Sharp ill-posedness for some nonlinear Dirac equations in one space dimension

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The equations

1. Thirring model:

$$(-i\gamma^\mu \partial_\mu + m)\psi = (\overline{\psi}\gamma^\mu \psi)\gamma_\mu \psi.$$

2. Maxwell-Dirac equations:

$$\begin{aligned}(-i\gamma^\mu \partial_\mu + m)\psi &= A_\mu \gamma^\mu \psi,\\ \Box A_\mu &= -\overline{\psi}\gamma_\mu \psi,\\ \partial^\mu A_\mu &= 0.\end{aligned}$$

Thirring model

Cauchy problem

$$\begin{aligned}(-i\gamma^\mu \partial_\mu + m)\psi &= (\overline{\psi}\gamma^\mu\psi)\gamma_\mu\psi, \\ \psi|_{t=0} &= \psi_0 \in X_0.\end{aligned}$$

- Well posed or ill posed?
- The space $X_0 = L^2(\mathbb{R})$ turns out to play a special role:
  - Scaling invariance of the eqs. (when $m = 0$)

$$\psi(x,t) \longrightarrow \psi^{(\lambda)}(x,t) = \frac{1}{\lambda^{1/2}}\psi\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right) \qquad (\lambda > 0)$$

  - Conservation of charge ($\partial_\mu j^\mu = 0$, $j^\mu = \overline{\psi}\gamma^\mu\psi$)

$$\int_{\mathbb{R}} |\psi(x,t)|^2 \, dx = \int_{\mathbb{R}} |\psi(0,t)|^2 \, dx$$

Global well-posedness

- First global result due to Delgado (1978) in $X_0 = H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$.
- Candy (2011) proved global well-posedness in the critical space $X_0 = L^2(\mathbb{R})$.
- What happens below the L^2 regularity? For example, for
  - $X_0 = L^p(\mathbb{R})$, $1 \le p < 2$, or
  - $X_0 = H^s(\mathbb{R})$, $s < 0$.
- Both have supercritical scaling: $\lambda \to 0$ means that
  - data norm tends to zero, and
  - existence time tends to zero,

  so heuristically one expects local well-posedness to fail.

What do we mean by local well-posedness

By *local well-posedness* in a Banach space X_0 of initial data, containing $L^2_{\mathrm{comp}}(\mathbb{R})$ as a dense subspace, we mean here the following: For any data

$$\psi_0 \in X_0$$

there exist

1. a neighborhood Ω of ψ_0 in X_0,
2. a time $T > 0$, and
3. a continuous map
$$S\colon \Omega \to C([0,T]; X_0)$$
which on $\Omega \cap L^2_{\mathrm{comp}}(\mathbb{R})$ agrees with the L^2 data-to-solution map obtained by Candy.

Our main result

Theorem (S. and Tesfahun, 2019)

The Cauchy problem

$$(-i\gamma^\mu \partial_\mu + m)\psi = (\overline{\psi}\gamma^\mu\psi)\gamma_\mu\psi,$$
$$\psi|_{t=0} = \psi_0 \in X_0$$

fails to be locally well posed in

$$X_0 = L^p(\mathbb{R}), \qquad 1 \le p < 2.$$

Moreover, if $m = 0$, *local well-posedness fails also in*

$$X_0 = H^s(\mathbb{R}), \qquad s < 0.$$

Choice of initial data

Consider

$$\psi_0(x) = \chi_{(-1,1)}(x) \frac{1}{|x|^{1/2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and approximations

$$\psi_0^\varepsilon(x) = \chi_{(-1,1)}(x) \frac{1}{(\varepsilon + |x|)^{1/2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then

- $\psi_0 \in L^p(\mathbb{R})$ for $1 \le p < 2$, but not for $p = 2$.
- $\psi_0 \in H^s(\mathbb{R})$ for $s < 0$.
- $\psi_0^\varepsilon \to \psi_0$ in the above spaces, as $\varepsilon \to 0$.
- $\psi_0^\varepsilon \in L^2(\mathbb{R})$, so has a global evolution $\psi^\varepsilon \in C(\mathbb{R}; L^2)$.
- To disprove local well-posedness we show that $\psi^\varepsilon(\cdot, t)$ cannot have a limit in L^p for $1 \le p < 2$, no matter how small $t > 0$ is.

Plan for the proof

1. Preliminaries
2. Massless case ($m = 0$). Explicit calculation.
3. Massless case alternative approach.
4. Massive case ($m > 0$).
5. Further remarks on the massless case.

