A cut discontinuous Galerkin framework for mixed-dimensional problems

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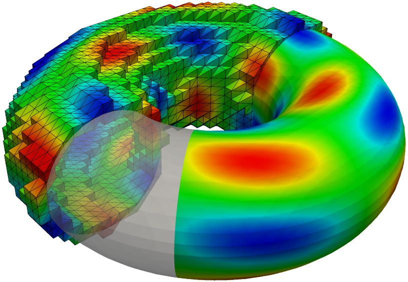
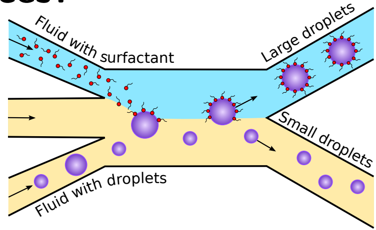
Thanks to
Ceren Gürkan,
Christoph Lehrenfeld, Fabian Heimann
Vetenskapsrådet, ESSENCE, Kempe foundation

Norwegian Meeting on PDEs

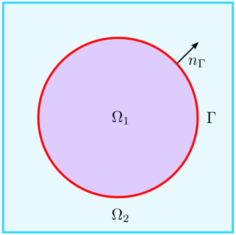
Trondheim, Norway
2019-06-07



How to model complex fluid behavior in microfluidic devices?



Microfluidic device



Single droplet fluid domains

G.K. Kurup and A.S. Basu, "Tensiophoresis: Migration and Sorting of Droplets in an Interfacial Tension Gradient," Micro Total Analysis Systems (MicroTAS).

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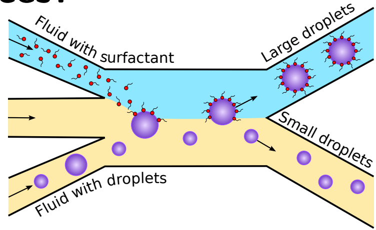

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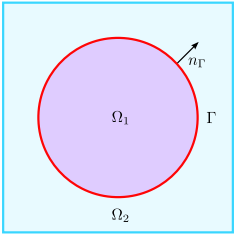

Single droplet fluid domains

$$\rho_i(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla\cdot\boldsymbol{\sigma}_i + \rho_i\boldsymbol{g} \quad \text{in } \Omega_i(t)$$
$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega_i(t)$$

$$[\boldsymbol{u}] = 0 \quad \text{on } \Gamma$$
$$[\boldsymbol{\sigma}\cdot\boldsymbol{n}_\Gamma] = -\tau\kappa\boldsymbol{n}_\Gamma \quad \text{on } \Gamma$$

Fluid equations

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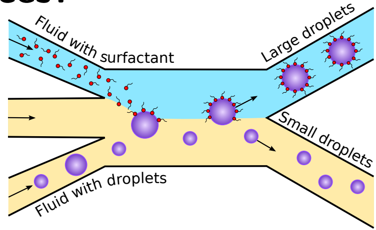

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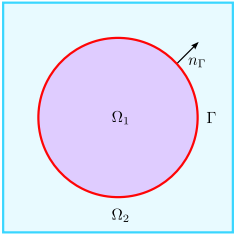

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Fluid equations

$$\partial_t c_\Omega + \boldsymbol{u}\cdot\nabla c_\Omega - \nabla\cdot(k_i\nabla c_\Omega) = 0 \quad \text{in } \Omega_i(t), \qquad [k\nabla c_\Omega\cdot\boldsymbol{n}] = j_{\text{coupling}} \quad \text{on } \Gamma(t)$$
$$\partial_t c_\Gamma + \boldsymbol{u}\cdot\nabla c_\Gamma + c_\Gamma\nabla_\Gamma\cdot\boldsymbol{u} - \nabla_\Gamma\cdot(k_\Gamma\nabla_\Gamma c_\Gamma) = j_{\text{coupling}} \quad \text{on } \Gamma(t)$$

Surfactant equations

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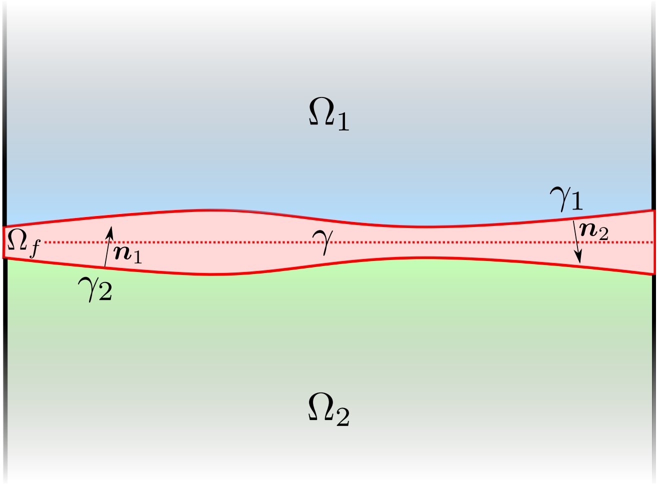
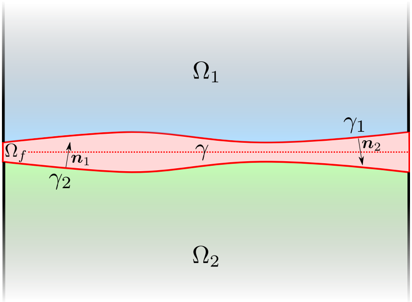
How to model flow and transport problems in porous media with large-scale fractures?



Fractured porous medium

Flow dynamics in fractured porous media: full 3D problem



Flow dynamics

$$\begin{cases} \nabla \cdot \boldsymbol{u}_i = f_v \\ \boldsymbol{u}_i = -\boldsymbol{K}_i \nabla p_i \end{cases} \qquad \text{in } \Omega_i$$

Coupling

$$\begin{cases} \boldsymbol{u}_j \cdot \boldsymbol{n}_i = \boldsymbol{u}_f \cdot \boldsymbol{n}_j \\ p_i = p_f \end{cases} \qquad \text{on } \gamma_i$$

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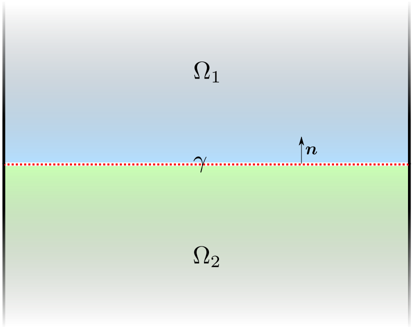

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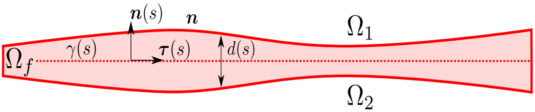
Flow dynamics in fractured porous media: Reduced interface problem

