Mean field games: at the crossroad between optimal control and optimal transport

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Mean field game theory: what is about ?

MFG theory was introduced since 2006 by J-M Lasry and P-L Lions.
A similar model developed independently by [Huang-Caines-Malhamé].

Goal: study Nash equilibria in large populations of rational agents with weak interaction

large population $\rightsquigarrow$ infinite number (a continuum) of similar players

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But everyone takes into account the collective behavior through the distribution law (empirical density) of the states.

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Basic idea: export the principle of statistical mechanics to (non cooperative) strategic interactions within rational particles

$\rightarrow$ Limit of Nash equilibria of symmetric N-players games will satisfy, as $N \rightarrow \infty$, a system of PDEs coupling the equation for the individual strategies with the equation for the distribution law

Macroscopic (mean-field) description

$$\begin{cases} dX_s = \alpha(X_s)ds + \sqrt{2}\, dB_s\,, \\ X_t = x \end{cases}$$

dynamics of each agent

$$u(t,x) = \inf_{\{\alpha(\cdot)\}} \mathbb{E}\int_t^T L(X_s, \alpha(X_s), \mu_s) + G(X_T, \mu_T)$$

where $\{\mu_t\}$ is an exogenous family of measures

$$\Big\downarrow \text{HJB}$$

$$-\partial_t u - \Delta u + H(x, \mu, Du) = 0$$

$$\underbrace{\alpha^*(\cdot) = -H_p(\cdot, \mu_t, Du(t,\cdot))}_{\text{optimal policy}},$$

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$$m(t) = \mathcal{L}(X_t)$$

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Nash equilibrium: $\mathcal{L}(X_t^*) = \mu_t$

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Nash equilibrium: $\mathcal{L}(X_t^*) = \mu_t$

$$\partial_t m - \Delta m - \operatorname{div}(m\, H_p(x, m, Du)) = 0 \qquad -\partial_t u - \Delta u + H(x, m, Du) = 0$$

The MFG system of PDEs

Model case (here H_p stands for $\frac{\partial H(x,p)}{\partial p}$)

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = F(x, m) & \text{in } (0, T) \times \Omega \\ \partial_t m - \Delta m - \mathrm{div}(m\, H_p(x, Du)) = 0 & \text{in } (0, T) \times \Omega\,, \end{cases}$$

usually complemented with initial-terminal conditions:

-$m(0) = m_0$ (initial distribution of the agents)

-$u(T) = G(x, m(T))$ (final pay-off)

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Rmk 1: This is not the most general structure.
Cost criterion $L(X_t, \alpha_t, m(t)) \to H(x, m, Du)$.

Rmk 2: The special structure $H = H(x, Du) - F(x, m)$ gives to the system a variational structure $\to$ optimality system

Link with optimal control systems

MFG as optimality system (optimal control with Fokker-Planck state eq.).

Ex: Optimize in terms of the field α

$$\partial_t m = \Delta m + \operatorname{div}(\alpha\, m), \qquad m(0) = m_0$$

$$\longrightarrow \quad \inf_\alpha \int_0^T \int_\Omega \{L(x,\alpha)m + \Phi(m(s))\}\, dt + \mathcal{G}(m(T))$$

where $\Phi'(m) = F(m)$ and $\mathcal{G}'(m) = G(m)$.

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First order optimality conditions give the adjoint state u:

$$\begin{cases} Du + L_\alpha(x,\alpha) = 0 \quad (m - q.o.) \\ -\partial_t u - \Delta u - \alpha \cdot Du - L(x,\alpha) = F(m) \end{cases} \quad \Leftrightarrow \quad \begin{aligned} &\alpha_{opt} = -H_p(x, Du(t,x)) \\ &-\partial_t u - \Delta u + H(x, Du) = F(m) \end{aligned}$$

Rmk: $\underline{F(m), G(m) \text{ nondecreasing} \Rightarrow \text{convexity of the functional}}$

$$
\begin{cases}
-\partial_t u - \Delta u + H(x, Du) = F(x, m) & \text{in } (0, T) \times \Omega \\
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Key-assumption: F, G nondecreasing $\to$ uniqueness, stability...

