Unconditional uniqueness of solutions to the derivative nonlinear Schrödinger equation

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The derivative nonlinear Schrödinger equation

Consider the initial-value problem:

$$\begin{cases} \partial_t u - i\partial_x^2 u = \partial_x(|u|^2 u)\,,\ t \in \mathbb{R}, x \in \mathbb{R}\,, \\ u|_{t=0} = u_0 \in H^s(\mathbb{R})\,. \end{cases} \tag{DNLS}$$

Model in plasma physics: Mjølhus, Røgister, Mio, Ogino, Takeda, ... (1970s)
propagation of Alfvén waves in magnetized plasma with const. magnetic field

Kaup-Newell '78: DNLS admits Lax pair.

Aim: Study the uniqueness of low-regularity solutions ($s < 1$).

Def. **Solution**: $u \in C([0,T]; H^s(\mathbb{R})) =: C_T H^s$ which satisfies

$$u(t) = e^{it\partial_x^2} u_0 + \int_0^t e^{i(t-t')\partial_x^2} \partial_x \big(|u(t')|^2 u(t')\big)\, dt'$$

in the sense of (spatial) distributions, for all t.

Usually, the well-posedness results yield *conditional uniqueness* of solutions, i.e. in $C_T H^s \cap X_T$.

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The 1-d cubic nonlinear Schrödinger equation (NLS)

$$\partial_t u - i\partial_x^2 u = i|u|^2 u \qquad \text{(NLS)}$$

Local well-posedness (LWP) in $H^s(\mathbb{M})$:

1. $s > \frac{1}{2}$: H^s is an algebra $\implies$ LWP in H^s
2. Tsutsumi '87: LWP in $L^2(\mathbb{R})$ via Strichartz estimates
3. Bourgain '93: LWP in $L^2(\mathbb{T})$ via the Fourier restriction norm method

- Uniqueness in **2** and **3** holds *conditionally*, i.e. in

$$\mathbf{2}: \quad C_T L_x^2 \cap L_T^8 L_x^4$$

$$\mathbf{3}: \quad X_T^{0,\frac{1}{2}+\varepsilon} \subset C_T L_x^2 \cap L_{T,x}^4$$

4. NLS is ill-posed in negative regularity Sobolev spaces

Unconditional uniqueness (UU):

5. Kato '95: UU in $H^s(\mathbb{R})$, for any $s \geq \frac{1}{6}$
6. Guo-Kwon-Oh '13: UU in $H^s(\mathbb{T})$, for any $s \geq \frac{1}{6}$

- *infinitely-many* normal form reductions.
- Kwon-Oh-Yoon '18: extended the Guo-Kwon-Oh method to the non-periodic setting.

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- Takaoka '99: LWP, $s \geq \frac{1}{2}$ [Fourier restriction norm method];
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- Herr '05: LWP, $s \geq \frac{1}{2}$;
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Let $s > \frac{1}{2}$ and $u_0 \in H^s(\mathbb{R})$. Then, there is a unique sol. to (DNLS) in $C_T H^s$.

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Let $s > \frac{1}{2}$ *and* $u_0 \in H^s(\mathbb{R})$. *Then, there is a unique sol. to* (DNLS) *in* $C_T H^s$.

Set-up for the normal form method

$$\partial_t u - i\partial_x^2 u = \partial_x(|u|^2 u) \tag{DNLS}$$

• use a gauge transformation (Lee '89, Hayashi '92, Hayashi-Ozawa '92):

$$u \longmapsto w := \mathcal{G}(u)(x) = u(x) \exp\left(-i \int_{-\infty}^{x} |u(y)|^2 dy\right).$$

[Lemma: For $s \geq 0$, $\mathcal{G}$ is bi-Lipschitz on bounded subsets of $H^s(\mathbb{R})$.]