Differential operators in local coordinates

$$\boldsymbol{N} := \boldsymbol{n} \otimes \boldsymbol{n} \qquad \boldsymbol{T} := \boldsymbol{I} - \boldsymbol{N}$$
$$\nabla_{\boldsymbol{n}} g := \boldsymbol{N} \nabla g \qquad \nabla_{\boldsymbol{\tau}} g := \boldsymbol{T} \nabla g$$
$$\nabla_{\boldsymbol{n}} \cdot \boldsymbol{u} := \boldsymbol{N} : \nabla \boldsymbol{u} \qquad \nabla_{\boldsymbol{\tau}} \cdot \boldsymbol{u} := \boldsymbol{T} : \nabla \boldsymbol{u}$$



Assumption on local representation of $\boldsymbol{K}_f$

$$\boldsymbol{K}_f = K_{f,\boldsymbol{n}} \boldsymbol{N} + K_{f,\boldsymbol{\tau}} \boldsymbol{T}$$

Bulk flow

$$\begin{cases} \nabla \cdot \boldsymbol{u}_i = f_v \\ \boldsymbol{u}_i = -\boldsymbol{K}_i \nabla p_i \end{cases} \quad \text{in } \Omega_i$$

Fracture flow

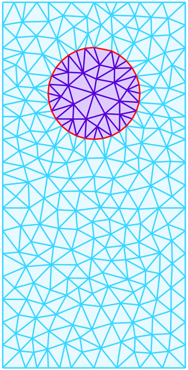
$$\begin{cases} \nabla_{\boldsymbol{\tau}} \cdot \widehat{\boldsymbol{u}}_f = \widehat{f}_v + [\boldsymbol{u} \cdot \boldsymbol{n}]_\gamma \\ \widehat{\boldsymbol{u}}_f = -d \boldsymbol{K}_{f,\boldsymbol{\tau}} \nabla_{\boldsymbol{\tau}} \widehat{p} \end{cases} \quad \text{on } \gamma$$

Coupling

$$\begin{cases} \xi_0 \eta_\gamma [\boldsymbol{u} \cdot \boldsymbol{n}]_\gamma = \{p\}_\gamma - \widehat{p}_f \\ \eta_\gamma \{\boldsymbol{u} \cdot \boldsymbol{n}\}_\gamma = [p]_\gamma \end{cases} \quad \text{on } \gamma$$

Martin/Jaffré/Roberts 2005, d'Angelo/Formaggia/Scotti/Fumagalli et al. 2011-2015

Unfitted discretization schemes can reduce the burden of mesh generation



Moving interfaces

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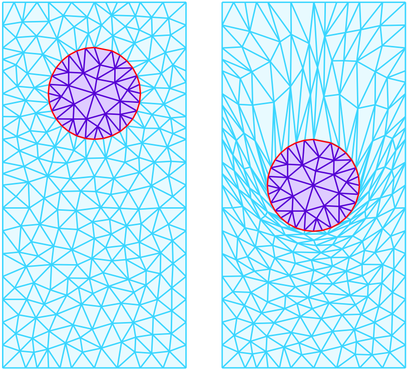

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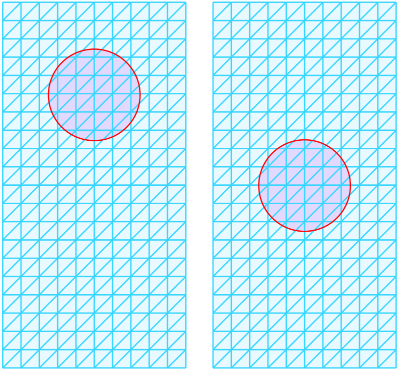

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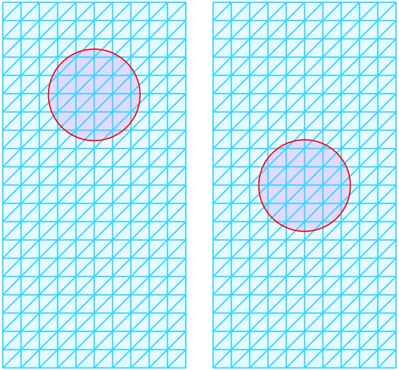

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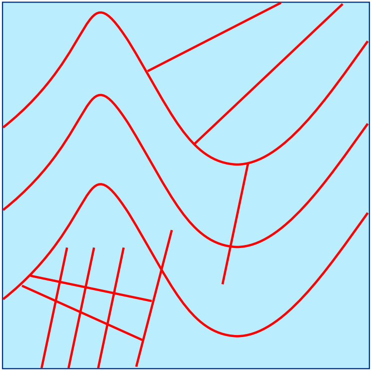

Embedded interfaces

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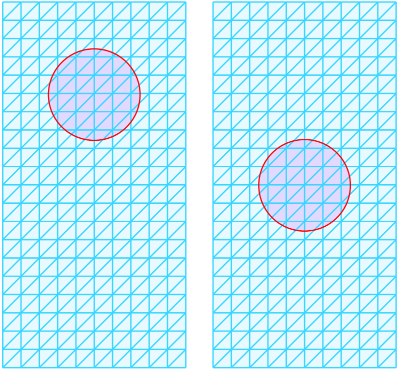
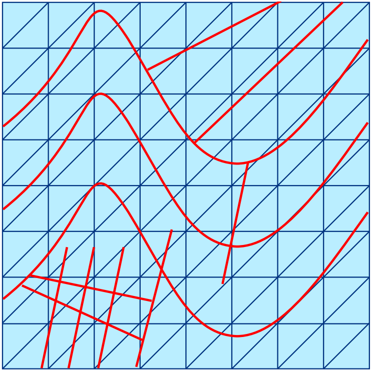


Moving interfaces



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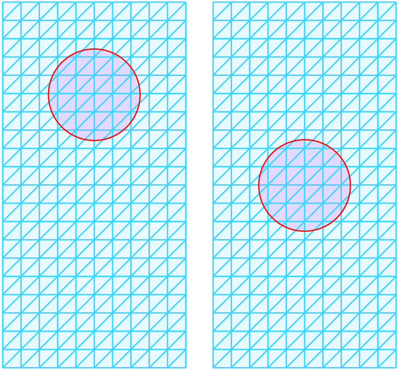
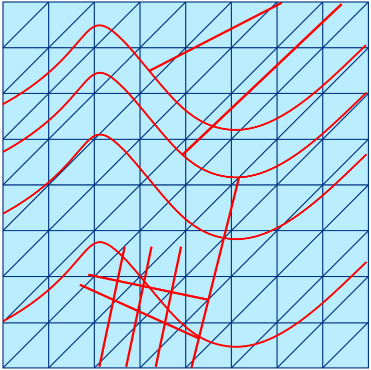