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Use the adjoint structure of the system: (u_1, m_1), (u_2, m_2) solutions,

$$-\frac{d}{dt}\left[\int_\Omega (u_1 - u_2)(m_1 - m_2)\right] = \int_\Omega [F(m_1) - F(m_2)]\,(m_1 - m_2)$$

$$\underbrace{+\int_\Omega [H(Du_1) - H(Du_2)](m_1 - m_2) - [m_1 H_p(Du_1) - m_2 H_p(Du_2)]D(u_1 - u_2)}_{\int_\Omega m_1\,[H(Du_2) - H(Du_1) - H_p(Du_1)\,D(u_2 - u_1)] \,+\, \int_\Omega m_2\,[H(Du_1) - H(Du_2) - H_p(Du_2)\,D(u_1 - u_2)]}$$

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$\rightsquigarrow$ H convex $+$ F nondecreasing $\Rightarrow$ all terms are ≥ 0 !!

$$\frac{d}{dt}\left[\int_\Omega (u_1 - u_2)(m_1 - m_2)\right] \leq 0$$

G nondecreasing $\Rightarrow$ $\int_\Omega (u_1 - u_2)(m_1 - m_2) \geq 0$ at time T.
But $m_1(0) = m_2(0)$.... $\to$ uniqueness.

Sample result on the MFG system: (smooth solutions, smoothing monotone couplings)

Assume H is smooth and satisfies

$$c_0\, I \leq H_{pp}(x,p) \leq C_0\, I$$

and the coupling F, G are smoothing and monotone operators:

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(i) [Lasry-Lions '06] there exists a unique classical solution (u,m)

(ii) [Cardaliaguet-Lasry-Lions-P. '12], [Cardaliaguet-P. '19]
In long horizon $T >> 1$, the solution (u^T, m^T) of the MFG system is nearly stationary for most of the time:

$\exists$ a (unique) stationary solution $(\bar u, \bar m)$ such that

$$\|Du^T(t) - D\bar u\|_{C^{0,\alpha}} + \|m^T(t) - \bar m\|_{C^{0,\alpha}} \leq C\left(e^{-\omega(T-t)} + e^{-\omega t}\right),$$

Rmk: The *long time behavior is formulated* as the *turnpike property* of optimality systems: boundary layers appear at initial and final time, yet for most of the time the strategies are almost stationary

From smooth to weak solutions

Even if the theory is understood for smooth couplings, new PDE questions arise for *local couplings*:

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- the existence of smooth solutions is known only in few cases:

  (i) if $m \mapsto F(x,m)$ or $p \mapsto H(x,p)$ have a mild growth ([Lasry-Lions], [Gomes-Pimentel-Sanchez Morgado])

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  (ii) in the homogeneous quadratic case $H(x,p) = |p|^2$, solutions are smooth *for every* $F(x,m) \geq 0$ ([Cardaliaguet-Lasry-Lions-P.])

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  Otherwise, regularity of solutions is not known.

- But it is not difficult to construct weak (distributional) solutions as soon as $F(x,m)$ is bounded below.

However: Weak solutions are in general unbounded !

A priori estimates of the system (valid with & without diffusion !):

$$
\left.\begin{array}{ll}
(1) & \int_0^T\int_\Omega F(x,m)m \\
(2) & \int_0^T\int_\Omega H(x,Du) \\
(3) & \int_0^T\int_\Omega m\, L(x,H_p(x,Du))
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Typical growth conditions

- $F(m) \simeq m^{p-1}$, $p > 1$:

$$(1) \Rightarrow \; m \in L^p \quad \Rightarrow \quad F(m) \in L^{p/p-1}$$

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- $L(x,\alpha)$, $H(x,p)$ with coercive quadratic growths:

$$(2)-(3) \Rightarrow \;\; Du \in L^2\,, \quad m\,|Du|^2 \in L^1$$

$\rightsquigarrow$ Fokker-Planck with L^2- drift

Main difficulties of a weak theory:

(i) Uniqueness may fail for unbounded solutions of HJB:

$$\exists \;\; u \in L^2(0,T;H^1_0)\,, \;\; u \neq 0 \text{ sol. of } \qquad \begin{cases} u_t - \Delta u + |Du|^2 = 0 \\ u(0) = 0 \end{cases}$$

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(ii) The typical setting of well-posedness of the Fokker-Planck

$$(FP) \qquad\qquad m_t - \Delta m + \operatorname{div}(m\, b) = 0$$

requires much more than L^2 drifts, usual theory needs $b \in L^\infty(0,T;L^d(\Omega))$, or $b \in L^{d+2}((0,T)\times\Omega)$ ([Aronson-Serrin], [Ladysenskaya-Solonnikov-Uraltseva])

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But......a full theory is still possible entirely relying on the estimate

$$m|b|^2 \in L^1$$

In mean field games this is indeed the estimate $m|Du|^2 \in L^1$ which comes from optimization !!

