Homogenization of convolution type operators in periodic and random media

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Outline

- Convolution type operators with periodic and random coefficients. Assumptions
- Homogenization and G-convergence. Main notions.
- Symmetric operators in periodic environment. Homogenization result
- Corrector and effective coefficients.
- Non-symmetric operator in periodic environment.
- Effective velocity and effective diffusion. Homogenization result.
- Symmetric operators in random media.

Convolution type operators

We consider a zero order convolution type non-local operator A in $L^2(\mathbb{R}^d)$, $d \geq 1$, with periodic or random statistically homogeneous coefficients. This operator is defined by

$$Au(x) = \int_{\mathbb{R}^d} \Lambda(x,y)a(x-y)\big(u(y)-u(x)\big)\,dy,$$

where the convolution kernel $a = a(z)$ is a (deterministic) non-negative integrable function, $a : \mathbb{R}^d \mapsto \mathbb{R}^+$, and $\Lambda(x,y)$ is a periodic function or a stationary random field that satisfies the uniform ellipticity conditions. This function Λ represents the local characteristics of the environment.

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Under the above assumptions A is a bounded linear operator in $L^2(\mathbb{R}^d)$. The corresponding evolution equation describes the dynamics of a continuous time jump Markov process in a periodic or random stationary medium.

Homogenization problem

When studying the large time behaviour of the said Markov evolution it is natural to make the diffusive scaling of spatial and temporal variables:

$$x \longrightarrow \varepsilon x, \qquad t \longrightarrow \varepsilon^2 t,$$

where ε is a small positive parameter. The generator of the scaled dynamics takes the form

$$(A^\varepsilon u)(x) \;=\; \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)\big(u(y)-u(x)\big)\,dy. \tag{1}$$

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The talk focuses on the homogenization problem for this family of operators, as $\varepsilon \to 0$.

Our goal is to obtain the homogenization results and to study the properties of the limit problem.

Homogenization

Given a function $f \in L^2(\mathbb{R}^d)$ and $\lambda > 0$, consider a problem

$$-A^\varepsilon u(x) + \lambda u(x) \; = \; f(x) \qquad \text{in } \mathbb{R}^d. \tag{2}$$

We will show that for each $\varepsilon > 0$ this problem has a unique solution $u^\varepsilon \in L^2(\mathbb{R}^d)$.

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Definition

We say that the family A^ε admits homogenization, as $\varepsilon \to 0$, if there exists an operator A^0 in $L^2(\mathbb{R}^d)$ such that the problem

$$-A^0 u(x) + \lambda u(x) = f(x) \qquad \textit{in } \mathbb{R}^d.$$

has a unique solution u^0 for any $f \in L^2(\mathbb{R}^d)$, and

$$u^\varepsilon \longrightarrow u^0 \qquad \textit{in } L^2(\mathbb{R}^d), \quad \textit{as } \varepsilon \to 0.$$

A^0 is called the effective operator.

Motivations and possible applications.

Zero order convolution type operators appear in many applications, such as models of population dynamics and the continuous contact model, where they describe the evolution of the density of a population. Several models of porous media are also described in terms of convolution type operators with integrable kernels.

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Studying the large time behaviour of such systems in periodic and stationary random media naturally leads to homogenization problems for the corresponding non-local operators.

Existing results for differential and difference operators

Periodic coefficients • For periodic operators first homogenization results were obtained by E. De Giorgi ('67) and S.Spagnolo ('70). It was shown that for divergence form second order elliptic operators of the form

$$L^{\varepsilon}u(x) = \operatorname{div}\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla u(x)\right)$$

with periodic coefficients and the corresponding parabolic operators the homogenization result holds, and the limit elliptic (parabolic) operator has constant coefficients.

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- Div-curl method - F.Murat, L.Tartar ('78–'79),

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- Estimates for the rate of convergence - V. Yurinskii ('81) . Sharp estimates - A. Gloria, F. Otto ('11 – '17) .