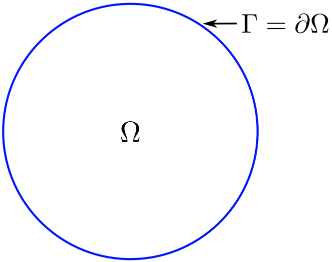


Moving interfaces

Nitsche's orginal method weakly imposes Dirichlet boundary conditions and inter-element continuity

Poisson problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
u &= g \quad \text{on } \Gamma
\end{aligned}
$$



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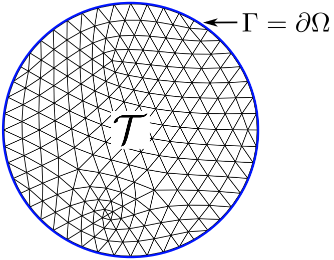

Variational formulation

Find u_h in $V_h = \bigoplus_{T \in \mathcal{T}_h} P_k(T)$ s.t.

$a_h(u_h, v) = l(v) \forall\, v \in V$.

$$a_h(u_h, v) = (\nabla u_h, \nabla v)_\Omega - \underbrace{\left(\{\nabla u_h \cdot \boldsymbol{n}\}, [v]\right)_{\mathcal{F}_h^i}}_{\text{Consistency}} - \underbrace{\left(\{\nabla v \cdot \boldsymbol{n}\}, [u_h]\right)_{\mathcal{F}_h^i}}_{\text{Symmetrization}} + \underbrace{\gamma \left(h^{-1}[u_h], [v]\right)_{\mathcal{F}_h^i}}_{\text{Penalization}}$$

$$l_h(v) = (f, v)_\Omega$$

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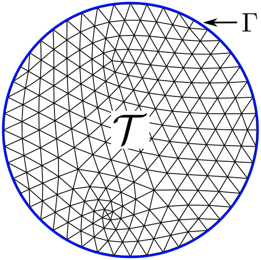

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$$l_h(v) = (f, v)_\Omega \qquad - (\nabla v \cdot \boldsymbol{n}, g)_\Gamma + \gamma(h^{-1} g, v)_\Gamma$$

Nitsche'71, Arnold '82

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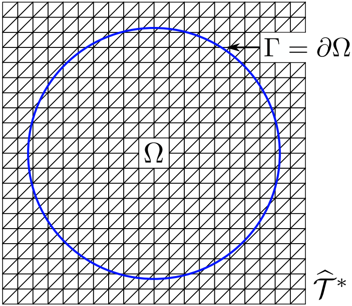

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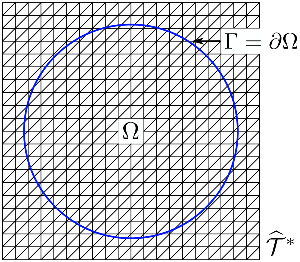
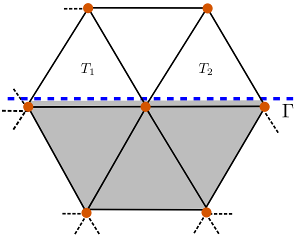
Cut discontinuous Galerkin methods encounter several theoretical challenges





Form stability

$$\|h^{1/2} n \cdot \nabla u\|_{\Gamma \cap T} \leqslant C \|\nabla u\|_{T \cap \Omega} \qquad C \to \infty$$

$$\|h^{1/2} n \cdot \nabla u\|_{F \cap T} \leqslant C \|\nabla u\|_{T \cap \Omega} \qquad C \to \infty$$

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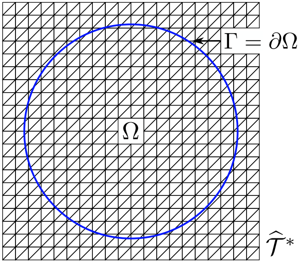

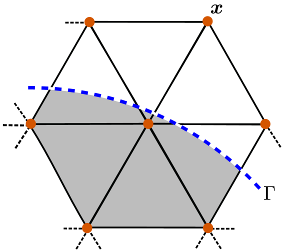

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$$\kappa(\mathcal{A}) \leqslant C h^2 \qquad C \to \infty$$

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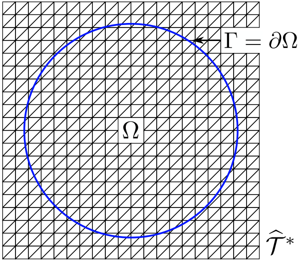

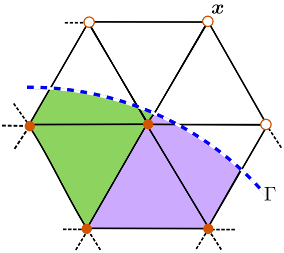

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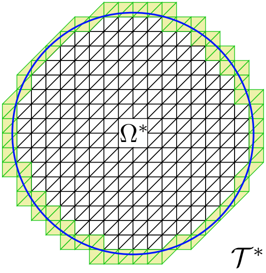
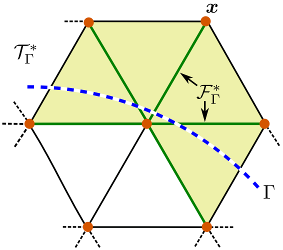
Clarke/Hassan/Salas'86, Quirk'94, Bastian/Engwer'09, Sollie/Bokhove/van der Vegt'11, Heimann/Engwer/Ippisch/Bastian'13, Qin/Krivodonava'13 Johansson/Larson'13, Müller/Kummer/Krämer-Eis/Oberlack'16, Badia/Verdugo/Martin'17 (cell-merging)

...but can be made robust by adding ghost penalties in the boundary zone





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Ghost penalty

$$|||v_h|||_{\Omega^*}^2 \sim |||v_h|||_{\Omega}^2 + g_h(v_h, v_h)$$

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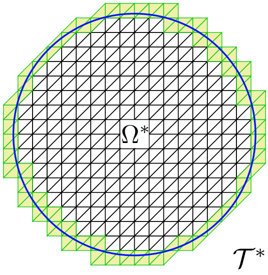

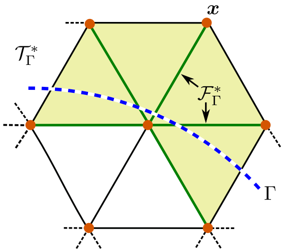

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Burman/Hansbo 2011

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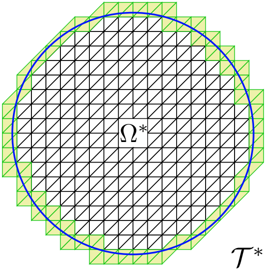

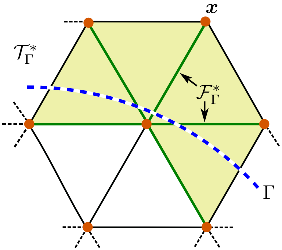

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Burman/Hansbo 2011/12, M./Logg/Larson/Rognes 2012/13, Hansbo/Larson/Zahedi 2014, Burman/Claus/M. 2015, M./Schott/Wall 2016, Burman/Hansbo/Larson 2015, Guzman/Sanchez/Sarkis 2015, Sticko/Kreiss 2016 ...