(DNLS) $\iff$ $$\partial_t w - i\partial_x^2 w = -w^2 \partial_x \overline{w} + \frac{i}{2}|w|^4 w \tag{gDNLS}$$

Why? A.: this eliminates the bad trilinear term $|u|^2 \partial_x u$.

• use the van der Pole change of variable:

$$w \longmapsto v := e^{-it\partial_x^2}(w(t))$$

(DNLS) $\iff$ $$\boxed{\partial_t v = \mathcal{T}(v) + \mathcal{Q}(v)} \tag{$\star$}$$

Why? A.: we can exploit an oscillatory factor (of the form $e^{it\Phi}$) in $\mathcal{T}(v)$.

Set-up for the normal form method

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The nonlinearities of $(\star)$

$$\boxed{\partial_t v = \mathcal{T}(v) + \mathcal{Q}(v)} \qquad (\star)$$

where $\mathcal{T}(v)$ is cubic with derivative on the complex-conjugate factor, i.e.

$$\widehat{\mathcal{T}(v)}(t,\xi) = \int_{\xi_1-\xi_2+\xi_3=\xi} e^{i\Phi(\vec{\xi})t}\xi_2 \widehat{v}(t,\xi_1)\overline{\widehat{v}(t,\xi_2)}\widehat{v}(t,\xi_3)\, d\xi_1 d\xi_2\,,$$

$$\Phi(\vec{\xi}) = \xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2 = 2(\xi_2-\xi_1)(\xi_2-\xi_3)\,,$$

and $\mathcal{Q}(v)$ is quintic, i.e. $\mathcal{Q}(v)(t) := \frac{i}{2}|e^{it\partial_x^2}v(t)|^4 e^{it\partial_x^2}v(t)$.

The quintic term is easy to control in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$:

$$\|\mathcal{Q}(v)\|_{H^s} \lesssim \|e^{it\partial_x^2}v\|_{H^s}^5 = \|v\|_{H^s}^5\,.$$

Henceforth, let's ignore $\mathcal{Q}(v)$ in $(\star)$.

Two key ideas for $\mathcal{T}(v)$:

(i) If we restrict the phase $\Phi(\vec{\xi})$, then we can establish a useful estimate.

(ii) If $|\Phi(\vec{\xi})| \gg 1$, then we exploit the highly oscillating nature of the integrand.

Infinitely-many normal form reductions: bird's eye view

Algorithmic procedure:

(i) Separate the problematic nonlinear terms into *nearly resonant* and *non-resonant* parts; Rmk.: *the threshold may change at each step*!

(ii) *Eliminate the non-resonant parts* via integration by parts in time $\Longrightarrow$ introduces *higher* order terms;

(iii) Repeat (or terminate the process at some finite step). Rmk.: for DNLS, infinitely-many iterations are needed.

Initially:
$$\partial_t v = \mathcal{T}(v) + \mathcal{Q}(v) \qquad (\star)$$

At the Jth step:
$$\partial_t v = \sum_{j=2}^{J} \partial_t \mathcal{T}_0^{(j)}(v) + \sum_{j=1}^{J} \mathcal{T}_1^{(j)}(v) + \mathcal{T}_2^{(J)}(v) + \sum_{j=2}^{J} \mathcal{T}_{\mathcal{Q}}^{(j)}(v) + \mathcal{Q}(v)$$

Key: the "remainder" term vanishes in the limit: **for any fixed** N,
$$\left\| \mathcal{T}_2^{(J)}(v) \right\|_{H^{s-1}} \to 0 \text{ as } J \to \infty .$$

Thus, any solution to $(\star)$ is also a solution to
$$\partial_t v = \sum_{j=2}^{\infty} \partial_t \mathcal{T}_0^{(j)}(v) + \sum_{j=1}^{\infty} \mathcal{T}_1^{(j)}(v) + \sum_{j=2}^{\infty} \mathcal{T}_{\mathcal{Q}}^{(j)}(v) + \mathcal{Q}(v). \qquad \text{(NFeq)}$$

