Existence and uniqueness of solutions for a conservation law arising in magnetohydrodynamics

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Study conservation law

$$\left.\begin{array}{rcl}\partial_t u + \partial_x f(u,v) & = & 0 \\ \partial_t v + \partial_x g(u,v) & = & 0\end{array}\right\} \qquad (\star)$$

Riemann problem:

$$u(x,0) = u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

$$v(x,0) = v_0(x) = \begin{cases} v_l, & x < 0 \\ v_r, & x > 0 \end{cases}$$

Weak solutions:

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(u\partial_t\varphi + f(u,v)\partial_x\varphi\right)\, dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\, dx = 0$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(v\partial_t\varphi + g(u,v)\partial_x\varphi\right)\, dxdt + \int_{\mathbb{R}} v_0(x)\varphi(x,0)\, dx = 0$$

For some initial data, the Riemann problem may not have a solution.

Singular shocks: early work

- Korchinski (PhD thesis, Adelphi University, 1978)

$$\begin{aligned} u_t + \left(\tfrac{1}{2}u^2\right)_x &= 0 \\ v_t + \left(\tfrac{1}{2}uv\right)_x &= 0 \end{aligned}$$

- Keyfitz & Kranzer (JDE, 1995)

$$\begin{aligned} u_t + (u^2 - v)_x &= 0 \\ v_t + (\tfrac{1}{3}u^3 - u)_x &= 0 \end{aligned}$$

- Hayes & Le Floch (Nonlinearity, 1996)

$$\begin{aligned} u_t + \tfrac{1}{2}\left(u^2 + v^2\right)_x &= 0 \\ v_t + (uv - u)_x &= 0 \end{aligned}$$

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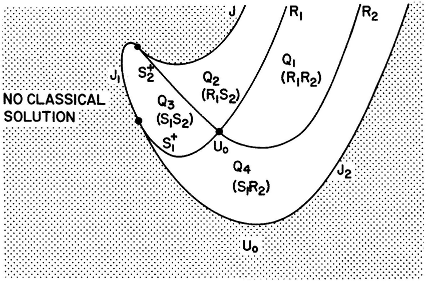

Introduce **Rankine-Hugoniot deficit:**

$$c[u] - [u^2 - v] = 0,$$
$$c[v] + \left[\tfrac{1}{3}u^3 - u\right] = \alpha'(t),$$

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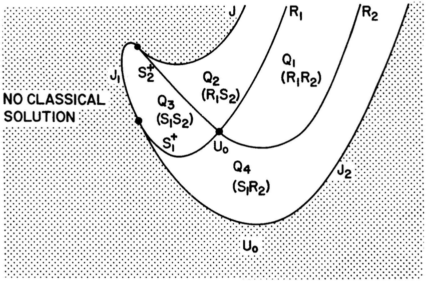

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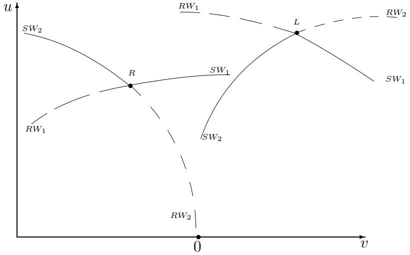
$$\begin{aligned} u_t + \tfrac{1}{2}\left(u^2 + v^2\right)_x &= 0 \\ v_t + (uv - u)_x &= 0 \end{aligned}$$

Not genuinly nonlinear at $v = 0$.



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Similar system:

$$
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v_t + (uv - u)_x &= 0
\end{aligned}
$$

If $u_l > u_r + 2$, the solutions contains singular shocks with Dirac delta distributions:

$$
\begin{aligned}
u(x,t) &= u_l + (u_r - u_l)H(x - ct) \\
v(x,t) &= v_l + (v_r - v_l)H(x - ct) + \alpha(t)\delta(x - ct)
\end{aligned}
$$

Singular shocks: early work

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To understand the singular solutions, use $V_x = v$, and solve the system

$$\begin{aligned} u_t + (\tfrac{1}{2}u^2)_x &= 0 \\ V_t + (u - 1)V_x &= 0 \end{aligned}$$