How to design the ghost penalty?

To guarantee geometrically robust optimal a priori error estimates g_h needs to satisfy 2 assumptions:

- **EP1** H^1 semi-norm extension property for $v \in V_h$,

$$\|\nabla v\|_{\mathcal{T}_h} \lesssim \|\nabla v\|_\Omega + |v|_{g_h}$$

- **EP2** Weak consistency for $v \in H^s(\Omega)$ and $r = \min\{s, k+1\}$,

$$|\pi_h^e v|_{g_h} \lesssim h^{r-1} \|v\|_{r,\Omega}$$

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To guarantee geometrically robust condition number estimates g_h needs to satisfy 2 more assumptions:

- **EP3** L^2 norm extension property for $v \in V_h$,

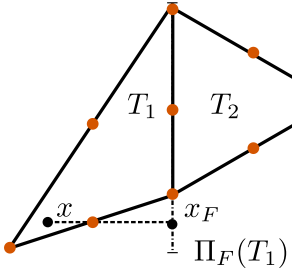
$$\|v\|_{\mathcal{T}_h} \lesssim \|v\|_\Omega + |v|_{g_h}$$

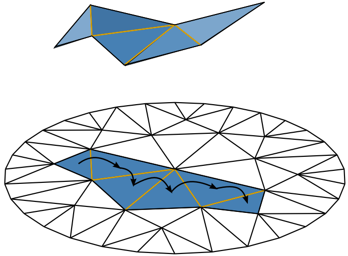
- **EP4** Inverse inequality for $v \in V_h$,

$$|v|_{g_h} \lesssim h^{-1} \|v\|_{\mathcal{T}_h}$$

Gürkan/M. 2019

Face-based jump penalties give a valid ghost-penalty





Lemma

Let v be a piecewise polynomial function defined on macro-element $\overline{T} = T_1 \cup T_2$ and $p = \max(\operatorname{ord}(v_1), \operatorname{ord}(v_2))$ of v. Then

$$\|v\|_{T_1}^2 \leqslant C\Big(\|v\|_{T_2}^2 + \sum_{j \leqslant p} h^{2j+1}([\partial_{\boldsymbol{n}}^j v], [\partial_{\boldsymbol{n}}^j v])_F\Big)$$

Proof: Scaling argument or Taylor-expansion

$$v_i(\boldsymbol{x}) = \sum_{|\alpha| \leqslant p} \frac{D^\alpha v_i(\boldsymbol{x}_F)}{\alpha!}(|\boldsymbol{x} - \boldsymbol{x}_F|\boldsymbol{n})^\alpha \Rightarrow v_1(\boldsymbol{x}) = v_2(\boldsymbol{x}) + \sum_{|\alpha| \leqslant p} \frac{[D^\alpha v(\boldsymbol{x}_F)]}{\alpha!}(|\boldsymbol{x} - \boldsymbol{x}_F|\boldsymbol{n})^\alpha$$

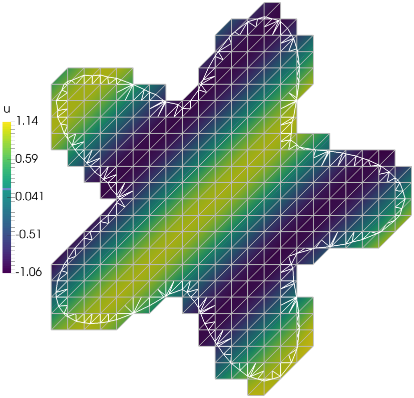
Burman '10, M. '12, M./Logg/Larson/Rognes '13

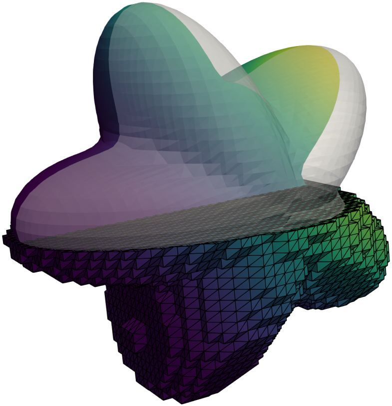
Ghost penalty is necessary to obtain geometrically robust optimal convergence rates

Theorem

Assume that $u \in H^{k+1}(\Omega)$, $V_h = \mathbb{P}^k_{\mathrm{dc}}(\mathcal{T}_h)$ *then*

$$|||u - u_h|||_{a_h,*} \lesssim h^k \|u\|_{k,\Omega}, \qquad \|u - u_h\|_\Omega \lesssim h^{k+1} \|u\|_{k,\Omega}.$$





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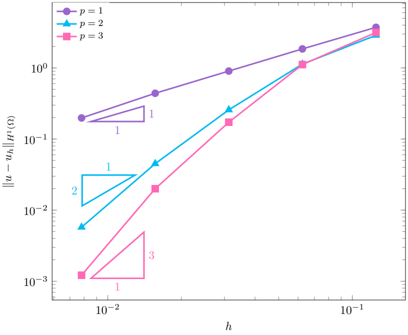

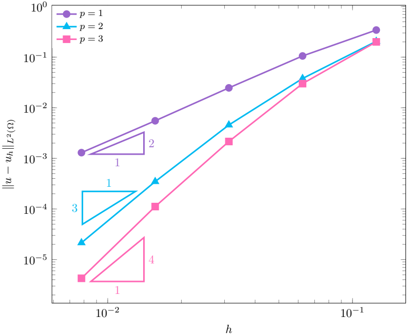

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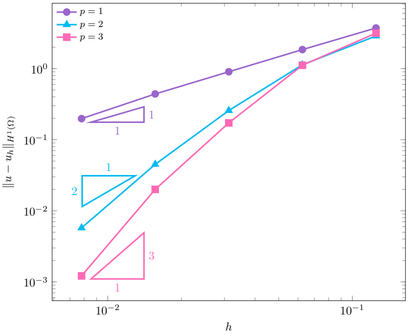
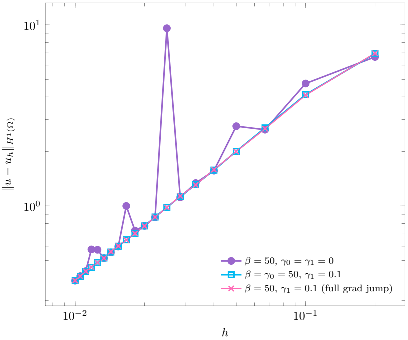


