The fundamental solution of a class of ultra-hyperbolic operators on pseudo-H-type Lie groups

Irina Markina

University of Bergen

Join work with

W. Bauer and A. Froehly

Leibniz Universität Hannover, Germany.

Some history

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Pseudo H-type Lie algebras

A Lie algebra $\mathfrak{n}_{r,s} = \big(\mathfrak{n}, [\,.\,,\,.\,]\big)$ is called pseudo H-type Lie algebra if

- it is 2-step nilpotent: $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$;
- $\mathfrak{n} = \mathfrak{v} \oplus_{\perp} \mathfrak{z}$, where $\mathfrak{z}$ is the centre and the direct sum is orthogonal with respect to non-degenerate scalar product

$$\langle .\,,.\rangle = \langle .\,,.\rangle_{\mathfrak{v}} + \langle .\,,.\rangle_{r,s}$$

$$\langle z, w\rangle_{r,s} = \sum\nolimits_{i=1}^{r} z_i w_i - \sum\nolimits_{j+1}^{s} z_{r+j} w_{s+j} \quad \text{and}$$

$$\langle x, y\rangle_{\mathfrak{v}} = \begin{cases} \sum_{i=1}^{n} x_i y_i - \sum_{j+1}^{n} x_{n+j} y_{n+j}, & \text{if} \quad s > 0 \\ \sum_{i=1}^{2n} x_i y_i, & \text{if} \quad s = 0 \end{cases}$$

Pseudo H-type Lie algebras

- The map $J_z\colon \mathfrak{v} \to \mathfrak{v}$ is defined by

$$\langle J_z x, y\rangle_{\mathfrak{v}} = \langle z, [x,y]\rangle_{r,s}, \quad z \in \mathfrak{z}, \quad x, y \in \mathfrak{v},$$

and satisfies one of the equivalent conditions

$$\langle J_z x, J_z y\rangle_{\mathfrak{v}} = \langle z, z\rangle_{r,s}\langle x, y\rangle_{\mathfrak{v}},$$ composition of q.forms

$$J_z^2 = -\langle z, z\rangle_{\mathfrak{v}} Id_{\mathfrak{v}}$$ Clifford algebra property.

Directly from the definition:

- $\langle J_z \bullet, y\rangle = \langle z, -ad_{\bullet} y\rangle$;
- $\langle J_z x, y\rangle_{\mathfrak{v}} = -\langle x, J_z y\rangle_{\mathfrak{v}}$

Pseudo H-type Lie algebras

$\mathfrak{n}_{1,0} \cong \mathfrak{n}_{0,1}$ is the Heisenberg algebra:

$$\mathfrak{z} \cong \mathbb{R}, \quad \mathfrak{v} = \mathrm{span}\{X_i, Y_j;\ i, j = 1, \ldots n\}, \quad [X_i, Y_j] = \delta_{ij}$$

$\mathfrak{n}_{r,0}$ are the H(eisenberg)-type algebras introduced by A.Kaplan (1980) to study properties and solutions

$$\Delta_{sub} u = \sum_{i=1}^{2n} X_i^2 u = -\sum_{i=1}^{2n} X_i^* X_i u = f$$

$\mathfrak{n}_{r,s}$, $s > 0$ are the pseudo H(eisenberg)-type algebras introduced by P.Ciatti (2000), Godoy, Korolko, M. (2013)

Pseudo H-type Lie groups

Let $\mathfrak{n}_{r,s}$ be a pseudo-H-type algebra. The exponential map

$$\exp\colon \mathfrak{n}_{r,s} \to N_{r,s}$$

is the global diffeomeorphism: $\mathfrak{n}_{r,s} \leftrightarrow N_{r,s}$. The product is given by Baker-Campbell-Hausdorff formula

$$\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y]).$$

Left translation of the scalar product defines a non-degenerate indefinite metric.

Thus $N_{r,s}$ is a pseudo-Riemannian manifold.

