What are the compact subsets of Lebesgue spaces?

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Outline of talk

(1) Show that Kolmogorov–Riesz only requires tightness and Lp equi-continuity

(2) Show that a common (technical) lemma implies both Kolmogorov–Riesz and Arzelà–Ascoli

The question

Consider $\mathcal{F} \subset L^p(\mathbb{R}^n), \quad p \in [1, \infty)$

When is $\mathcal{F}$ pre-compact or totally bounded?

Why?

Consider $\{f_j\}_j \subset L^p(\mathbb{R}^n)$

When can one find a strongly convergent subsequence $\{f_{j_i}\}_{j_i}$ such that

$\|f_{j_i} - f\|_{L^p(\mathbb{R}^n)} \to 0$ for some $f \in L^p(\mathbb{R}^n)$

Just some definitions…

Definition: A metric space is called totally bounded if it admits an epsilon-cover for every positive epsilon.

Definition: A subset is pre-compact if its closure is compact.

Theorem: In metric spaces sequential compactness equals compactness

Theorem: A subset of a complete metric space is pre-compact if and only if it is totally bounded.

Theorem: A metric space is compact if and only if it is complete and totally bounded.

The Kolmogorov–Riesz theorem

A subset $\mathcal{F} \subset L^p(\mathbb{R}^n), \quad p \in [1, \infty)$ is totally bounded if and only if:

(1) $\mathcal{F}$ is bounded

[bounded]

(2) For every ϵ positive, there exists an R such that

$$\int_{|x|>R} |f(x)|^p \, dx \leq \epsilon^p$$ uniformly in $\mathcal{F}$ [tightness]

(3) For every ϵ positive, there exists a ρ positive such that for all $|y| < \rho$

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \, dx \leq \epsilon^p$$ uniformly in $\mathcal{F}$

[L^p equi-continuity]

Historical background

Kolmogorov (1931) proved the case with $p > 1$ and bounded subsets of $\mathbb{R}^n$

Tamarkin (1932) extended the result to all of $\mathbb{R}^n$

Tulajkov (1933) extended the result to $p = 1$

At the same time M. Riesz (1933) had similar results.

Fréchet (1937) replaced boundedness and tightness by the single condition «equi-summability» and extended it to general positive p

Sudakov (1957) showed that the boundedness assumption is redundant

Sudakov's result

Tamarkin claimed in his paper that all three conditions were necessary

Sudakov's result is virtually unknown, and was only published in Russian

He uses a different approach compared to the standard, contemporary approach

$$\mathcal{F} = \{f_n\}_n \in L^p(\mathbb{R})$$

$$f_n(x) = (f(x) + n)\mathbf{1}_{(0,1)}(x)$$

$$f \in L^p(\mathbb{R})$$

$\mathcal{F}$ satisfies tightness, but is neither bounded nor Lp equi-continuous, and is not totally bounded

Boundedness is redundant

We already have the standard Kolmogorov–Riesz result that total boundedness is equivalent to (1) boundedness; (2) tightness; and (3) Lp equi-continuity

Suffices to show that tightness and Lp equi-continuity imply boundedness

Write $\displaystyle\int_\Omega |f(x)|^p dx = \|f\mathbf{1}_\Omega\|^p_{L^p(\mathbb{R}^n)}$

Need to show

$$\|f\|_p \leq \left\|f\mathbf{1}_{B_R(0)}\right\|_p + \left\|f\mathbf{1}_{\mathbb{R}^n\setminus B_R(0)}\right\|_p < \text{constant}$$

uniformly

Boundedness is redundant (cont'd)

Shift: $T_y f(x) = f(x+y)$

$$\left\|f\mathbf{1}_{B_R(z)}\right\|_p \leq \left\|(T_y f - f)\mathbf{1}_{B_R(z)}\right\|_p + \left\|f\mathbf{1}_{B_R(z+y)}\right\|_p$$
$$\leq \|(T_y f - f)\|_p + \left\|f\mathbf{1}_{B_R(z+y)}\right\|_p$$
$$\leq 1 + \left\|f\mathbf{1}_{B_R(z+y)}\right\|_p$$

(Choose $\epsilon = 1$ and $|y| < \rho$)

By induction: $\left\|f\mathbf{1}_{B_R(0)}\right\|_p \leq N + \left\|f\mathbf{1}_{B_R(Ny)}\right\|_p$

Choose $N|y| > 2R$ and $\epsilon = 1$ in tightness definition

$$\|f\|_p \leq \left\|f\mathbf{1}_{B_R(0)}\right\|_p + \left\|f\mathbf{1}_{\mathbb{R}^n \setminus B_R(0)}\right\|_p \leq N + 2$$

The Kolmogorov–Riesz–Sudakov theorem

A subset $\mathcal{F} \subset L^p(\mathbb{R}^n), \quad p \in [1, \infty)$ is totally bounded if and only if:

(1) For every ϵ positive, there exists an R such that

$$\int_{|x|>R} |f(x)|^p \, dx \leq \epsilon^p$$ uniformly in $\mathcal{F}$ [tightness]

(2) For every ϵ positive, there exists a ρ positive such that for all $|y| < \rho$

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \, dx \leq \epsilon^p$$ uniformly in $\mathcal{F}$