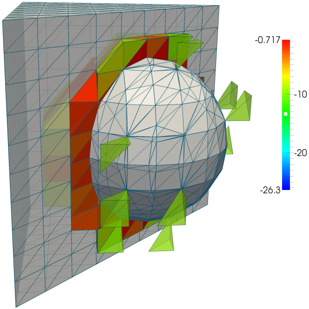


Ghost penalty is necessary to obtain geometrically robust optimal condition number estimates

Theorem

The condition number of the stiffness matrix satisfies the estimate

$$\kappa(\mathcal{A}) \lesssim h^{-2}$$



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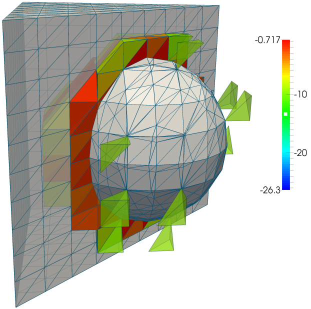

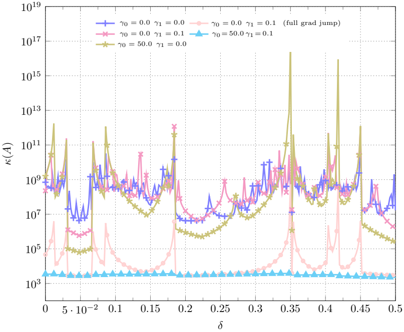

Take home message: small guys are unstable, big big guys are stable, ghost-penalties make smalls guys big again

Not so stable ...

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but with ghost penalties:

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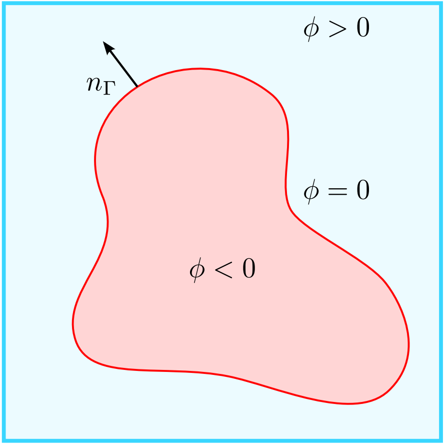
Not so stable ...

but with ghost penalties:

pretty stable!

The Poisson problem on a surface



$$\Gamma = \{\phi = 0\} \qquad n_\Gamma = \frac{\nabla \phi}{|\nabla \phi|}$$

Tangential gradient

$$\nabla_\Gamma u = (I - n_\Gamma \otimes n_\Gamma)\nabla \overline{u} = P_\Gamma \nabla u$$

Laplace-Beltrami operator

$$\Delta_\Gamma u = \nabla_\Gamma \cdot \nabla_\Gamma u$$

Poisson problem

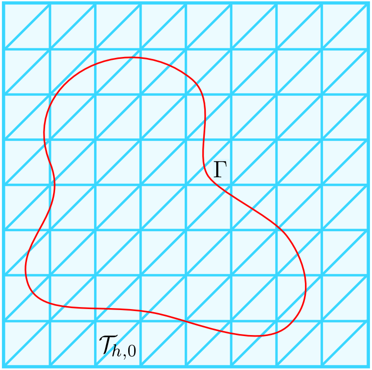
$$-\Delta_\Gamma u = f$$

Weak problem

$$(\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma = (f, v)_\Gamma$$

Dziuk '88, Olshanskii/Reusken/Grande '09, Dedner/Madhavan/Stinner '13, Burman/Hansbo/Larson '15, Burman/Hansbo/Larson/M. '16

Computational domains and level-set discretization



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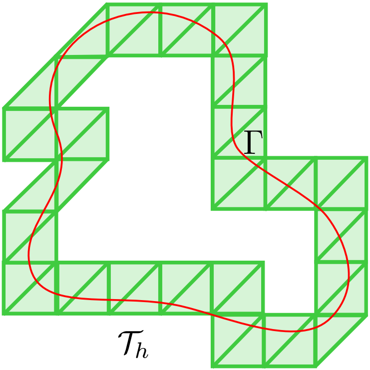

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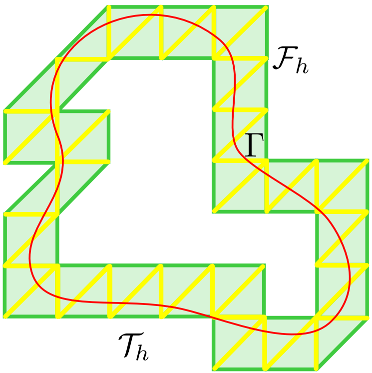
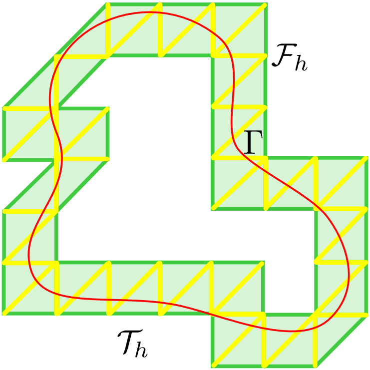
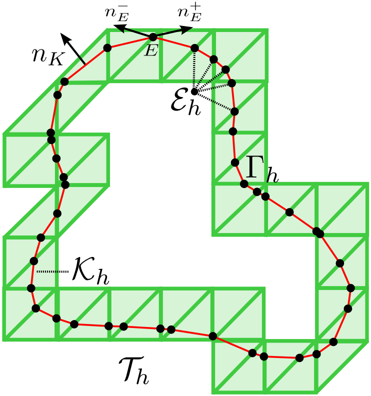


$$\mathcal{T}_h = \{ T \in \mathcal{T}_{h,0} : T \cap \Gamma \neq \emptyset \}$$

$$\mathcal{F}_h = \{ F = T^+ \cap T^- : T^+, T^- \in \mathcal{T}_h \}$$

Computational domains and level-set discretization





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$$\Gamma_h = \{ x \in \Omega : \rho_h(x) = 0 \}$$