Ultra-hyperbolic operator on $N_{r,s}$

Take orthonormal bases

$\mathcal{X} = \{X_j : j = 1, \ldots, 2n\}$ for $\mathfrak{v}$, $\quad \mathcal{Z} = \{Z_k : k = 1, \ldots, r+s\}$ for $\mathfrak{z}$

Identify $\mathcal{X}$ and $\mathcal{Z}$ with left-invariant v.f-ds on $N_{r,s}$ by

$$X_j := \frac{\partial}{\partial x_j} + \sum_{m=1}^{2n} \sum_{k=1}^{r+s} a_{mj}^k x_m \frac{\partial}{\partial z_k}, \qquad j = 1, \ldots, 2n,$$

$$Z_k := \frac{\partial}{\partial z_k}, \qquad k = 1, \ldots, r+s.$$

Here $[X_m, X_j] = 2\sum_{k=1}^{r+s} a_{mj}^k Z_k$ are the structure constants of $\mathfrak{n}_{r,s}$.

Ultra-hyperbolic operator on $N_{r,s}$

Let $r, s \in \mathbb{Z}^+ \cup \{0\}$ and $s > 0$. We call

$$\Delta_{r,s} := \sum_{i=1}^{n} X_i^2 - \sum_{i=1}^{n} X_{i+n}^2 = -\sum_{i=1}^{n} X_i^* X_i + \sum_{i=1}^{n} X_{i+n}^* X_{i+n}$$

an ultra-hyperbolic operator on $N_{r,s}$. (notice the similarity with the classical ultra-hyperbolic operator $\mathcal{L} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_j^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_{j+n}^2}$

Example: Let $N_{0,1}$ be the 3-dimensional Heisenberg group. A corresponding uh-operator is:

$$\Delta_{0,1} = \left(\frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial z}\right)^2 - \left(\frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial z}\right)^2$$

Aim of the project

(a) Characterize the pairs (r, s) for which the u-h. operator $\Delta_{r,s}$ admits an inverse (= fundamental solution).

(b) Derive a class of fundamental solutions of $\Delta_{r,s}$ in the space of tempered distributions, (whenever the existence is guaranteed): determine $K \in S'(\mathbb{R}^{2n+r+s})$ explicitly such that $\Delta_{r,s} K = \delta_0$.

(c) Characterize (r, s) for which the u-h operator $\Delta_{r,s}$ is locally solvable. (a left-invariant differential operator L on $N_{r,s}$ is called locally solvable at $p_0 \in N_{r,s}$ if there is an open neighborhood U of p_0 with $L(C^\infty)(U) \supset C_0^\infty(U)$.)

Strategy to find the inverse of $\Delta_{r,s}$

1. Perform a formal change of variables in the symbol of the u.h. operator to obtain the symbol of a sub-Laplacian Δ_{sub} on an H-type group.

2. Integrate the time-variable in the well-know expression of the heat kernel of $\frac{\partial}{\partial t} + \Delta_{sub}$ to obtain a fundamental solution of the sub-Laplace operator Δ_{sub}.

3. Formally reverse the change of variables in the fundamental solution of Δ_{sub}.

Note: The operator $\Delta_{r,s}$ does not have constant coefficients. Therefore the existence of a fundamental solution is not guaranteed (e.g. by the theorem of Malgrange and Ehrenpreis.)

Form of fundamental solution to $\Delta_{r,s}$

This recipe led to a meaningful distribution, which then rigorously was shown to be the fundamental solution of $\Delta_{r,s}$ for $r = 0$. For $\vartheta \neq 0$ write the kernel

$$q(\xi, \vartheta) := \frac{i}{(2\pi)^{n+\frac{s}{2}}} \int_0^\infty \frac{1}{|\vartheta| \cosh^n t} \exp\left\{ i \frac{\tanh t}{|\vartheta|} P(\xi) \right\} dt.$$

With the Fourier transform $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}^{2n+s})$ put:

$$K_{0,s}(\varphi) := \int_{\mathbb{R}^{2n+s}} q(\xi, \vartheta) \big[\mathcal{F}\varphi\big](\xi, \vartheta)\, d\xi\, d\vartheta, \quad \varphi \in \mathcal{S}(\mathbb{R}^{2n+s})$$

Then $K_{0,s}$ is a tempered distribution and it defines a fundamental solution to $\Delta_{0,s}$: $\Delta_{0,s} K_{0,s}(\varphi) = \varphi(0)$