Physical explanation of singular shocks

Mazzotti, Tarafder, Cornel, Gritti, Guiochon
Journal of Chromatography A (2010)

Experiments: "unbounded" concentration

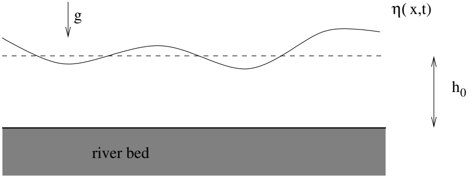
Hilden, Nilsen, Raynaud
Transport in Porous Media (2016)

Numerics: accumulation of mass

Kalisch, Mitrovic, Teyekpiti
Physics Letters A (2017)

Modelling: Conservation of momentum vs. conservation of energy

Example: shallow-water equations



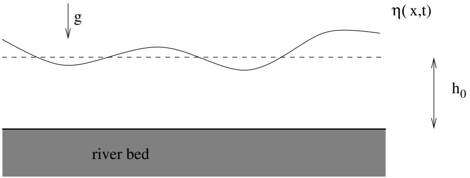
Shallow-water equations:

$$\begin{aligned} \eta_t + h_0 u_x + (\eta u)_x &= 0 \\ u_t + g\eta_x + u u_x &= 0 \end{aligned}$$

Assumptions:

- $p = \rho g (\eta - z)$ (hydrostatic)

Example: shallow-water equations



Shallow-water equations:

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Assumptions:

- $p = \rho g(\eta - z)$ (hydrostatic)
- $u = u(x,t)$ (no vertical acceleration)

Momentum conservation:

$$\Big[(h_0+\eta)u\Big]_t \;+\; \Big[(h_0+\eta)u^2 + \frac{1}{2}g(h_0+\eta)^2\Big]_x = 0$$

Energy conservation:

$$\frac{1}{2}\Big[(h_0+\eta)u^2 + (h_0+\eta)^2\Big]_t \;+\; \Big[\frac{1}{2}(h_0+\eta)u^3 + gu(h_0+\eta)^2\Big]_x = 0$$

Shallow-water equations:

Mass conservation:

$$\big[h\big]_t \;+\; \big[uh\big]_x = 0 \tag{1}$$

Conservation of total head:

$$\big[u\big]_t \;+\; \big[gh + \tfrac{u^2}{2}\big]_x = 0 \tag{2}$$

Momentum balance:

$$\big[hu\big]_t \;+\; \Big[hu^2 + \tfrac{1}{2}gh^2\Big]_x = 0 \tag{3}$$

Energy conservation:

$$\Big[\tfrac{1}{2}hu^2 + \tfrac{1}{2}h^2\Big]_t \;+\; \Big[\tfrac{1}{2}u^3 + guh^2\Big]_x = 0 \tag{4}$$

Traveling hydraulic jump

A traveling hydraulic jump must respect *conservation of mass and momentum*.

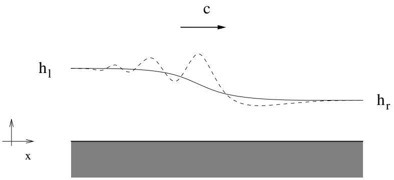
In shallow-water theory, we consider the jump as a discontinuity.

Rankine-Hugoniot conditions:

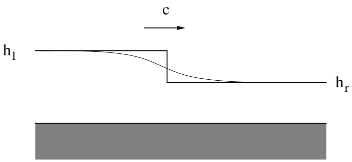
$$c[h] = [uh]$$
$$c[uh] = \left[u^2 h + \tfrac{1}{2} g h^2\right]$$

The velocity can be expressed as

$$c = \frac{u_r h_r - u_l h_l}{h_r - h_l}$$



Surface profile of a traveling hydraulic jump



Shallow-water approximation

Traveling hydraulic jump: induced Rankine-Hugoniot deficit

Relative velocity:

$$m = h_r(u_r - c) = h_l(u_l - c) = \mp h_r h_l \sqrt{\frac{g}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right)}$$