Preliminaries

Adopting the particular representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for the Dirac matrices, and writing $\psi = (u, v)^\intercal$, the problem takes the form

$$\begin{cases} (\partial_t + \partial_x)u = -imv + 2i\,|v|^2\,u, & u(x,0) = f(x), \\ (\partial_t - \partial_x)v = -imu + 2i\,|u|^2\,v, & v(x,0) = g(x). \end{cases}$$

Key fact: local form of conservation of charge,

$$\int_0^t 2\,|u(x+t-\sigma,\sigma)|^2\,d\sigma + \int_0^t 2\,|v(x-t+\sigma,\sigma)|^2\,d\sigma \\ = \int_{x-t}^{x+t} \left(|f(y)|^2 + |g(y)|^2\right)\,dy.$$

Massless case: $m = 0$

System reads

$$(\partial_t + \partial_x)u = 2i\,|v|^2\,u, \qquad u(x,0) = f(x),$$
$$(\partial_t - \partial_x)v = 2i\,|u|^2\,v, \qquad v(x,0) = g(x).$$

Multiply by integrating factors $e^{-i\phi_+}$ and $e^{-i\phi_-}$, where

$$(\partial_t + \partial_x)\phi_+ = 2\,|v|^2\,, \qquad \phi_+(x,0) = 0,$$
$$(\partial_t - \partial_x)\phi_- = 2\,|u|^2\,, \qquad \phi_-(x,0) = 0,$$

that is,

$$\phi_+(t,x) = \int_0^t 2\,|v(x - t + \sigma, \sigma)|^2\,d\sigma,$$
$$\phi_-(t,x) = \int_0^t 2\,|u(x + t - \sigma, \sigma)|^2\,d\sigma.$$

Massless case: $m = 0$

Then

$$u(x,t) = f(x-t)e^{i\phi_+(t,x)},$$
$$v(x,t) = g(x+t)e^{i\phi_-(t,x)}.$$

In particular, $|u(x,t)| = |f(x-t)|$ and $|v(x,t)| = |g(x+t)|$, so

$$\phi_+(x,t) = \int_0^t 2\left|g(x-t+2\sigma)\right|^2 \, d\sigma = \int_{x-t}^{x+t} |g(s)|^2 \, ds,$$
$$\phi_-(x,t) = \int_0^t 2\left|f(x+t-2\sigma)\right|^2 \, d\sigma = \int_{x-t}^{x+t} |f(s)|^2 \, ds.$$

For data $f_\varepsilon(x) = g_\varepsilon(x) = \frac{1}{(\varepsilon+|x|)^{1/2}}$, we get

$$\phi_+^\varepsilon(x,t) = \phi_-^\varepsilon(x,t) = \int_{x-t}^{x+t} \frac{dy}{\varepsilon + |y|}.$$

Massless case: $m = 0$

In particular, in the region $t > |x|$,

$$e^{2i\log\varepsilon} u_\varepsilon(x,t) = \frac{1}{(\varepsilon + t - x)^{1/2}} e^{i[\log(\varepsilon+t-x)+\log(\varepsilon+t+x)]}.$$

Implies that $u_\varepsilon(\cdot, t)$ cannot converge in L^p or in H^s as $\varepsilon \to 0$, no matter how small $t > 0$ is taken.
On the other hand, the initial data do converge in those spaces if $p < 2$ (respectively $s < 0$), so we have the proof of ill-posedness.

Massless case, alternative approach

Motivation: In massive case, cannot calculate ϕ_+^ε and ϕ_-^ε explicitly.

- But by conservation of charge (also in massive case)

$$\phi_+^\varepsilon(x,t) + \phi_-^\varepsilon(x,t) = \int_{x-t}^{x+t} \frac{dy}{\varepsilon + |y|}.$$

- To make use of this, it is desirable to work with the product $u_\varepsilon v_\varepsilon$.
- Illustrate on the massless case:

$$e^{4i\log\varepsilon} u_\varepsilon v_\varepsilon(x,t) = \frac{e^{2i\log(\varepsilon+x+t)}}{(\varepsilon+x+t)^{1/2}} \frac{e^{2i\log(\varepsilon+t-x)}}{(\varepsilon+t-x)^{1/2}}.$$

- Choose positive $\varepsilon_n, \varepsilon_n' \to 0$ such that $e^{4i\log\varepsilon_n} = 1$ and $e^{4i\log\varepsilon_n'} = -1$.
- Assuming well-posedness in L^p implies convergence a.e. of a subsequence, hence

$$+uv(x,t) = -uv(x,t) = \frac{e^{2i\log(x+t)}}{(x+t)^{1/2}} \frac{e^{2i\log(t-x)}}{(t-x)^{1/2}}$$

for almost every $x \in (-t,t)$, for any fixed $t > 0$.

