\chi(t,\eta)$ might change in height, but remain always at the same position.
- Intervals where $\chi(t,\eta)$ is constant disappear immediately again.

Example

$$\mathcal{U}_t(t,\eta) = \frac{1}{2}\eta - \frac{1}{4}$$

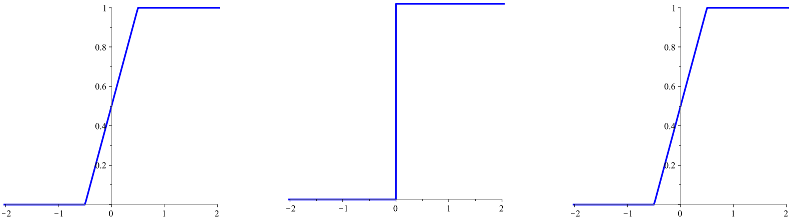


The function u.

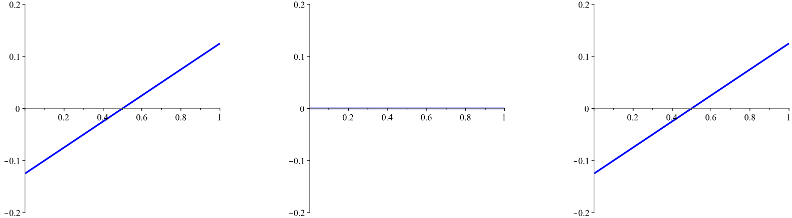


The function $\mathcal{U}$.

Example $\quad \chi_t(t,\eta) = \mathcal{U}(t,\eta)$



The function F.



The function χ.

Lipschitz metric

Let

$$
\begin{aligned}
d((u_1, \mu_1), (u_2, \mu_2)) = {} & \|\chi_1(\cdot) - \chi_2(\cdot)\|_{L^1([0,1])} \\
& + \|u_1(\chi_1(\cdot)) - u_2(\chi_2(\cdot))\|_{L^\infty([0,1])},
\end{aligned}
$$

then

$$
\begin{aligned}
d((u_1(t), \mu_1(t)), & (u_2(t), \mu_2(t))) \\
& \leq (1 + t) d((u_1(0), \mu_1(0)), (u_2(0), \mu_2(0))).
\end{aligned}
$$

Lipschitz metric

Problems:

- The support depends on the total energy C, i.e.

$$\chi,\ \mathcal{U} : [0, C] \to \mathbb{R}.$$

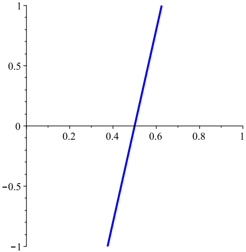
- If $(u, \mu) = (0, 0)$, then χ is not well-defined and the same applies to $\mathcal{U}$.

Lipschitz metric

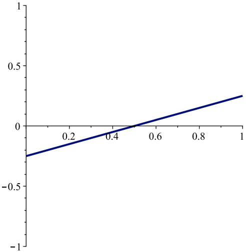
Solution:

- Introduce a rescaling

$$\tilde{\chi}(t,\eta) = C\chi(t, C\eta) \quad \text{and} \quad \tilde{\mathcal{U}}(t,\eta) = C\mathcal{U}(t, C\eta).$$

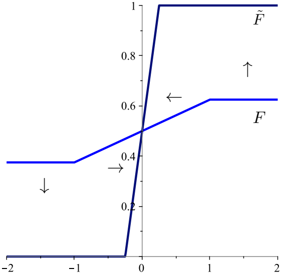


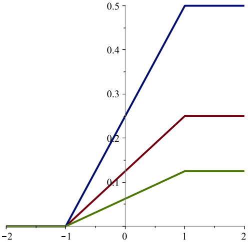
$\chi(\eta - \frac{1-C}{2})$



$C\chi(C\eta)$

Lipschitz metric





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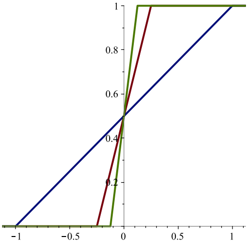
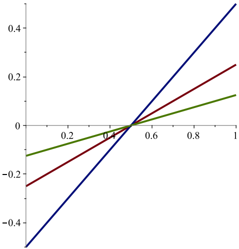


$\tilde{F}$



$\tilde{\chi}$

Lipschitz metric in the general case

Let $\mu_1(t,\mathbb{R}) = C_1$ and $\mu_2(t,\mathbb{R}) = C_2$ and introduce

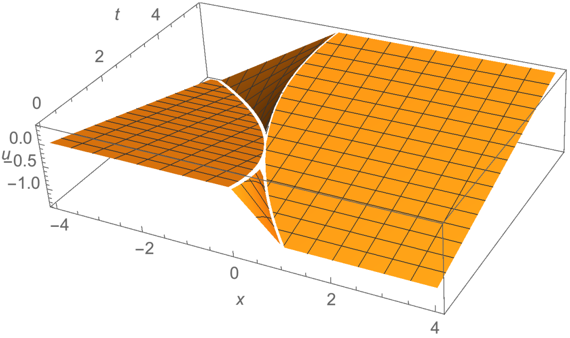
$$\begin{aligned} d((u_1,\mu_1),(u_2,\mu_2)) = {} & \|C_1\chi_1(C_1\cdot) - C_2\chi_2(C_2\cdot)\|_{L^1([0,1])} \\ & + \|C_1u_1(\chi_1(C_1\cdot)) - C_2u_2(\chi_2(C_2\cdot))\|_{L^\infty([0,1])} \\ & + |C_1^2 - C_2^2|, \end{aligned}$$

then

$$\begin{aligned} d((u_1(t),\mu_1(t)),&(u_2(t),\mu_2(t))) \\ & \leq (1 + t + \frac{1}{8}t^2)(d((u_1(0),\mu_1(0)),(u_2(0),\mu_2(0)))). \end{aligned}$$

Remarks

- To derive the time evolution in the new variables rigorously, one has to go via Lagrangian coordinates.
- The system in the new variables offers the possibility to plot solutions in Eulerian coordinates.



Remark

In the case of equations with infinite speed of propagation, one cannot expect the above approach to work without adaptations.

Some references

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Thanks for your attention!