1st iteration:

(1.i) Decompose:
$$\mathcal{T}(v) = \underbrace{\mathcal{T}_1^{(1)}(v)}_{|\Phi(\vec{\xi})|\leq N} + \underbrace{\mathcal{T}_2^{(1)}(v)}_{|\Phi(\vec{\xi})|>N}$$

(1.ii) Normal form reduction:

$$\widehat{\mathcal{T}_2^{(1)}(v)}(t,\xi) = \partial_t \int_{\xi_1-\xi_2+\xi_3=\xi} \mathbf{1}_{|\Phi(\vec{\xi})|>N} \frac{e^{it\Phi(\vec{\xi})}\xi_2}{\Phi(\vec{\xi})} \widehat{v}(t,\xi_1)\overline{\widehat{v}(t,\xi_2)}\widehat{v}(t,\xi_3)d\xi_1 d\xi_2$$
$$- \int_{\xi_1-\xi_2+\xi_3=\xi} \mathbf{1}_{|\Phi(\vec{\xi})|>N} \frac{e^{it\Phi(\vec{\xi})}\xi_2}{\Phi(\vec{\xi})} \partial_t\Big(\widehat{v}(t,\xi_1)\overline{\widehat{v}(t,\xi_2)}\widehat{v}(t,\xi_3)\Big)\, d\xi_1 d\xi_2$$

$$= \partial_t \widehat{\mathcal{T}_0^{(2)}(v)}(t,\xi)$$
$$- \int_{\xi_1-\xi_2+\xi_3=\xi} \mathbf{1}_{|\Phi(\vec{\xi})|>N} \Bigg(\int_{\xi_{11}-\xi_{12}+\xi_{13}=\xi_1} \frac{e^{it\left(\Phi(\vec{\xi})+\Phi(\vec{\xi_1})\right)}\xi_2\xi_{12}}{\Phi(\vec{\xi})} \widehat{v}(\xi_{11})\overline{\widehat{v}(\xi_{12})}\widehat{v}(\xi_{13})\Bigg)\overline{\widehat{v}(\xi_2)}\widehat{v}(\xi_3)$$
$$- \int_{\xi_1-\xi_2+\xi_3=\xi} \mathbf{1}_{|\Phi(\vec{\xi})|>N} \widehat{v}(\xi_1)\Bigg(\int_{\xi_{21}-\xi_{22}+\xi_{23}=\xi_2} \frac{e^{-it\left(\Phi(\vec{\xi})+\Phi(\vec{\xi_2})\right)}\xi_2\xi_{22}}{\Phi(\vec{\xi})} \overline{\widehat{v}(\xi_{11})}\widehat{v}(\xi_{12})\overline{\widehat{v}(\xi_{13})}\Bigg)\widehat{v}(\xi_3)$$
$$- \int_{\xi_1-\xi_2+\xi_3=\xi} \mathbf{1}_{|\Phi(\vec{\xi})|>N} \widehat{v}(\xi_1)\overline{\widehat{v}(\xi_2)}\Bigg(\int_{\xi_{31}-\xi_{32}+\xi_{33}=\xi_3} \frac{e^{it\left(\Phi(\vec{\xi})+\Phi(\vec{\xi_3})\right)}\xi_2\xi_{32}}{\Phi(\vec{\xi})} \widehat{v}(\xi_{11})\overline{\widehat{v}(\xi_{12})}\widehat{v}(\xi_{13})\Bigg)$$

It is convenient to keep track of the various terms by using tree diagrams. Schematically, we rewrite the above step as:

$$\mathcal{T}_2^{(1)} = \partial_t \mathcal{T}_0^{(2)} - \int_* \frac{e^{it\Phi}\xi_2}{\Phi} \partial_t \Big(\quad \Big)$$

$$= \partial_t \mathcal{T}_0^{(2)} - \underbrace{\int_* \frac{e^{it(\Phi+\Phi^{(1)})}\xi_2 \xi_2^{(1)}}{\Phi} \Big(\quad + \quad + \quad \Big)}_{=:\mathcal{T}^{(2)}}$$