The typical statement is the following (adapted to Dirichlet, Neumann, or to entire space $\mathbb{R}^N$ under suitable modifications)

Theorem (P. '15)

Let $b \in L^2(Q_T)^N$ and $m_0 \in L^1$. Then the problem

$$\begin{cases} m_t - \Delta m - \mathrm{div}(m\, b) = 0\,, & \text{in } (0,T)\times\Omega, \\ m(0) = m_0 & \text{in } \Omega. \\ +\, BC & \end{cases} \tag{1}$$

admits at most one weak sol. $m \in L^1(Q_T)_+$: $m|b|^2 \in L^1(Q_T)$.

Moreover, in this case any weak solution is a renormalized solution, belongs to $C^0([0,T];L^1)$ and satisfies (for a suitable truncation $T_k(\cdot)$):

$$(T_k(m))_t - \Delta T_k(m) - \mathrm{div}(T_k'(m) m\, b) = \omega_k\,, \qquad \text{in } Q_T \tag{2}$$

where $\omega_k \in L^1(Q_T)$, and $\omega_k \overset{k\to\infty}{\to} 0$ in $L^1(Q_T)$.

Main idea: *a nonlinear look at a linear equation*

- for general convection-diffusion problems (possibly nonlinear)

$$\begin{cases} m^\varepsilon_t + A m^\varepsilon = \operatorname{div}\left(\phi(t,x,m^\varepsilon)\right) & \text{in } Q_T \\ m^\varepsilon(0) = m^\varepsilon_0\,, \ +\text{BC} \end{cases}$$

we have that if

$$|\phi(t,x,m)| \leq c(1+\sqrt{m})\,k(t,x)\,, \qquad k \in L^2(Q_T) \tag{3}$$

then

$$m^\varepsilon \to m \quad \text{in } C^0([0,T];L^1)$$

and m is renormalized solution relative to m_0.

- One can apply this idea even in the Di Perna-Lions approach, regularizing m by convolution:

$$m_t - \Delta m - \mathrm{div}(m\, b) = 0 \qquad \star\, \rho_\varepsilon$$

$$\Rightarrow m^\varepsilon := m \star \rho_\varepsilon \quad \text{solves}$$

$$m_t^\varepsilon - \Delta m_\varepsilon - \mathrm{div}((mb) \star \rho_\varepsilon) = 0$$

where Schwartz's inequality $+$ $m \geq 0$ imply

$$|(mb) \star \rho_\varepsilon| \leq \underbrace{(m \star \rho_\varepsilon)^{\frac{1}{2}}}_{\sqrt{m^\varepsilon}} \underbrace{((m|b|^2) \star \rho_\varepsilon)^{\frac{1}{2}}}_{B^\varepsilon}$$

with B^ε converging in $L^2(Q_T)$.

$\rightarrow$ for purely second order operators, no need of commutators

$$\begin{cases} -u_t - \Delta u + H(x, Du) = F(x, m)\,, \\ m_t - \Delta m - \text{ div } (m\, H_p(x, Du)) = 0\,, \\ u(T) = G(x, m(T))\,,\ m(0) = m_0 \end{cases}$$

- $F, G \in C^0(\overline{\Omega} \times \mathbb{R})$
- $p \mapsto H(x, p)$ is convex and satisfies structure conditions

Ex: $H \simeq \gamma(t, x)|\nabla u|^2$.

Def. of weak solutions:

- $u, m \in C^0([0, T]; L^1)$, $m\,|Du|^2 \in L^1$

-$G(x, m(T)) \in L^1$, $H(x, Du) \in L^1$, $F(x, m) \in L^1$,

- the equations hold in the sense of distributions.