Energy loss:

$$\frac{1}{\rho Y}\Delta E = -\frac{mg(h_r - h_l)^3}{4h_r h_l}$$

Head loss:

$$g\Delta H = \frac{g(h_l - h_r)^3}{4h_l h_r}$$

Traveling hydraulic jump

Mass conservation:

$$c\left[h\right] \;-\; \left[uh\right] = 0$$

Head loss:

$$c\left[u\right] \;-\; \left[g(h+b) + \tfrac{u^2}{2}\right] = g\Delta H$$

Momentum balance:

$$c\left[hu\right] \;-\; \left[hu^2 + \tfrac{1}{2}gh^2\right] = 0$$

Energy loss:

$$c\left[\tfrac{1}{2}hu^2 + \tfrac{1}{2}h^2 + bh\right] \;-\; \left[\tfrac{1}{2}u^3 + guh(h+b)\right] = \frac{1}{\rho Y}\Delta E$$

Weak asymptotic method

Weak asymptotic method

Definition 1

Let $f_\varepsilon(x) \in \mathcal{D}'(\mathbb{R})$ be a family of distributions depending on $\varepsilon \in (0,1)$, We say that $f_\varepsilon = o_{\mathcal{D}'}(1)$ if for any test function $\phi(x) \in \mathcal{D}(\mathbb{R})$, we have

$$\langle f_\varepsilon, \phi \rangle = o(1), \quad \text{as} \quad \varepsilon \to 0$$

Link between RH-deficit and δ solutions?

Weak asymptotic method

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Definition 2

The collection of smooth **complex-valued** *distributions (u_ε) and (v_ε) represent a weak asymptotic solution to $(\star)$ if there exist real-valued distributions $u, v \in C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}))$, such that for every fixed $t \in \mathbb{R}_+$*

$$u_\varepsilon \rightharpoonup u, \quad v_\varepsilon \rightharpoonup v \quad \text{as} \quad \varepsilon \to 0,$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$, and

$$\begin{aligned} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon, v_\varepsilon) &= o_{\mathcal{D}'}(1), \\ \partial_t v_\varepsilon + \partial_x g(u_\varepsilon, v_\varepsilon) &= o_{\mathcal{D}'}(1). \end{aligned}$$

In addition, we need

$$u_\varepsilon(x,0) \rightharpoonup u(x,0) \quad \text{and} \quad v_\varepsilon(x,0) \rightharpoonup v(x,0).$$

Weak asymptotics for traveling hydraulic jump

Let $\rho \in C_c^\infty(\mathbb{R})$ be non-negative, smooth, compactly supported even function with $\text{supp}\rho \subset (-1,1)$ and $\int_{\mathbb{R}} \rho(z)dz = 1$
Define $C = \int_{\mathbb{R}} \rho^2(z)dz$, and

$$\delta_\varepsilon(x,t) = \frac{1}{2\varepsilon}\rho\left(\frac{x-ct-4\varepsilon}{\varepsilon}\right) + \frac{1}{2\varepsilon}\rho\left(\frac{x-ct+4\varepsilon}{\varepsilon}\right),$$

$$R_\varepsilon(x,t) = \frac{i}{2\varepsilon}\rho\left(\frac{x-ct-2\varepsilon}{\varepsilon}\right) - \frac{i}{2\varepsilon}\rho\left(\frac{x-ct+2\varepsilon}{\varepsilon}\right),$$

$$S_\varepsilon(x,t) = \frac{1}{\sqrt{\varepsilon}}\frac{1}{\sqrt{C}}\rho\left(\frac{x-ct}{\varepsilon}\right)$$

$$U_\varepsilon(x,t) = \begin{cases} u_l, & x < ct - 20\varepsilon, \\ 0, & ct - 10\varepsilon \le x \le ct + 10\varepsilon, \\ u_r, & x \ge ct + 20\varepsilon, \end{cases}$$

$$H_\varepsilon(x,t) = \begin{cases} h_l, & x < ct - 20\varepsilon, \\ 0, & ct - 10\varepsilon \le x \le ct + 10\varepsilon, \\ h_r, & x \ge ct + 20\varepsilon. \end{cases}$$

Now make the ansatz

$$h_\varepsilon(x,t) = H_\varepsilon(x-ct),$$

$$u_\varepsilon(x,t) = U_\varepsilon(x-ct) + \alpha(t)(\delta_\varepsilon(x-ct) + R_\varepsilon(x-ct)) + \sqrt{c\alpha(t)}S_\varepsilon(x-ct)$$