Upshot: $$\partial_t v = \mathcal{T}_1^{(1)}(v) + \partial_t \mathcal{T}_0^{(2)}(v) + \mathcal{T}^{(2)}(v) \qquad (\star\star)$$

2nd iteration:

(2.i) Decompose: $$\mathcal{T}^{(2)}(v) = \underbrace{\mathcal{T}_1^{(2)}(v)}_{|\Phi+\Phi^{(1)}|\le\beta_1|\Phi|} + \underbrace{\mathcal{T}_2^{(2)}(v)}_{|\Phi+\Phi^{(1)}|>\beta_1|\Phi|} \qquad (\beta_1 \ge 2)$$

(2.ii) Normal form reduction:

$$\mathcal{T}_2^{(2)} = \partial_t \mathcal{T}_0^{(3)} - \int_* \frac{e^{it(\Phi+\Phi^{(1)})}\xi_2 \xi_2^{(1)}}{\Phi} \partial_t \Big(\quad + \quad + \quad \Big)$$

$$= \partial_t \mathcal{T}_0^{(3)} - \int_* \frac{e^{it(\Phi+\Phi^{(1)}+\Phi^{(2)})}\xi_2 \xi_2^{(1)} \xi_2^{(2)}}{\Phi(\Phi+\Phi^{(1)})} \Big(\quad \Big)$$

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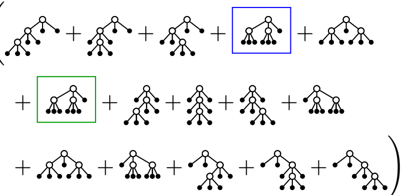

Bookkeeping of terms

Difficulty: When we apply integration by parts, the time derivative may fall on any of the factors $\widehat{v}(t, \xi_j)$. *The structure of the resulting terms depends on where the time derivative falls.*

Christ '07, *Power series solution of a nonlinear Schrödinger equation*:

- introduced *ordered trees* for indexing such terms arising at the Jth step.
- ordered trees = (ternary) trees "with memory"
  NB: keep track of the order in which we have used the $\partial_t v$-equation.
- $\#\{\text{ordered trees of generation } J\} = 1 \cdot 3 \cdot 5 \cdots (2J-1) = (2J-1)!!$.
  NB: we need to overcome this rapidly growing cardinality.
- For any tree T of generation J:

$$\# \{\text{root (non-terminal) nodes of } T\} = J,$$
$$\# \{\text{terminal nodes of } T\} = 2J+1.$$

The analytical part

Consider the trilinear operator $\mathcal{T}_\Phi$ defined by

$$\mathcal{F}\Big[\mathcal{T}_{|\Phi|^{\frac{1}{2}}}(v_1, v_2, v_3)\Big](\xi) = \int_{\xi=\xi_1-\xi_2+\xi_3} \frac{|\xi_2|}{|\Phi(\vec{\xi})|^{\frac{1}{2}}} \widehat{v_1}(\xi_1)\overline{\widehat{v_2}(\xi_2)}\widehat{v_3}(\xi_3) d\xi_1 d\xi_2$$

Recall: $\Phi(\vec{\xi}) = 2(\xi_2 - \xi_1)(\xi_2 - \xi_3)$.