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- Periodic homogenization for nonlinear integro-differential equations. – R. Schwab ('10).

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- Normalization and second moments

C3

$$\int_{\mathbb{R}^d} a(x)dx = 1, \quad \int_{\mathbb{R}^d} |x|^2 a(x)dx < \infty.$$

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- Uniform ellipticity.

P2

$$0 < \Lambda^- \leq \mu(y) \leq \Lambda^+$$

ε-problem

Let m be a positive number. We consider a family of problems

$$-A^{\varepsilon}u + mu = f, \quad f \in L^2(\mathbb{R}^d) \tag{3}$$

with

$$(A^{\varepsilon}u)(x) \;=\; \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon}\Big)\mu\Big(\frac{y}{\varepsilon}\Big)\big(u(y)-u(x)\big)\,dy.$$

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Lemma

For any $m > 0$ and any $f \in L^2(\mathbb{R}^d)$ equation (3) has a unique solution $u^{\varepsilon} \in L^2(\mathbb{R}^d)$. Moreover,

$$\|u^{\varepsilon}\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{m}\|f\|_{L^2(\mathbb{R}^d)}.$$

Periodic symmetric case. Results

Theorem

There exists a positive definite symmetric $d \times d$ matrix a^{hom} such that for any $f \in L^2(\mathbb{R}^d)$ the solution u^ε converges in $L^2(\mathbb{R}^d)$, as $\varepsilon \to 0$, to a solution u^0 of the following homogenized problem

$$-\mathrm{div}\big(a^{\mathrm{hom}}\nabla u^0\big) + mu^0 = f \qquad \text{in } \mathbb{R}^d.$$

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By the Hille-Yosida theorem for any $\varepsilon > 0$ the operator A^ε is the generator of a contraction semigroup $\mathcal{S}_t^\varepsilon = e^{A^\varepsilon t}$ in $L^2(\mathbb{R}^d)$. The semigroup with the generator $\mathrm{div}\big(a^{\mathrm{hom}} \nabla \cdot\,\big)$ is denoted by $\mathcal{S}_t^0$.

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Theorem

The semigroup $\mathcal{S}_t^\varepsilon$ strongly converges to $\mathcal{S}_t^0$. The convergence is uniform on any finite time interval.

Framework of continuous functions

Here we suppose that two additional conditions are fulfilled:

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- Continuity.

C4

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- Uniform decay.

C5

$$a(z) \le \frac{C}{(1+|z|)^{d+2+\varkappa}}$$

for some $\varkappa > 0$.

Invariance principle

Under the above conditions there exists a jump Markov process ξ_t^ε with the generator A^ε. Its trajectories belong to the Skorokhod space $D([0,\infty),\mathbb{R}^d$.

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The space of continuous functions in $\mathbb{R}^d$ that vanish at infinity is denoted by $C_0(\mathbb{R}^d)$. One can check that for $f \in C_0(\mathbb{R}^d)$ the solution u^ε is an element of $C_0(\mathbb{R}^d)$.

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Theorem

Let $f \in C_0(\mathbb{R}^d)$. Then u^ε converges to u^0 in the uniform convergence norm. Furthermore, the process $\xi_\cdot^\varepsilon$ converges in law, as $\varepsilon \to 0$, in the Skorokhod topology to a Brownian motion with the covariance matrix $2a^{\mathrm{hom}}$.

Auxiliary cell problem. Corrector.

Cell periodic problem

$$\int_{\mathbb{R}^d} a(\xi-\eta)\mu(\xi)\mu(\eta)\,\big(\chi(\eta)-\chi(\xi)\big)\,d\eta =$$

$$= -\int_{\mathbb{R}^d} a(\xi-\eta)\mu(\xi)\mu(\eta)\big(\eta-\xi\big)\,d\eta.$$

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Lemma

There exists a unique up to an additive constant periodic solution $\chi \in L^2([0,1]^d, \mathbb{R}^d)$ *of the above cell problem.*

Observe that χ is a vector-function.