We can show that

$$\partial_t U_\varepsilon + \frac{1}{2}\partial_x U_\varepsilon^2 + \partial_x H_\varepsilon + \alpha'(t)\delta_\varepsilon \underbrace{- c\alpha(t)\delta' + c\alpha\partial_x S_\varepsilon^2} = o_{\mathcal{D}'}(1)$$

$$\partial_t U_\varepsilon + \frac{1}{2}\partial_x U_\varepsilon^2 + \partial_x H_\varepsilon + \alpha'(t)\delta_\varepsilon = o_{\mathcal{D}'}(1)$$

Choosing

$$\alpha'(t) = (u_r - u_l)c + \frac{1}{2}(u_l^2 - u_r^2) + g(h_r - h_l)$$

h_ε and u_ε are solutions of shallow water (1), (2) in the sense of Definition 2.

Note that the Rankine-Hugoniot deficit is nonzero:

$$\alpha'(t) = g\Delta \neq 0$$

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Note that the Rankine-Hugoniot deficit is nonzero:

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Define an energy

$$q(u, v) = \frac{u^2 + v^2}{2}$$

Then the system can be rewritten as

$$\begin{aligned} \partial_t u + \partial_x q &= 0 \\ \partial_t q + \partial_x \left((2u - 1)q + \tfrac{u^2}{2} - \tfrac{2u^3}{3} \right) &= 0 \end{aligned}$$

Flux function of the transformed system:

$$F = \begin{pmatrix} q \\ (2u - 1)q + \frac{u^2}{2} - \frac{2u^3}{3} \end{pmatrix}$$

Flux Jacobian:

$$DF = \begin{pmatrix} 0 & 1 \\ 2q + u - 2u^2 & 2u - 1 \end{pmatrix}$$

Characteristic velocities: $\lambda_{-,+} = \frac{2u-1\mp\sqrt{8q-4u^2+1}}{2}$

Eigenvectors:

$$r_- = \begin{pmatrix} 1 \\ u - \frac{1}{2} - \sqrt{2q - u^2 + \frac{1}{4}} \end{pmatrix}, \; r_+ = \begin{pmatrix} 1 \\ u - \frac{1}{2} + \sqrt{2q - u^2 + \frac{1}{4}} \end{pmatrix}$$

Genuinely nonlinear characteristic fields:

$$\nabla\lambda_- \cdot r_- = 2 + \frac{1}{\sqrt{8q - 4u^2 + 1}}\,, \quad \nabla\lambda_+ \cdot r_+ = 2 - \frac{1}{\sqrt{8q - 4u^2 + 1}}$$

If $u_R < u_L$:

Shock wave of the first family (SW1):

$$q_R = q_L - \frac{1}{2}(u_L - u_R)(2u_R - 1) + \\ \mid u_L - u_R \mid \left(2q_L + \tfrac{1}{2}(u_L - u_R) - \tfrac{1}{3}(2u_L^2 + 2u_L u_R - u_R^2) + \tfrac{1}{4}\right)^{\frac{1}{2}}$$

Shock wave of the second family (SW2):

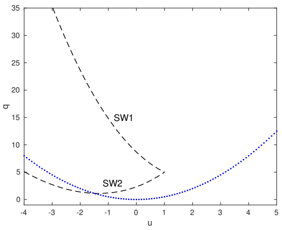
$$q_R = q_L - \frac{1}{2}(u_L - u_R)(2u_R - 1) - \\ \mid u_L - u_R \mid \left(2q_L + \tfrac{1}{2}(u_L - u_R) - \tfrac{1}{3}(2u_L^2 + 2u_L u_R - u_R^2) + \tfrac{1}{4}\right)^{\frac{1}{2}}$$

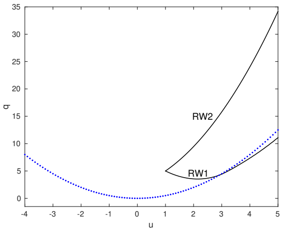
If $u_R > u_L$:
Rarefaction wave of the first family (RW1):

$$\frac{dq}{du} = \frac{2u - 1 - \sqrt{8q - 4u^2 + 1}}{2} = \lambda_-(u, q), \quad q(u_L) = q_L$$