[L^p equi-continuity]

Sudakov's result

Introduce the *Steklov mean*

$$S_h f(x) = |B_h|^{-1} \int_{B_h} f(x+y)dy = |B_h|^{-1} f * \mathbf{1}_{B_h}(x)$$

Then Sudakov shows that the Lp equi-continuity

$$\int_{\mathbb{R}^n} |f(x) - T_y f(x)|^p dx < \epsilon^p$$

can be replaced by

$$\int_{\mathbb{R}^n} |f(x) - S_h f(x)|^p dx < \epsilon^p$$

The relationship with the Arzelà–Ascoli theorem

Proved by Arzelà (1894/95) and Ascoli (1884)

Let Ω be a compact topological space

Then a subset of $C(\Omega)$ is totally bounded if and only if

(i) it is pointwise bounded

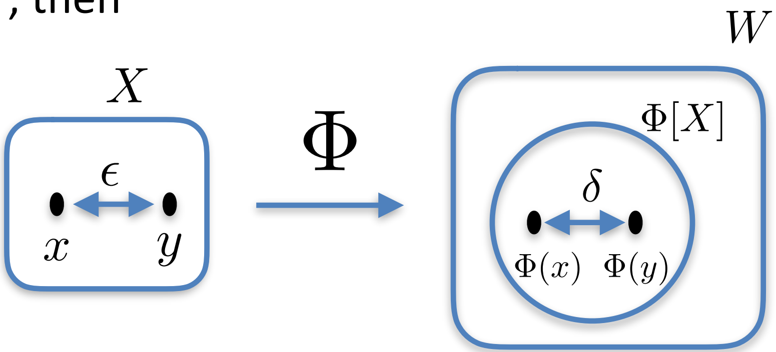
(ii) it is equi-continuous.

Both Kolmogorov–Riesz and Arzelà–Ascoli follow from the following theorem

Let X be a metric space.

Assume that for every $\epsilon > 0$ there exists a $\delta > 0$ and a metric space W and a mapping $\Phi\colon X \to W$ such that $\Phi[X]$ is totally bounded, and whenever $x, y \in X$ are such that $d\big(\Phi(x), \Phi(y)\big) < \delta$, then $d(x,y) < \epsilon$

Then X is totally bounded.



Arzelà–Ascoli theorem

Proof (one direction):

Assume $\mathcal{F} \subset C(\Omega)$ is pointwise bounded and equi-continuous. Let $\epsilon > 0$

We can find points $x_1, \ldots, x_n \in \Omega$ with neighborhoods $V_1, \ldots, V_n$

such that $|f(x) - f(x_j)| < \epsilon$ for all $x \in V_j$ and $f \in \mathcal{F}$

$\cup_{j=1}^{n} V_j = \Omega$

Define $\Phi \colon \mathcal{F} \to \mathbb{R}^n$

$$\Phi(f) = \big(f(x_1), \ldots, f(x_n)\big)$$

$\Phi[\mathcal{F}]$ is bounded, and hence totally bounded in $\mathbb{R}^n$

Let $f, g \in \mathcal{F}$ with $\|\Phi(f) - \Phi(g)\|_\infty < \epsilon$

Then $|f(x) - g(x)| \leq 3\epsilon$ Thus $\|f - g\|_\infty \leq 3\epsilon$ $\mathcal{F}$ totally bounded

Kolmogorov–Riesz theorem

Proof (one direction):

$Q \subset B_{\rho/2}(0)$ open cube centered at the origin

$Q_1, \ldots, Q_N$ non-overlapping translates such that $\bigcup_i Q_i \supset B_R(0)$

Define projection $$Pf(x) = \begin{cases} \dfrac{1}{|Q_i|} \displaystyle\int_{Q_i} f(z)\,dz, & x \in Q_i, \quad i = 1, \ldots, N, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f - Pf\|_p^p \le (2^n + 1)\epsilon^p$ Assume $\|Pf - Pg\|_p < \epsilon$

Then $\|f - g\|_p < \left((2^n + 1)^{1/p} + 1\right)\epsilon$

$P[\mathcal{F}]$ is bounded, and thus totally bounded in $\mathbb{R}^N$

$\mathcal{F}$ is totally bounded

Summary I

Both the Arzelà–Ascoli theorem and the Kolmogorov–Riesz theorem are consequences of a common, simple lemma.

Hanche-Olsen, H.: Expo. Math. 28 (2010) 385–394

Addendum, Expo. Math. 34 (2016) 243–245

Summary II

The Kolmogorov–Riesz–Sudakov theorem

A subset $\mathcal{F} \subset L^p(\mathbb{R}^n), \quad p \in [1, \infty)$ is totally bounded if and only if:

(1) For every ϵ positive, there exists an R such that

$$\int_{|x|>R} |f(x)|^p \, dx \leq \epsilon^p$$ uniformly in $\mathcal{F}$ [tightness]

(2) For every ϵ positive, there exists a ρ positive such that for all $|y| < \rho$

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \, dx \leq \epsilon^p$$ uniformly in $\mathcal{F}$

[L^p equi-continuity]

Hanche-Olsen, H., Malinnikova: Expo. Math. (https://doi.org/10.1016/j.exmath.2018.03.002)

Thank you for your attention!