Massive case

Defining ϕ_+ and ϕ_- as before, one obtains

$$e^{-i\phi_+}u(x,t) = f(x-t) - im\int_0^t \left(e^{-i\phi_+}v\right)(x-t+\sigma,\sigma)\,d\sigma,$$

$$e^{-i\phi_-}v(x,t) = g(x+t) - im\int_0^t \left(e^{-i\phi_-}u\right)(x+t-\sigma,\sigma)\,d\sigma.$$

Thus,

$$e^{-i(\phi_+ + \phi_-)}uv(x,t) = f(x-t)g(x+t) + \sum_{j=1}^{3} R_j(x,t),$$

where

$$R_1(x,t) = f(x-t)\left(-im\int_0^t \left(e^{-i\phi_-}u\right)(x+t-\sigma,\sigma)\,d\sigma\right),$$

$$R_2(x,t) = g(x+t)\left(-im\int_0^t \left(e^{-i\phi_+}v\right)(x-t+\sigma,\sigma)\,d\sigma\right),$$

$$R_3(x,t) = \left(-im\int_0^t \left(e^{-i\phi_-}u\right)(x+t-\sigma,\sigma)\,d\sigma\right)\left(-im\int_0^t \left(e^{-i\phi_+}v\right)(x-t+\sigma,\sigma)\,d\sigma\right).$$

Massive case

Now take $f_\varepsilon(x) = g_\varepsilon(x) = \frac{1}{(\varepsilon+|x|)^{1/2}}$, so by conservation of charge,

$$(\phi_+^\varepsilon + \phi_-^\varepsilon)(x,t) = \int_{x-t}^{x+t} \frac{dy}{\varepsilon + |y|}.$$

Then for $t > |x|$,

$$e^{4i\log\varepsilon} u_\varepsilon v_\varepsilon(x,t) = \frac{e^{2i\log(\varepsilon+x+t)}}{(\varepsilon+x+t)^{1/2}} \frac{e^{2i\log(\varepsilon+t-x)}}{(\varepsilon+t-x)^{1/2}} + R_\varepsilon(x,t),$$

where

$$R_\varepsilon(x,t) = e^{2i\log(\varepsilon+x+t)} e^{2i\log(\varepsilon+t-x)} \sum_{j=1}^{3} R_{j,\varepsilon}(x,t).$$

We then show that this remainder is negligible compared to the first term on right hand side, in the ball B centered at $(x,t) = (0,\delta)$ with radius $\delta/4$, for $\delta > 0$ sufficiently small and $\varepsilon < \delta$.

Massive case

Define, for $t > 0$

$$A(t) = \sup_{y\in\mathbb{R}} \int_0^t |u_\varepsilon(y-\sigma,\sigma)| \; d\sigma + \sup_{y\in\mathbb{R}} \int_0^t |v_\varepsilon(y+\sigma,\sigma)| \; d\sigma.$$

For $y \in \mathbb{R}$ and $\sigma > 0$,

$$|u_\varepsilon(y-\sigma,\sigma)| \le \frac{1}{|y-2\sigma|^{1/2}} + m\int_0^\sigma |v_\varepsilon(y-2\sigma+s,s)| \; ds,$$

$$|v_\varepsilon(y+\sigma,\sigma)| \le \frac{1}{|y+2\sigma|^{1/2}} + m\int_0^\sigma |u_\varepsilon(y+2\sigma-s,s)| \; ds,$$

and integrating this with respect to $\sigma \in (0,t)$ we get

$$A(t) \le 8t^{1/2} + 2m\int_0^t A(\sigma)\, d\sigma,$$

so by Grönwall we get

$$A(t) \le ct^{1/2}$$

for $t < 1$. The rest is easy.

Further remarks on the massless case

Although the solution $(u_\varepsilon, v_\varepsilon)$ does not have a limit as $\varepsilon \to 0$, one can nevertheless observe that by restricting ε to any sequence $\varepsilon_{\boldsymbol{n}} \to 0$ such that $e^{i \log \varepsilon_n}$ has a limit, then the solution does converge in $C([0,T]; L^p)$, $1 \leq p < 2$, to a valid solution in that space. In this way, one obtains a continuum of possible limiting solutions, depending on the limit of $e^{i \log \varepsilon_n}$.

Thank you for your attention!