Rarefaction wave of the second family (RW2):

$$\frac{dq}{du} = \frac{2u - 1 + \sqrt{8q - 4u^2 + 1}}{2} = \lambda_+(u, q), \quad q(u_L) = q_L$$



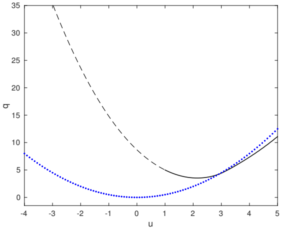


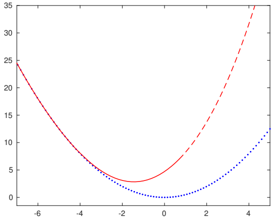
Critical curve: $q_{crit} = u^2/2$ so that

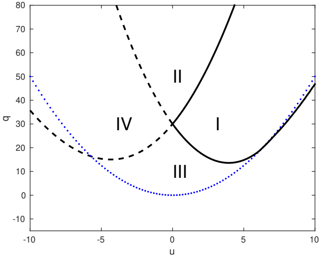
Theorem: Given a left state (u_L, q_L) and a right state (u_R, q_R), so that both are above the critical curve $q_{crit}(u) = \frac{u^2}{2}$, these states can be connected by Lax admissible shocks and rarefaction waves via a middle state also in the domain $q > u^2/2$.

Proof

For a given (u_L, q_L) ...







- (u_R, q_R) in region *I*: RW1 followed by RW2
- (u_R, q_R) in region *II*: SW1 followed by RW2
- (u_R, q_R) in region *III*: RW1 followed by SW2
- (u_R, q_R) in region *IV*: SW1 followed by SW2

The Brio system:

$$\partial_t u + \partial_x \left(\tfrac{u^2+v^2}{2} \right) = 0,$$
$$\partial_t v + \partial_x \big(v(u-1) \big) = 0.$$

Characteristic speeds:

$$\lambda_1(u,v) = u - 1/2 - \sqrt{v^2 + 1/4},$$
$$\lambda_2(u,v) = u - 1/2 + \sqrt{v^2 + 1/4}$$

Definition: The pair of distributions

$$u = U + \alpha(x,t)\delta(\Gamma),$$
$$v = V + \beta(x,t)\delta(\Gamma),$$

with $f(u,v) = \frac{u^2+v^2}{2}$ and $g(u,v) = v(u-1)$ is an admissible δ-type solution to the Brio System if

- The regular parts of the distributions u and v are such that the functions U and $q = (U^2 + V^2)/2$ represent Lax-admissible solutions to the Transformed system with the initial data

$$u\big|_{t=0} = U_0, \quad q\big|_{t=0} = q_0 = (U_0^2 + V_0^2)/2$$

- For every $t \geq 0$, the support of the δ-distributions appearing in u and v is of minimal cardinality.

Theorem

There exists a unique admissible δ-type solution to the Brio System with Riemann initial data

$$u\big|_{t=0} = \begin{cases} U_L, & x \leq 0 \\ U_R, & x > 0 \end{cases}, \quad v\big|_{t=0} = \begin{cases} V_L, & x \leq 0 \\ V_R, & x > 0 \end{cases}.$$

Proof

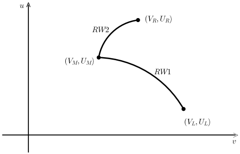
- Case I: both left and right states of the function V_0 have the same sign
- Case II: left and right states of the function V_0 have opposite signs

To compute α and β in the delta-solution, we compute the Rankine-Hugoniot deficit if it exists at all.