Homogenized operator

The formula for the homogenized matrix $a^{\rm hom}$ reads

Effective matrix

$$a^{\rm hom} = \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{R}^d} a(\xi-\eta)\mu(\xi)\mu(\eta)\Big(\frac{1}{2}(\xi-\eta)\otimes(\xi-\eta)-(\xi-\eta)\otimes\chi(\eta)\Big)\, d\eta d\xi.$$

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In order to make our technique work we should show that the effective matrix is positive definite.

Lemma

The matrix $a^{\rm hom}$ is positive definite.

Some ideas of the proof

We use the asymptotic expansion technique. Assuming that the limit function u^0 is smooth enough we write down the asymptotic expansion for u^ε of the form

Asymptotic expansion

$$u^\varepsilon = u^0(x) + \varepsilon\chi\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla u^0(x) + \varepsilon^2\kappa\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla\nabla u^0(x) + r^\varepsilon,$$

the functions $\chi(\xi)$ and $\kappa(\xi)$ are periodic.

We also introduce a new variable $z = \frac{x-y}{\varepsilon}$ and expand $u^0(y) = u^0(x - \varepsilon z)$ in Taylor series about x:

$$u^0(y) = u^0(x) - \varepsilon z\cdot\nabla u^0(x) + \frac{1}{2}z\otimes z\cdot\nabla\nabla u^0(x) + \dots$$

Some ideas of the proof

Then we substitute the above expansion for u^ε in the equation and collect power-like terms in the resulting relation.

Collecting the terms of order ε^{-1} results in the equation for the corrector χ.

The terms of order ε^0 give the equation for κ. The compatibility condition for this equation allows us to determine the effective coefficients.

Non-symmetric operators

We turn to the case of non-symmetric operators. We still assume that conditions C1, C3 and P1, P2 are fulfilled.
Here we consider the parabolic Cauchy problem

$$\partial_t u = A^\varepsilon u, \qquad u(x,0) = u_0(x).$$

For each $u_0 \in L^2(\mathbb{R}^d)$ this problem has a unique solution $u^\varepsilon \in L^\infty(0, T, L^2(\mathbb{R}^d))$.

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For each $u_0 \in L^2(\mathbb{R}^d)$ this problem has a unique solution $u^\varepsilon \in L^\infty(0, T, L^2(\mathbb{R}^d))$.

Theorem

There exist a constant vector $b \in \mathbb{R}^d$ and a positive definite $d \times d$ constant matrix a^{hom} such that

$$\int_0^T \int_{\mathbb{R}^d} \left(u^\varepsilon(x,t) - u^0\left(x - \frac{b}{\varepsilon}t, t\right)\right)^2 dxdt \longrightarrow 0,$$

as $\varepsilon \to 0$, where u^0 is a solution of the following Cauchy problem

$$\partial_t u^0 = \mathrm{div}\left(a^{\mathrm{hom}} \nabla u^0(\cdot, t)\right), \qquad u^0(x,0) = u_0(x).$$

Non-symmetric operators

Observe that in the non-symmetric case the homogenization result holds in moving coordinates (moving frame). This reflects the law of large numbers for the corresponding process, b being the effective velocity.

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The above theorem was formulated in a vague form. We did not specify so far the construction of b and a^{hom}.

Kernel of the adjoint periodic operator

We consider in $L^2(\mathbb{T}^d)$ the bounded operator

$$A_{\#} v(\xi) = \int_{\mathbb{R}^d} a(\xi - \eta)\Lambda(\xi, \eta)\big(v(\eta) - v(\xi)\big)\, d\eta$$

and its adjoint

$$A_{\#}^{\star} p(\xi) = \int_{\mathbb{R}^d} \big[a(\eta - \xi)\Lambda(\eta, \xi)p(\eta) - a(\xi - \eta)\Lambda(\xi, \eta)p(\xi)\big]\, d\eta.$$

Kernel of the adjoint periodic operator

Lemma

The kernel of the operator $A^\star_\#$ has dimension one. There exist $\mathbf{c}_- > 0$ and $\mathbf{c}_+ > 0$ such that under proper normalization the function p_0 satisfies the relation

$$0 < \mathbf{c}_- \le p_0(\xi) \le \mathbf{c}_+.$$

The equation $A_\# v(\xi) = g(\xi)$ is solvable in the space of periodic functions if and only if g is orthogonal to p_0 in $L^2(\mathbb{T}^d)$.