Theorem (P. '15)

Assume that $m \mapsto G(x,m)$ is nondecreasing, and let $m_0 \in L^\infty_+$.

(i) If F, G are bounded below, then there exists a weak solution.

(ii) If in addition $m \mapsto F(x,m)$ is nondecreasing, $p \mapsto H(x,p)$ is strictly convex (at infinity), then there is at most one weak solution (u,m) such that $m > 0$.

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Rmk: The coupling functions F, G have no growth restriction from above

- The case $F = F(x)$ is included !! $\rightsquigarrow$ new results for HJ equations with L^1-data

$$\begin{cases} u_t - \Delta u + H(x, Du) = F(x) \\ u_{\partial\Omega} = 0\,, \;\; u(0) = u_0 \end{cases}$$

Uniqueness $\iff$ $m_t - \Delta m -$ div $(H_p(x, Du)\, m) = 0$ admits a sol. m with $H_p(x, Du) \in L^2(m)$.

$\rightsquigarrow$ *uniqueness holds if the adjoint of the linearized admits nice solutions ....a Fredholm-type result !*

- Numerical schemes converge towards weak solutions [Achdou-P. '16]

We use finite differences approximations of the mean field games system as in [Achdou-Capuzzo Dolcetta]:

$$\begin{cases} \frac{u_{i,j}^{k+1}-u_{i,j}^{k}}{\Delta t} - (\Delta_h u^k)_{i,j} + g(x_{i,j}, [\nabla_h u^k]_{i,j}) = F(m_{i,j}^{k+1}), \\ \frac{m_{i,j}^{k+1}-m_{i,j}^{k}}{\Delta t} - (\Delta_h m^{k+1})_{i,j} + \mathcal{T}_{i,j}(u^k, m^{k+1}) = 0, \end{cases}$$

where g is a monotone approximation of the Hamiltonian H as in upwind schemes:
Ex (1-d): $g = g\left(\frac{u_{i+1}-u_i}{h}, \frac{u_i-u_{i-1}}{h}\right)$ with $g(p_1, p_2)$ increasing in p_2 and decreasing in p_1, $g(q, q) = H(q)$.

while $\mathcal{T}$ is the discrete adjoint of the associated linearized transport:

$$\mathcal{T}(v, m) \cdot w = m\, g_p([\nabla_h v]) \cdot [\nabla_h w]$$

Similar structure allows to have discrete estimates and compactness as in the continuous model.

- vanishing viscosity limit of weak solutions
  [Cardaliaguet-Graber-P.-Tonon '15]

$$\begin{cases} -\partial_t u - \varepsilon\Delta u + H(x, Du) = F(x, m) & \text{in } (0, T) \times \Omega \\ \partial_t m - \varepsilon\Delta m - \mathrm{div}(m\, H_p(x, Du)) = 0 & \text{in } (0, T) \times \Omega\,, \\ u(T) = G(x, m(T))\,,\ m(0) = m_0 \end{cases}$$

  Assume some coercivity on the coupling terms:

  - $F, G \simeq m^{p-1}$, with $p > 1$.

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Assume some coercivity on the coupling terms:

- $F, G \simeq m^{p-1}$, with $p > 1$.

$\Rightarrow$ as $\varepsilon \to 0$, weak solutions converge towards a relaxed formulation of the first order system:

(i) u is a distributional subsolution: $-u_t + H(x, \nabla u) \leq F(x, m)$

(ii) m is a distributional solution: $m_t - \ \mathrm{div}\ (m\, H_p(x, \nabla u)) = 0$

(iii) the energy equality holds

$$\int_0^T\int_\Omega m\, F(x, m) dxdt + \int_0^T\int_\Omega m \left\{H_p(x, Du)Du - H(x, Du)\right\} dxdt$$
$$= \int_\Omega m_0\, u(0) - \int_\Omega G(x, m(T))m(T)$$

Theorem (CGPT)

Assume in addition that $p \mapsto H(x,p)$ is strictly convex and $m \mapsto F(x,m)$ is increasing. Then the first order system

$$\begin{cases} -u_t + H(x, Du) = F(x,m)\,, \\ m_t - \ div\,(m\, H_p(x, Du)) = 0\,, \\ m(0) = m_0\,,\ \ u(T) = G(x, m(T)) \end{cases}$$

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- Existence is proved through vanishing viscosity limit. Ingredients: coercivity + weak limits + Minty's argument (convex Hamiltonian and monotone couplings...)
- Key point: duality between sub solutions of Hamilton-Jacobi and solutions of the continuity equation

MFG $\to$ optimal transport

Planning problem in Mean Field games: prescribe a final distribution law $m(T) = m_1$

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Here, no condition is assumed on u at time T.