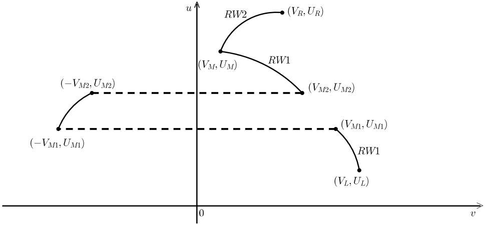
Region I:

$$(U_L, q_L) \xrightarrow{RW1} (U_M, q_M) \xrightarrow{RW2} (U_R, q_R)$$

No Rankine-Hugoniot deficit since (u, q) is a continuous solution to the transformed system. So $(u, v) = (u, \sqrt{2q - u^2})$



- u is unique since it is the Lax admissible solution to transformed system
- v is determined by the unique functions u and q via $v = \pm\sqrt{2q - u^2}$



The shock connecting (U_{M1}, V_{M1}) and $(U_{M1}, -V_{M1})$ cannot be singular, (i.e. no Rankine-Hugoniot deficit). No jump in $\partial_t u + \partial_x((u^2 + v^2)/2) = 0$ and therefore, the speed c of the shock must satisfy the Rankine-Hugoniot condition $-c[v] + [v(u-1)] = 0$.

The shock speed is

$$c = U_{M1} - 1.$$

On the other hand, the characteristic speeds of (U_{M1}, V_{M1}) and $(U_{M1}, -V_{M1})$ are

$$\lambda_1(U_{M1}, V_{M1}) = \lambda_1(U_{M1}, -V_{M1}) \neq c$$

A shock connection between (U_{M1}, V_{M1}) and $(U_{M1}, -V_{M1})$ is impossible with the Rankine-Hugoniot condition satisfied.

Similarly,

$$c = U_{M2} - 1.$$

$$\lambda_2(U_{M2}, V_{M2}) = \lambda_2(U_{M2}, -V_{M2}) \neq c$$

Conclusion: a δ-shock connection between (U_{M2}, V_{M2}) and $(U_{M2}, -V_{M2})$ is impossible with the Rankine-Hugoniot condition satisfied.

- The only possible connection of (U_L, V_L) and (U_R, V_R) is

$$(U_L, q_L) \xrightarrow{RW1} (U_M, q_M) \xrightarrow{RW2} (U_R, q_R)$$

- RW1 and RW2 corresponding to the transformed system are transformed via

$$(u, q) \mapsto (u, \sqrt{2q - u^2})$$

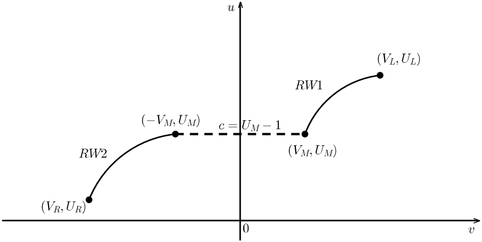
  into RW1 and RW2 corresponding to the Brio System.

- This is possible since q is the entropy function for the Brio System and RW1 and RW2 are smooth solutions to the transformed system

Case II: $V_L > 0$ and $V_R < 0$

- To get an admissible δ-type solution, we solve the transformed system with (U_0, q_0) as the initial data
- The obtained solution connects (U_L, q_L) with (U_R, q_R) by Lax admissible waves through a middle state (U_M, q_M)
- Solve the Brio System by connecting (U_L, V_L) with $(U_M, \sqrt{2q_M - U_M^2})$ by an elementary wave containing the Rankine-Hugoniot deficit and corresponding δ-shock wave
- Connect $(U_M, \sqrt{2q_M - U_M^2})$ with $(U_M, -\sqrt{2q_M - U_M^2})$ by the shock wave whose speed is $c = U_M - 1$
- Finally, connect $(U_M, -\sqrt{2q_M - U_M^2})$ with (U_R, V_R) by an elementary wave containing the corresponding Rankine-Hugoniot deficit and δ-shock wave

Region *I*: The states (U_L, q_L) and (U_R, q_R) are connected by RW1 and RW2 via the middle state (U_M, q_M).



since there is no jump in u, and there is no jump in v^2, the first equation of the Brio System is satisfied in the classical sense and the shock speed is determined by the Rankine-Hugoniot condition: $-c[v] + [v(u-1)] = 0 \;\Rightarrow\; c = U_{M1} - 1$

$$U_M - \frac{1}{2} - \sqrt{V_M^2 + \tfrac{1}{4}} \leq U_M - 1 \leq U_M - \frac{1}{2} + \sqrt{V_M^2 + \tfrac{1}{4}}$$

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