Lemma 1 (Basic trilinear estimate in the H^s-norm)

Let $s > \frac{1}{2}$. Then there exists $C = C(s) > 0$ such that

$$\left\|\mathcal{T}_{|\Phi|^{\frac{1}{2}}}(v_1, v_2, v_3)\right\|_{H^s} \leq C \prod_{j=1}^{3} \|v_j\|_{H^s} .$$

Schematically:

$$\left\| \int_* \frac{|\xi_2|}{|\Phi|^{\frac{1}{2}}} \pitchfork \right\|_{H^s} \leq C \|\bullet\|_{H_s}^3$$

$$\implies \quad \left\|\mathcal{T}_1^{(1)}(v)\right\|_{H^s} \lesssim \left\| \int_* \mathbf{1}_{|\Phi|\leq N} |\xi_2| \pitchfork \right\|_{H^s} \lesssim N^{\frac{1}{2}} \|v\|_{H^s}^3$$

$$\implies \quad \left\|\mathcal{T}_0^{(1)}(v)\right\|_{H^s} \sim \left\| \int_* \mathbf{1}_{|\Phi|>N} \frac{|\xi_2|}{|\Phi|} \pitchfork \right\|_{H^s} \lesssim N^{-\frac{1}{2}} \|v\|_{H^s}^3$$

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Terms in the 2nd iteration:

$$|\Phi| > N \text{ and } |\Phi + \Phi^{(1)}| \leq \beta_1 |\Phi| \quad \Longrightarrow \quad |\Phi^{(1)}| \leq 2\beta_1 |\Phi|.$$

$$\begin{aligned}
\Longrightarrow \quad \left\| \int_* \frac{|\xi_2| \cdot |\xi_2^{(1)}|}{|\Phi|} \right\|_{H^s} &\leq (2\beta_1)^{\frac{1}{2}} \left\| \int_* \frac{|\xi_2|\,|\xi_2^{(1)}|}{|\Phi|^{\frac{1}{2}} |\Phi^{(1)}|^{\frac{1}{2}}} \right\|_{H^s} \\
&\leq (2\beta_1)^{\frac{1}{2}} C \left\| \int_* \frac{|\xi_2^{(1)}|}{|\Phi^{(1)}|^{\frac{1}{2}}} \right\|_{H^s} \|v\|_{H^s}^2 \\
&\leq (2\beta_1)^{\frac{1}{2}} C^2 \|v\|_{H^s}^3 \|v\|_{H^s}^2
\end{aligned}$$

$$\Longrightarrow \quad \left\| \mathcal{T}_1^{(2)}(v) \right\|_{H^s} \leq 3(2\beta_1)^{\frac{1}{2}} C^2 \|v\|_{H^s}^5$$

Terms in the 3rd iteration:

$|\Phi| > N$, $|\Phi + \Phi^{(1)}| > \beta_1 |\Phi|$, and $|\Phi + \Phi^{(1)} + \Phi^{(2)}| \le \beta_2 |\Phi + \Phi^{(1)}|$

$$\implies \quad |\Phi^{(1)}| \sim |\Phi + \Phi^{(1)}| > N \text{ and } |\Phi^{(2)}| \le 2\beta_2 |\Phi + \Phi^{(1)}|.$$

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$$\implies \quad \left\| \mathcal{T}_1^{(3)}(v) \right\|_{H^s} \le (3 \cdot 5) \beta_2^{\frac{1}{2}} C^3 N^{-\frac{1}{2}} \|v\|_{H^s}^7$$

Terms in the Jth iteration:

We postuale that the *nearly resonant* contribution corresponds to:

$$|\Phi + \Phi^{(1)} + \ldots + \Phi^{(J)}| \le \beta_J |\Phi + \ldots + \Phi^{(J-1)}|$$

$$\implies \|\mathcal{T}_1^{(J+1)}(v)\|_{H_x^s(\mathbb{R})} \le \frac{((2J+1)!!) \beta_J^{\frac{1}{2}} C^{J+1}}{\beta_2^{\frac{1}{2}} \cdots \beta_{J-1}^{\frac{1}{2}}} N^{-\frac{1}{2}(J-1)} \|v\|_{H^s}^{2J+3}$$