Solution is not unique

Let $\mu(\vartheta), \lambda(\vartheta) \colon \mathbb{R}^s \to \mathbb{C}$ be measurable functions and

$$q_{0,s}^{\lambda,\mu}(\xi,\vartheta) = \frac{i}{(2\pi)^{n+\frac{s}{2}}|\vartheta|}\int_0^1 (1-\rho^2)^{\frac{n-2}{2}}\left\{\lambda e^{\frac{i\|\xi\|_{r,s}^2}{|\vartheta|}\rho} - \mu e^{\frac{-i\|\xi\|_{r,s}^2}{|\vartheta|}\rho}\right\}d\rho$$

Then

$$K_{0,s}^{\lambda,\mu}(\varphi) := \int_{\mathbb{R}^{2n+s}} q_{0,s}^{\lambda,\mu}(\xi,\vartheta)\big[\mathcal{F}\varphi\big](\xi,\vartheta)d\xi d\vartheta$$

is the fundamental solution of $\Delta_{0,s}$ whenever $\lambda + \mu = 1$.

For some parameters λ and μ we recovered known solutions on the Heisenberg groups.

Solution is not unique

The result follows from the observation that any fundamental solution has to satisfy the ODE

$$(2\pi)^{-\frac{2n+s}{2}} = -vf(v) - n|\vartheta|^2 f'(v) - |\vartheta| v f''(v),$$

if we set $q_{0,s}(\xi,\vartheta) = f(\|\xi\|_{r,s}^2, \vartheta) = f(v,\vartheta)$.

Then it was shown that

$$q_{0,s}^{\lambda,\mu}(\xi,\vartheta) = \frac{i}{(2\pi)^{n+\frac{s}{2}}|\vartheta|} \int_0^1 (1-\rho^2)^{\frac{n-2}{2}} \left\{ \lambda e^{\frac{i\|\xi\|_{r,s}^2}{|\vartheta|}\rho} - \mu e^{\frac{-i\|\xi\|_{r,s}^2}{|\vartheta|}\rho} \right\} d\rho$$

indeed the kernel for fundamental solution.

No fundamental solutions of $\Delta_{r,s}$ for $r > 0$

Encouraging observation: For $r > 0$ the formal change of variables still transforms $\Delta_{r,s}$ to Δ_{sub} defined on an pseudo H-type group.

Problem: The corresponding change of variables in the integral expression of a fundamental solution of Δ_{sub} produces an expression which cannot be interpreted as a distribution in an obvious way.

THEOREM. Let $r > 0$, then the u.h. operator $\Delta_{r,s}$ does not have a fundamental solution in $\mathcal{S}'(\mathbb{R}^{2n+r+s})$.

Proof made by a counterexample.

Question!

Is it possible that $\Delta_{r,s}$ has a fundamental solution in the larger space $\mathcal{D}'(\mathbb{R}^{2n+r+s})$ of Schwartz distributions?

Our counterexample does not work in this case but we found general theorem (F. Battesti):

Let L be a left-invariant and homogenous differential operator on $N_{r,s}$. Then the following are equivalent:

(a) L is locally solvable,

(b) L has a fundamental solution in $\mathcal{D}'(N_{r,s})$.

$\Delta_{r,s}$ *is not locally solvable*

Theorem (D. Müller, 1991)

Let L be a left-invariant homogeneous differential operator on a homogeneous, simply connected nilpotent Lie group G.

Assume there exists a Schwartz functions on G satisfying
(i) $\psi(0) = 1$
(ii) For every continuous semi-norm $\| \cdot \|$ on the Schwartz space $\mathcal{D}(G)$ it holds: $\|\psi\| \cdot \|L^T \psi\| = 0$

Then L is not locally solvable.

$\Delta_{r,s}$ *has not fundamental solution*

By constructing the function ψ we proved

(1) If $r > 0$ then $\Delta_{r,s}$ is not locally solvable. In particular, $\Delta_{r,s}$ does not even admit a fundamental solution in the space of Schwartz distributions $\mathcal{D}(N_{r,s})$ and

$$\Delta_{r,s}\Big(C^\infty(\mathbb{R}^{2n+r+s})\Big) \subsetneq C^\infty(\mathbb{R}^{2n+r+s}).$$

(2) If $r = 0$, then we have a positive result. The operators $\Delta_{r,s}$, $s > 0$, are locally solvable and

$$\Delta_{r,s}\Big(C^\infty(\mathbb{R}^{2n+s})\Big) = C^\infty(\mathbb{R}^{2n+s}).$$

The end

Thank you for the attention