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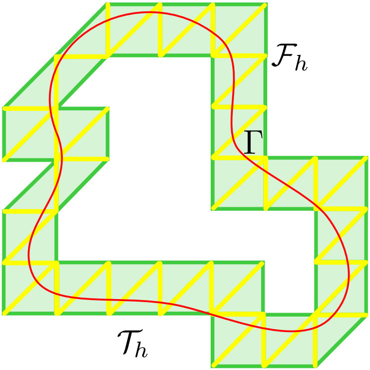
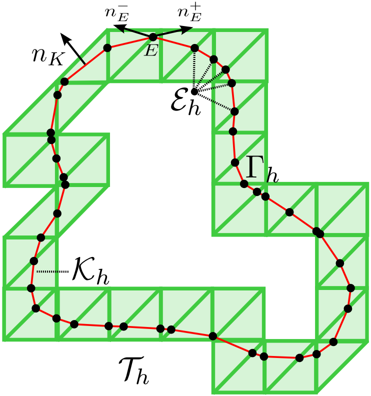
The following estimates hold

$$\|\rho\|_{L^\infty(\Gamma_h)} \lesssim h^2, \qquad \|n^e - n_K\|_{L^\infty(\mathcal{K}_h)} \lesssim h$$

Computational domains and level-set discretization





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The following estimates hold

$$\| g - \widetilde{g}_h \|_{L^\infty(\Gamma)} \lesssim h^{k+1}$$

A Cut Discontinuous Galerkin Method for the Poisson problem on Surfaces

Define $V^h = \bigoplus_{T \in \mathcal{T}^h} P_1(T) \qquad H^s(\mathcal{T}^h) = \bigoplus_{T \in \mathcal{T}^h} H^s(T)$ and the DG operators

$$\{n_E \cdot \nabla w\}|_E = \frac{1}{2}(n_E^+ \cdot \nabla w^+ - n_E^- \cdot \nabla w^-), \quad [w]|_E = w_E^+ - w_E^-, \quad [w]|_F = w_F^+ - w_F^-$$

The cutDGM takes the form: find $u^h \in V^h$ such that

$$\underbrace{a^h(v,w) + s^h(v,w)}_{A^h} = l^h(v) \quad \forall v \in V^h$$

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$$l^h(v) = (f^e, v)_{\Gamma^h}.$$

Norm: $\|v\|^2_{A^h} = \|\nabla_{\Gamma_h} v\|^2_{\mathcal{K}^h} + \|h^{-1/2}[v]\|^2_{\mathcal{E}^h} + s^h(v,v)$

Burman/Hansbo/Larson/M. 2016

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Burman/Hansbo/Larson/M. 2016

Heimann/Lehrenfeld/M. 2019

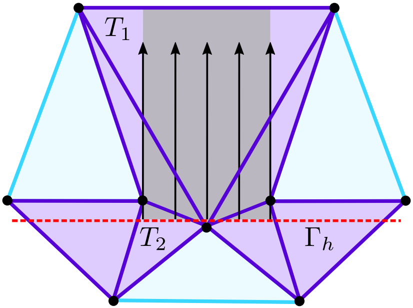
Mechanism behind the normal gradient stabilization

Lemma

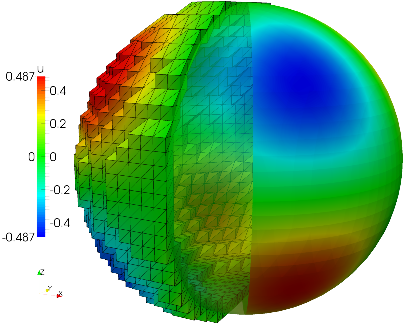
For $v \in V_h$ it holds

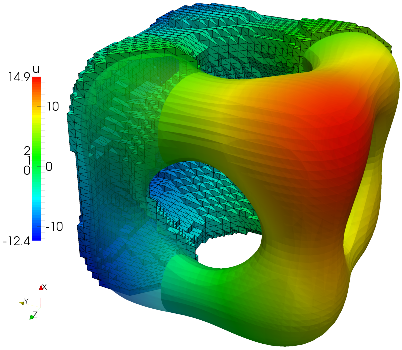
$$h^{-1}\|v\|^2_{\mathcal{T}_h} \lesssim \|v\|^2_{\Gamma_h} + \|\,[v]\,\|_{\mathcal{F}_h} + h\|n^h \cdot \nabla v\|^2_{\mathcal{T}_h},$$

where the hidden constant depends only on quasi-uniformity.



Ghost penalty is necessary to obtain geometrically robust optimal convergence rates





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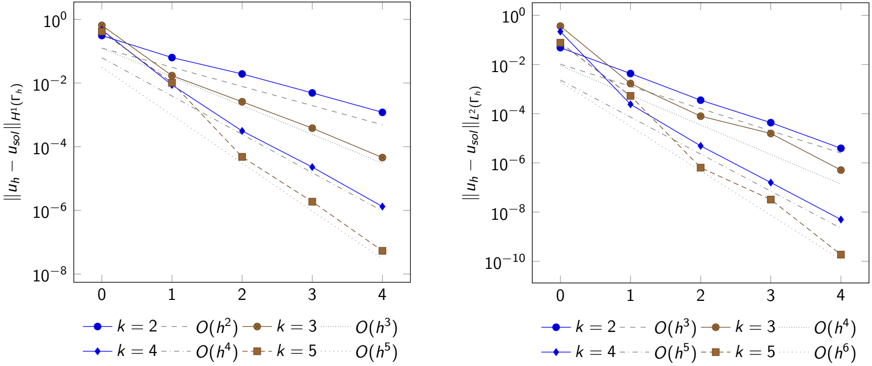

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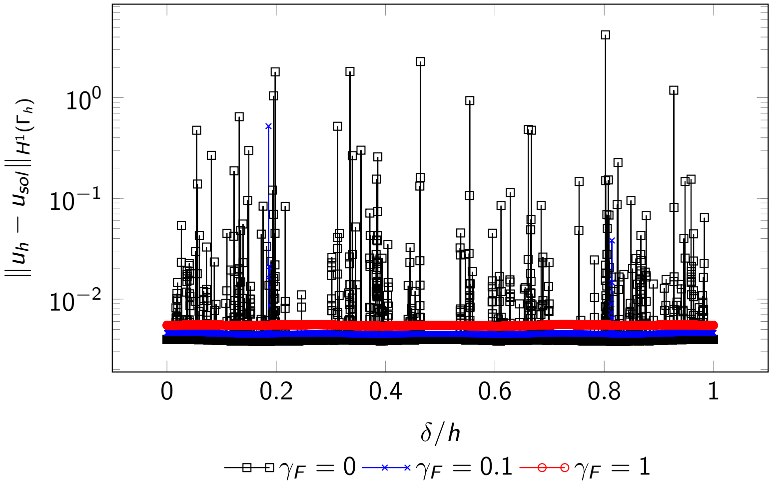
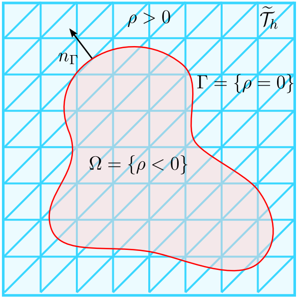