Proposition

Let $s > \frac{1}{2}$ and $J \ge 2$. Then,

$$\|\mathcal{T}_1^{(J+1)}(v) - \mathcal{T}_1^{(J+1)}(w)\|_{H^s} \lesssim N^{-\frac{1}{2}(J-1)} \left(\|v\|_{H^s}^{2J+2} + \|w\|_{H^s}^{2J+2} \right) \|v - w\|_{H^s}.$$

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Terms in the Jth iteration:

We postuale that the *nearly resonant* contribution corresponds to:

$$|\Phi + \Phi^{(1)} + \ldots + \Phi^{(J)}| \leq \beta_J|\Phi + \ldots + \Phi^{(J-1)}|$$

$$\implies \|\mathcal{T}_1^{(J+1)}(v)\|_{H_x^s(\mathbb{R})} \leq \frac{((2J+1)!!)\beta_J^{\frac{1}{2}} C^{J+1}}{\beta_2^{\frac{1}{2}} \cdots \beta_{J-1}^{\frac{1}{2}}} N^{-\frac{1}{2}(J-1)} \|v\|_{H^s}^{2J+3}$$

Proposition

Let $s > \frac{1}{2}$ and $J \geq 2$. Then,

$$\|\mathcal{T}_1^{(J+1)}(v) - \mathcal{T}_1^{(J+1)}(w)\|_{H^s} \lesssim N^{-\frac{1}{2}(J-1)}\left(\|v\|_{H^s}^{2J+2} + \|w\|_{H^s}^{2J+2}\right)\|v - w\|_{H^s}.$$

Convergence to zero of the remainder term

The estimate $\|\partial_t v\|_{H^s_x} = \left\|v^2\partial_x\overline{v}\right\|_{H^s_x} \lesssim \|v\|^3_{H^s_x}$ fails. However, for $s > \frac{1}{2}$,

$$\left\|\partial_t v\right\|_{H^{s-1}_x} = \left\|v^2\partial_x\overline{v}\right\|_{H^{s-1}_x} \lesssim \|v^2\|_{H^s_x}\|\partial_x\overline{v}\|_{H^{s-1}} \lesssim \|v\|^3_{H^s_x}\,.$$

Consider the trilinear operator $\mathcal{T}_{|\Phi|>M}$ defined by

$$\mathcal{F}\Big[\mathcal{T}_{|\Phi|>M}(v_1,v_2,v_3)\Big](\xi) = \int_{\substack{\xi=\xi_1-\xi_2+\xi_3\\ |\Phi(\vec{\xi})|>M}} \frac{|\xi_2|}{|\Phi(\vec{\xi})|}\widehat{v_1}(\xi_1)\overline{\widehat{v_2}(\xi_2)}\widehat{v_3}(\xi_3)d\xi_1 d\xi_2\,.$$

Lemma 2. (Basic trilinear estimate in the H^{s-1}-norm)

Let $s > \frac{1}{2}$ and $M \geq 1$. Then, for $\theta = \theta(s) := \min\{2s-1, \frac{1}{2}\}$,

$$\left\|\mathcal{T}_{|\Phi|>M}(v_1,v_2,v_3)\right\|_{H^{s-1}} \leq CM^{-\theta}\|v_j\|_{H^{s-1}}\|v_k\|_{H^s}\|v_\ell\|_{H^s}\,,$$

for any j,k,ℓ such that $\{j,k,\ell\} = \{1,2,3\}$.