Lemma

The kernel of the operator $A_{\#}^{\star}$ has dimension one. There exist $\mathbf{c}_{-} > 0$ and $\mathbf{c}_{+} > 0$ such that under proper normalization the function p_0 satisfies the relation

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The equation $A_{\#} v(\xi) = g(\xi)$ is solvable in the space of periodic functions if and only if g is orthogonal to p_0 in $L^2(\mathbb{T}^d)$.

The proof relies on the Fredholm and Krein-Rutman theorems.

In what follow $\int_{\mathbb{R}^d} p_0(\xi)\, d\xi = 1$.

Cell problem in the non-symmetric case

Asymptotic expansion

$$u^\varepsilon = u^0(x - \frac{b}{\varepsilon}t, t) + \varepsilon\chi\Big(\frac{x}{\varepsilon}\Big) \cdot \nabla u^0(x - \frac{b}{\varepsilon}t, t) + \dots$$

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Playing the same game as in the symmetric case and collecting power-like terms we arrive at the following equation for the corrector χ:

Cell problem

$$\int\limits_{\mathbb{R}^d} a(\xi-\eta)\Lambda(\xi,\eta)\big(\chi(\eta)-\chi(\xi)\big)\,d\eta = -\int\limits_{\mathbb{R}^d} a(\xi-\eta)\Lambda(\xi,\eta)\big(\eta-\xi-b\big)\,d\eta.$$

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Clearly, there is a unique b such that the compatibility condition holds and the equation has a periodic solution.

Effective velocity and diffusion

For b we obtain the following expression:

Effective drift

$$b = \int\limits_{\mathbb{R}^d} a(\xi-\eta)\Lambda(\xi,\eta)\big(\eta-\xi\big)\,d\eta d\xi.$$

Collection the terms of order ε^0 we obtain an equation for the second corrector $\kappa(\cdot)$. The compatibility condition for this equation allows us to determine the effective matrix a^{hom}.

One can show that a^{hom} is positive definite.

Random coefficients

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space, and assume that T_x is an ergodic dynamical system on this probability space that is an ergodic group of measurable transformations of Ω such that

- $T_{x+y} = T_x \circ T_y \quad \text{for all } x,\, y \in \mathbb{R}^d, \quad T_0 = \mathrm{Id},$

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Ergodicity of $T.$ means that for any set $S \in \mathcal{F}$ such that $T_x S = S$ for all $x \in \mathbb{R}^d$, we have either $\mathbf{P}(S) = 0$ or $\mathbf{P}(S) = 1$.

Random coefficients

In the random case our condition on μ reads

- Stationarity and ellipticity.

R1

$$\mu(x,\omega) = \boldsymbol{\mu}(T_x\omega),$$

where a random variable $\boldsymbol{\mu}$ satisfies the estimate

$$0 < \alpha_1 \le \boldsymbol{\mu}(\omega) \le \alpha_2 < \infty.$$

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Then

$$\Lambda(x,y) = \mu(x,\omega)\mu(y,\omega) = \boldsymbol{\mu}(T_x\omega)\boldsymbol{\mu}(T_y\omega).$$

and

$$(A^\varepsilon_\omega u)(x) \; = \; \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon},\omega\Big)\mu\Big(\frac{y}{\varepsilon},\omega\Big)\Big(u(y)-u(x)\Big)dy.$$

Main result

For a constant $d \times d$ matrix Θ denote

$$A^{0,\Theta} u(x) = \Theta_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u(x).$$

Theorem (Homogenization theorem)