Coupled surface-bulk problems can be easily discretized using cutDG methods

Surface-bulk problem



$$
\begin{aligned}
-\nabla \cdot (k_\Omega \nabla u_\Omega) + u_\Omega &= f_\Omega \quad \text{in } \Omega \\
c_\Omega u_\Omega - c_\Gamma u_\Gamma + k_\Omega \nabla u_\Omega \cdot n &= 0 \quad \text{on } \Gamma \\
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\end{aligned}
$$

Ranner/Elliot'13 (fitted),
Groß/Olshanskii/Reusken'15 ,
Burman/Hansbo/Larson/Zahedi'16
(CG-Trace/CutFEM) M.'18,
Heiman/Lehrenfeld/M. (in prep)
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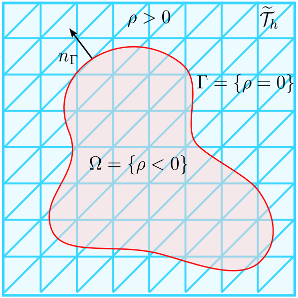

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Weak formulation

Find $u = (u_\Gamma, u_\Omega) \in H^1(\Omega) \times H^1(\Gamma)$ s.t.
$a(u, v) = l(v) \; \forall\, v \in V$ where

$$
\begin{aligned}
a(u, v) &= a_\Omega(u, v) + a_\Gamma(u, v) + a_{\Omega\Gamma}(u, v) \\
a_\Omega(u_\Omega, v_\Omega) &= c_\Omega \left(k_\Omega \nabla u_\Omega, \nabla v_\Omega\right)_\Omega \\
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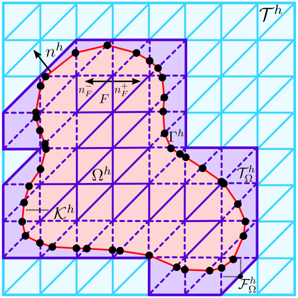
Coupled surface-bulk problems can be easily discretized using cutDG methods

Surface-bulk problem



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Discrete formulation

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$a^h(u^h, v) + s^h(u^h, v) = l(v) \forall\, v \in V^h$.

Ghost penalties

$$
s_\Omega(v_\Omega, w_\Omega) = h^{-1}([v_\Omega], [w_\Omega])_{\mathcal{F}_\Omega} + h([\nabla v_\Omega \cdot n], [\nabla w_\Omega \cdot n])_{\mathcal{F}_\Omega^h}
$$

$$
s_\Gamma(v_\Gamma, w_\Gamma) = h^{-2}([v_\Gamma], [w_\Gamma])_{\mathcal{F}_\Gamma^h} + ([\nabla v_\Gamma \cdot n], [\nabla w_\Gamma \cdot n])_{\mathcal{F}_\Gamma^h}
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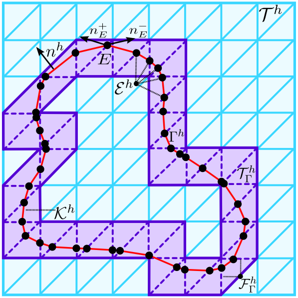
Coupled surface-bulk problems can be easily discretized using cutDG methods

Surface-bulk problem



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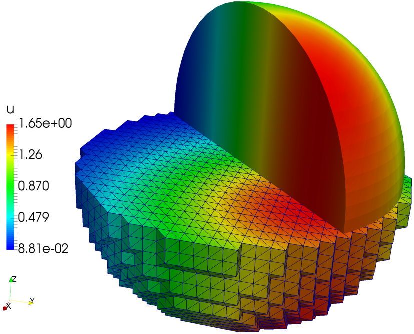
$a^h(u^h, v) + s^h(u^h, v) = l(v) \forall \, v \in V^h$.

Ghost penalties

$$s_\Omega(v_\Omega, w_\Omega) = h^{-1}([v_\Omega], [w_\Omega])_{\mathcal{F}_\Omega} + h([\nabla v_\Omega \cdot n], [\nabla w_\Omega \cdot n])_{\mathcal{F}_\Omega^h}$$

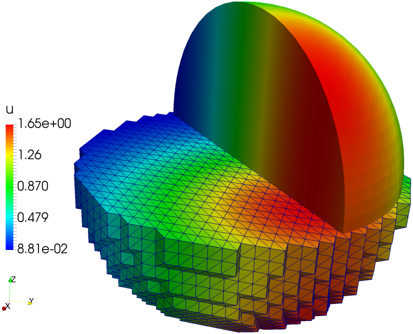
$$s_\Gamma(v_\Gamma, w_\Gamma) = h^{-2}([v_\Gamma], [w_\Gamma])_{\mathcal{F}_\Gamma^h} + ([\nabla v_\Gamma \cdot n], [\nabla w_\Gamma \cdot n])_{\mathcal{F}_\Gamma^h}$$

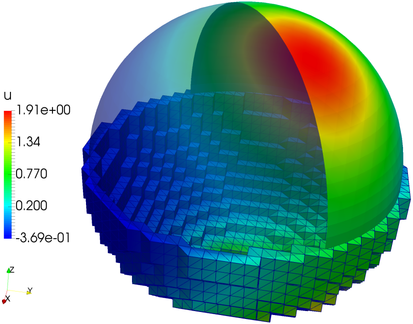
Coupled surface-bulk problems can be easily discretized using cutDG methods



Manufactured solution from Elliot/Ranner '13

Coupled surface-bulk problems can be easily discretized using cutDG methods





Manufactured solution from Elliot/Ranner '13

Coupled surface-bulk problems can be easily discretized using cutDG methods

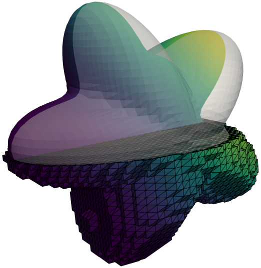
k	$\|e^h\|_{H^1(\Omega^h)}$	EOC	$\|e^k\|_{L^2(\Omega^h)}$	EOC	$\|e^k\|_{H^1(\Gamma^h)}$	EOC	$\|e^k\|_{L^2(\Gamma^h)}$	EOC
0	$5.28 \cdot 10^{-1}$	–	$8.60 \cdot 10^{-2}$	–	$2.17 \cdot 10^{0}$	–	$2.73 \cdot 10^{-1}$	–
1	$3.44 \cdot 10^{-1}$	$+0.62$	$3.04 \cdot 10^{-2}$	$+1.50$	$1.12 \cdot 10^{0}$	$+0.96$	$7.38 \cdot 10^{-2}$	$+1.89$
2	$1.84 \cdot 10^{-1}$	$+0.90$	$7.34 \cdot 10^{-3}$	$+2.05$	$5.80 \cdot 10^{-1}$	$+0.94$	$1.80 \cdot 10^{-2}$	$+2.04$
3	$9.35 \cdot 10^{-2}$	$+0.98$	$1.83 \cdot 10^{-3}$	$+2.00$	$2.76 \cdot 10^{-1}$	$+1.07$	$4.63 \cdot 10^{-3}$	$+1.96$
4	$4.71 \cdot 10^{-2}$	$+0.99$	$4.66 \cdot 10^{-4}$	$+1.98$	$1.39 \cdot 10^{-1}$	$+0.99$	$1.07 \cdot 10^{-3}$	$+2.12$