$\rightsquigarrow$ this is a singular limit of MFG systems with terminal condition $u_\varepsilon(T) = \frac{m_\varepsilon(T) - m_1}{\varepsilon}$, $\varepsilon \to 0$.

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• This is an optimal transport problem: a generalization of the Benamou-Brenier dynamic characterization of the Wassernstein distance

$$W_2^2(m_0, m_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} |v|^2 dm(t,x) \;: \right.$$

$$\left. \partial_t m + \text{ div } (vm) = 0,\; m_i = m|_{t=i}, i = 0, 1 \right\}.$$

The mean field planning problem can be characterized in terms of optimal transport [Orrieri- P.- Savaré '19]

$$\mathcal{B}(m,v) := \inf \left[\int_0^T \int_{\mathbb{R}^d} L(x,v)\, m \, dxdt + \int_0^T \int_{\mathbb{R}^d} \Phi(x,m) \ : \ \begin{cases} \partial_t m + \text{ div } (vm) = 0, \\ m(0) = m_0, m(T) = m_1 \end{cases} \right].$$

where $\Phi_m = F(x,m)$.

Assume as before:
$F(x,m) \simeq m^{p-1}$ and increasing
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Marginal measures $m_0, m_1 \in L^p$.

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- Dual problem:

$$\mathcal{A}(u,\alpha) := \sup \left\{ \int_{\mathbb{R}^d} u(0) m_0 dx - \int_{\mathbb{R}^d} u(T) m_1\, dx - \int_0^T \int_{\mathbb{R}^d} \Phi^*(x,\alpha)\, dxdt \ : \ \partial_t u + H(x,Du) \le \alpha\,, \ \ \alpha \in L^{p/p-1} \right\}.$$

where Φ^* is the Legendre transform of Φ.

- There exists a (unique) minimizer (m, v) of the optimal transport problem, and $v = -H_p(x, Du)$, where u is a maximizer of the dual problem $\mathcal{A}$

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- The dual problems have the same value $\mathcal{A}(u, \alpha) = \mathcal{B}(m, v)$ if and only if
  (i) $\alpha = f(x, m)$ a.e.
  (ii) $v = -D_p H(x, Du)$ m-a.e.
  (iii) u is a "renormalized" solution to

$$\partial_t(um) - \operatorname{div}\,(um\,H_p(x, Du)) = \Big(H(x, Du) - H_p(x, Du)\cdot Du - F(x, m)\Big)m$$

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- The above condition is equivalent to (u, m) being a weak (relaxed) solution of the MFG-planning system, i.e.

$$\begin{cases} -u_t + H(x, Du) \le F(x, m) \\ m_t - \text{ div } (m\, H_p(x, dau)) = 0\,, \quad m(0) = m_0\,, m(T) = m_1 \\ \int_0^T\!\!\int_\Omega m\, F(x, m) dxdt + \int_0^T\!\!\int_\Omega m\, \{H_p(x, Du)Du - H(x, Du)\}\, dxdt \\ \qquad = \int_\Omega m_0\, u(0)\, dx - \int_\Omega m_1\, u(T)\, dx \end{cases}$$

Conclusions

- So far, the analysis of mean field game systems enhanced a deeper investigation of the duality between Hamilton-Jacobi and Fokker-Planck (or transport) equations
- Duality methods proved to be crucial in order to build a robust theory of weak solutions for both second order and first order systems.
- Mean field game theory is built on the interaction between optimal control and transport. This is currently stimulating new directions of research in both fields. Ex:

  - optimal control problems on the Wassernstein space

  - optimal transport problems with additional entropic regularization: the mean field planning problem with coercive coupling is one such example.

  - the study of the long time behavior of MFG systems renewed the interest in the turnpike property of optimal control problems

Thanks for the attention !