Let conditions C1–C3 and R1 be fulfilled. Then there exists a constant deterministic symmetric positive definite matrix Θ such that almost surely for any $f \in L^2(\mathbb{R}^d)$ and any $m > 0$ the solution u^ε of the problem $-A^\varepsilon_\omega u + mu = f$ converges in $L^2(\mathbb{R}^d)$, as $\varepsilon \to 0$, to the solution of the effective problem

$$-A^{0,\Theta} u(x) + mu(x) = f(x).$$

that is

$$\|(A^\varepsilon_\omega - m)^{-1} f - (A^{0,\Theta} - m)^{-1} f\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Asymptotic expansion

We first assume that $f \in C_0^\infty(\mathbb{R}^d)$ and that

$$u^0 \;=\; (A^{0,\Theta} - \lambda)^{-1} f \in C_0^\infty(\mathbb{R}^d).$$

We denote this set by $\mathcal{S}_0(\mathbb{R}^d)$. Observe that this set is dense in $L^2(\mathbb{R}^d)$.

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Given $u^0 \in C_0^\infty(\mathbb{R}^d)$, we write down the ansatz

$$u^\varepsilon(x) = u^0(x) + \varepsilon\theta\big(\frac{x}{\varepsilon},\omega\big)\nabla u^0(x) + u_{\varepsilon,R}(x,\omega),$$

here u^0 is the leading term of the expansion and $\theta(z,\omega)$ is the so-called corrector. Denote

$$v^\varepsilon(x) = u^0(x) + \varepsilon\theta\big(\frac{x}{\varepsilon},\omega\big)\nabla u^0(x).$$

Asymptotic analysis

We introduce a new variable $z = \frac{x-y}{\varepsilon}$ and substitute for u^ε the two leading terms of the asymptotic expansion. This yields

$$(A^\varepsilon_\omega v^\varepsilon)(x) =$$

$$\frac{1}{\varepsilon^2}\int\limits_{\mathbb{R}^d} a(z)\mu\big(\frac{x}{\varepsilon}\big)\mu\big(\frac{x}{\varepsilon}-z\big)\big(u^0(x-\varepsilon z)-u^0(x)\big)\,dz+$$

$$\frac{1}{\varepsilon^2}\int\limits_{\mathbb{R}^d} a(z)\mu\big(\frac{x}{\varepsilon}\big)\mu\big(\frac{x}{\varepsilon}-z\big)\Big(\varepsilon\theta\big(\frac{x}{\varepsilon}-z\big)\nabla u^0(x-\varepsilon z)-\varepsilon\theta\big(\frac{x}{\varepsilon}\big)\nabla u^0(x)\Big)\,dz.$$

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The Taylor expansion of a function $u^0(x-\varepsilon z)$ reads

$$u^0(x-\varepsilon z)$$
$$= \ u^0(x) - \nabla u^0(x)\cdot z + \int_0^1 \nabla\nabla u^0(x-zt)z\cdot z(1-t)\ dt$$

and is valid for any $x, z \in \mathbb{R}^d$.

Collecting power-like terms

Writing down a similar expansion for $\nabla u^0(x-\varepsilon z)$ we obtain

$$(A^\varepsilon_\omega v^\varepsilon)(x)$$

$$= \frac{1}{\varepsilon}\mu\Big(\frac{x}{\varepsilon}\Big)\nabla u^0(x)\cdot\int\limits_{\mathbb{R}^d}\Big[\theta\Big(\frac{x}{\varepsilon}-z\Big)-\theta\Big(\frac{x}{\varepsilon}\Big)-z\Big]a(z)\mu\Big(\frac{x}{\varepsilon}-z\Big)\,dz$$

$$+\mu\Big(\frac{x}{\varepsilon}\Big)\nabla\nabla u^0(x)\cdot\int\limits_{\mathbb{R}^d}\Big[\frac{1}{2}z\otimes z-z\otimes\theta\Big(\frac{x}{\varepsilon}-z\Big)\Big]a(z)\mu\Big(\frac{x}{\varepsilon}-z\Big)dz$$

$$+\phi_\varepsilon(x)$$

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\end{aligned}
$$

Our first goal is to choose $\theta(\zeta,\omega)$ in such a way that the sum of the terms of order ε^{-1} vanishes. This leads to the following equation for θ:

The corrector

$$\int\limits_{\mathbb{R}^d} \Big(-z + \theta(\zeta - z, \omega) - \theta(\zeta, \omega) \Big) a(z) \mu(\zeta - z, \omega) \ dz \ = \ 0.$$

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$$\int_{\mathbb{R}^d} \Big(-z + \theta(\zeta - z, \omega) - \theta(\zeta, \omega) \Big) a(z) \mu(\zeta - z, \omega) \; dz \; = \; 0.$$

Also, we want the term $\varepsilon\theta\left(\frac{x}{\varepsilon}\right)\nabla u^0$ to be asymptotically small as $\varepsilon \to 0$. Thus, $\theta(\zeta, \omega)$ should be a.s. of sublinear growth.

$$\int_{\mathbb{R}^d} \Big(-z + \theta(\zeta - z, \omega) - \theta(\zeta, \omega) \Big)\, a(z)\mu(\zeta - z, \omega)\ dz \ = \ 0.$$

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Theorem

There exists a unique (up to an additive constant vector) solution $\theta(\cdot, \omega) \in L^2_{\rm loc}(\mathbb{R}^d)$ *such that*

- *the increments* $\zeta_z(\xi, \omega) = \theta(z + \xi, \omega) - \theta(\xi, \omega)$ *are stationary for any given* z*, i.e.* $\zeta_z(\xi, \omega) = \zeta_z(0, T_\xi\omega)$;
- $\varepsilon\theta\big(\frac{x}{\varepsilon}, \omega\big)$ *is a function of sub-linear growth in* $L^2_{\rm loc}(\mathbb{R}^d)$*: for any bounded Lipschitz domain* $Q \subset \mathbb{R}^d$

$$\Big\| \varepsilon\, \theta\Big(\frac{x}{\varepsilon}, \omega\Big) \Big\|_{L^2(Q)} \to 0 \quad a.s.$$

Zero order terms

We turn to the terms of order zero:

$$I^\varepsilon(x) = \mu\Big(\frac{x}{\varepsilon}\Big)\nabla\nabla u^0(x)\cdot\int\limits_{\mathbb{R}^d}\Big[\frac{1}{2}z\otimes z - z\otimes\theta\Big(\frac{x}{\varepsilon}-z\Big)\Big]a(z)\mu\Big(\frac{x}{\varepsilon}-z\Big)dz$$

Proposition

For any $\varphi\in C_0^\infty(\mathbb{R}^d)$ we have a.s.

$$(I^\varepsilon,\varphi)_{L^2(\mathbb{R}^d)} \longrightarrow \int\limits_{\mathbb{R}^d}(\Theta_1+\Theta_2)_{ij}\frac{\partial u^0(x)}{\partial x_i\partial x_j}\varphi(x)dx$$

where

$$\Theta_1 = \int\limits_{\mathbb{R}^d}\frac{1}{2}\,z\otimes z\,a(z)\,\mathbf{E}\{\mu(0,\omega)\mu(-z,\omega)\}dz$$

and

$$\Theta_2 = \frac{1}{2}\int\limits_{\mathbb{R}^d}a(z)z\otimes\mathbf{E}\{\zeta_{-z}(0,\omega)\mu(0,\omega)\mu(-z,\omega)\}dz.$$

Proposition

The matrix $\Theta = \frac{1}{2}[(\Theta_1 + \Theta_2) + (\Theta_1 + \Theta_2)^t]$ *is positive definite.*

Works in progress

- Non-local operators in high contrast media.

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- Homogenization of convolution type operators in perforated domains.
- Non-symmetric operators with random coefficients (in the case of finite range of dependence).