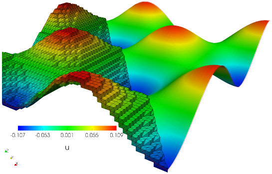
Coupled surface-bulk problems can be easily discretized using cutDG methods

k	$\|e^h\|_{H^1(\Omega^h)}$	EOC	$\|e^k\|_{L^2(\Omega^h)}$	EOC	$\|e^k\|_{H^1(\Gamma^h)}$	EOC	$\|e^k\|_{L^2(\Gamma^h)}$	EOC
0	$5.28 \cdot 10^{-1}$	–	$8.60 \cdot 10^{-2}$	–	$2.17 \cdot 10^{0}$	–	$2.73 \cdot 10^{-1}$	–
1	$3.44 \cdot 10^{-1}$	$+0.62$	$3.04 \cdot 10^{-2}$	$+1.50$	$1.12 \cdot 10^{0}$	$+0.96$	$7.38 \cdot 10^{-2}$	$+1.89$
2	$1.84 \cdot 10^{-1}$	$+0.90$	$7.34 \cdot 10^{-3}$	$+2.05$	$5.80 \cdot 10^{-1}$	$+0.94$	$1.80 \cdot 10^{-2}$	$+2.04$
3	$9.35 \cdot 10^{-2}$	$+0.98$	$1.83 \cdot 10^{-3}$	$+2.00$	$2.76 \cdot 10^{-1}$	$+1.07$	$4.63 \cdot 10^{-3}$	$+1.96$
4	$4.71 \cdot 10^{-2}$	$+0.99$	$4.66 \cdot 10^{-4}$	$+1.98$	$1.39 \cdot 10^{-1}$	$+0.99$	$1.07 \cdot 10^{-3}$	$+2.12$

k	$\|e^h\|_{H^1(\Omega^h)}$	EOC	$\|e^k\|_{L^2(\Omega^h)}$	EOC	$\|e^k\|_{H^1(\Gamma^h)}$	EOC	$\|e^k\|_{L^2(\Gamma^h)}$	EOC
0	$7.27 \cdot 10^{-1}$	–	$1.32 \cdot 10^{-1}$	–	$5.38 \cdot 10^{0}$	–	$1.16 \cdot 10^{0}$	–
1	$8.88 \cdot 10^{-1}$	-0.29	$1.99 \cdot 10^{-1}$	-0.59	$8.46 \cdot 10^{0}$	-0.65	$1.82 \cdot 10^{0}$	-0.65
2	$1.14 \cdot 10^{0}$	-0.36	$2.72 \cdot 10^{-1}$	-0.45	$1.02 \cdot 10^{2}$	-3.59	$2.60 \cdot 10^{0}$	-0.51
3	$1.01 \cdot 10^{0}$	$+0.17$	$2.51 \cdot 10^{-1}$	$+0.11$	$1.87 \cdot 10^{1}$	$+2.44$	$2.25 \cdot 10^{0}$	$+0.21$

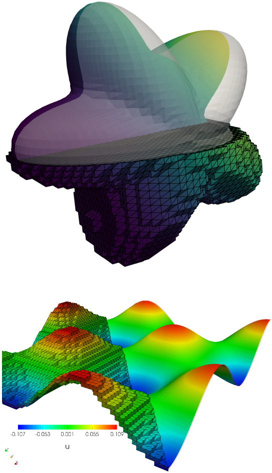
In conclusion, stabilized cutDGMs yield robust schemes for multidimensional multi-physics problems





Embedded domains

In conclusion, stabilized cutDGMs yield robust schemes for multidimensional multi-physics problems



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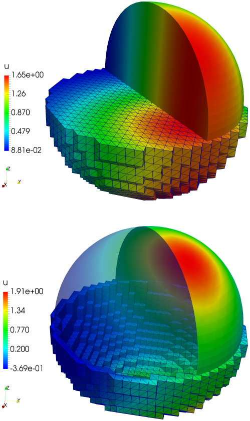
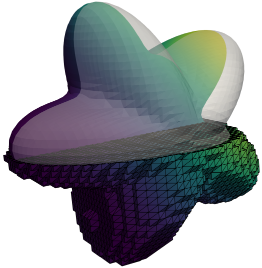
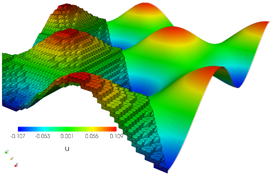


Multidimensional coupled problems

In conclusion, stabilized cutDGMs yield robust schemes for multidimensional multi-physics problems





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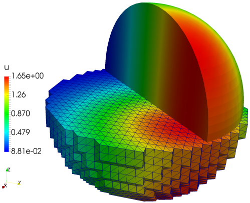
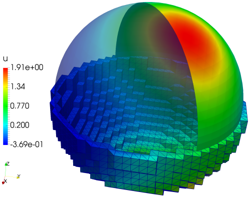
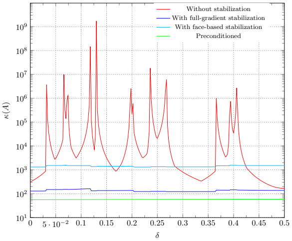




Multidimensional coupled problems



k	$\|u_k - u\|_{1,\Gamma_h}$	EOC	$\|u_k - u\|_{\Gamma_h}$	EOC
0	$9.39{\cdot}10^0$	–	$1.04{\cdot}10^0$	–
1	$5.00{\cdot}10^0$	0.91	$3.38{\cdot}10^{-1}$	1.62
2	$2.43{\cdot}10^0$	1.04	$8.43{\cdot}10^{-2}$	2.00
3	$1.21{\cdot}10^0$	1.00	$2.14{\cdot}10^{-2}$	1.98
4	$5.99{\cdot}10^{-1}$	1.02	$5.28{\cdot}10^{-3}$	2.02
5	$2.96{\cdot}10^{-1}$	1.01	$1.31{\cdot}10^{-3}$	2.01

Geometrically robust