Also, from Lemma 1, we immediately get

$$\left\|\mathcal{T}_{|\Phi|>M}(v_1,v_2,v_3)\right\|_{H^s} \leq CM^{-\theta}\prod_{j=1}^{3}\|v_j\|_{H^s}\,.$$

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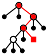
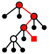
Proof of concept for the remainder term

Recall • $= v$, and now also ■ $= \partial_t v$:

$$\left\| \int_* \frac{|\xi_2| \cdot |\xi_2^{(1)}| \cdot |\xi_2^{(2)}| \cdot |\xi_2^{(3)}|}{|\Phi| \cdot |\Phi + \Phi^{(1)}| \cdot |\Phi + \Phi^{(1)} + \Phi^{(2)}| \cdot |\Phi + \ldots + \Phi^{(3)}|} \right\|_{H^{s-1}}$$



$$\leq 2^3 \left\| \int_* \frac{|\xi_2|}{|\Phi|} \cdot \frac{|\xi_2^{(1)}|}{|\Phi^{(1)}|} \cdot \frac{|\xi_2^{(2)}|}{|\Phi^{(2)}|} \cdot \frac{|\xi_2^{(3)}|}{|\Phi^{(3)}|} \right\|_{H^{s-1}}$$



$$\leq \frac{2^3 C}{\beta_1^\theta} N^{-\theta} \|v\|_{H^s}^2 \left\| \int_* \frac{|\xi_2^{(1)}|}{|\Phi^{(1)}|} \cdot \frac{|\xi_2^{(2)}|}{|\Phi^{(2)}|} \cdot \frac{|\xi_2^{(3)}|}{|\Phi^{(3)}|} \right\|_{H^{s-1}}$$

$$\leq \frac{2^3 C^2}{(\beta_1 \beta_2)^\theta} N^{-2\theta} \|v\|_{H^s}^2 \left\| \int_* \frac{|\xi_2^{(2)}|}{|\Phi^{(2)}|} \frac{|\xi_2^{(3)}|}{|\Phi^{(3)}|} \right\|_{H^{s-1}} \|v\|_{H^s}^2$$

$$\leq \frac{2^3 C^3}{(\beta_1 \beta_2 \beta_3)^\theta} N^{-3\theta} \|v\|_{H^s}^2 \left\| \int_* \frac{|\xi_2^{(3)}|}{|\Phi^{(3)}|} \right\|_{H^s} \|v\|_{H^s} \, \| ■ \|_{H^{s-1}} \|v\|_{H^s}^2$$

$$\leq \frac{2^3 C^4}{(\beta_1 \beta_2 \beta_3 \beta_4)^\theta} N^{-4\theta} \|v\|_{H^s}^{11}$$

Proposition

Let $s > \frac{1}{2}$ and $\theta = \min\{2s-1, \frac{1}{2}\}$. Then,

$$\left\|\mathcal{T}^{(J+1)}(v)\right\|_{H^{s-1}} \leq N^{-\theta J}\|v\|_{H^s}^{2J+3}.$$

The above estimates also allows us to *justify the other NF steps*, i.e.

- switching time derivatives and integrals in spatial frequencies, and
- the application of the product rule.

Consequently, if $v \in C_T H^s$ is a solution to $(\star)$ then

$$v(t) = v_0 + \sum_{j=2}^{\infty}\left[\mathcal{T}_0^{(j)}(v)\right]_{t'=0}^{t'=t} + \sum_{j=1}^{\infty}\int_0^t \mathcal{T}_1^{(j)}(v)dt'.$$

Choose $N = N(\|v_0\|_{H^s}) \gg 1$ and $T = T(N) \ll 1$.
$\implies$ the right-hand side is a contraction in $\left\{v \in C_T H^s : \|v\|_{C_T H^s} \leq 2\|v_0\|_{H^s}\right\}$.

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Remarks on the NF method with ∞ly-many reductions

- Usually, harmonic analytic methods consist of an *intricate functional setting* on a *simple Duhamel formula*.
- Here, we first derive an *intricate Duhamel formual* (NFeq) after which *the functional setting is simple*.
- The analytical part is easy: fixed point argument in $C_t H_x^s$ for the (Duhamel formulation of the) *normal form equation*.
- The relevant estimates can be established by recursively applying trilinear estimates (which in turn are proved simply with the Cauchy-Schwarz inequality).