# On the physics of the Navier-Stokes equations

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# The compressible Navier-Stokes equations

$$\begin{aligned}
\partial_t \rho + div_{\mathbf{x}}(\rho \mathbf{v}) &= 0 \\
\partial_t(\rho \mathbf{v}) + div_{\mathbf{x}}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p &= div_{\mathbf{x}} \mathbb{S} \\
\partial_t(E) + div_{\mathbf{x}}(E\mathbf{v} + p\mathbf{v}) &= div_{\mathbf{x}} \mathbb{S}\mathbf{v} + div_{\mathbf{x}}(\kappa \nabla_{\mathbf{x}} T) \\
p &= \rho R T \quad \text{ideal gas law}
\end{aligned}$$

$\mathbf{v} = (u, v, w)$

Total energy: $E = \frac{1}{2}\rho|\mathbf{v}|^2 + \rho c_v T$.

# Basic assumptions

- Ideal gas
  - Gas behaves as "billiard balls" with no interparticle forces.
  - Atoms have no rotational energy.
- Models macroscopic states in *local thermodynamic equilibrium*. Diffusion dominates at small scales.
- Stresses modelled as Newtonian fluid.
- No external forces.

# Physically problematic properties

- Positivity (not given by equations; essential for well-posedness)
- Adiabatic wall (non-zero diffusive entropy and energy)
- Far-field boundaries (inconsistent with Euler)
- Small-scale relaxation
  - Short wave length thermodynamic perturbation of fluid at rest should diffuse.
  - Instead, high-frequency convective velocity is induced.
- Multi-component fluids

# Multi-component fluids

$$
\begin{aligned}
(\rho_1+\rho_2)_t + ((\rho_1+\rho_2)u)_x &= 0 \qquad (\rho = \rho_1+\rho_2) \\
(\rho_1)_t + (\rho_1(u+j_1))_x &= 0 \\
(\rho_2)_t + (\rho_2(u+j_2))_x &= 0
\end{aligned} \tag{1}
$$

$j_{1,2}$ are mass-diffusion fluxes.

- Velocity has conflicting meanings.
- $j_1, j_2$ must not violate (1).
  - If only one component, $j_1 = 0$.
  - If two, also: $j_1 + j_2 = 0$.
- Fick's law has to be modified: $j_1 = \xi(\rho_1)_x$ to $j_1 = \xi(\frac{\rho_1}{\rho})_x$ etc.

# Lack of mass diffusion is the cause of these problems

$$\begin{aligned} \partial_t \rho + div_{\mathbf{x}}(\rho \mathbf{v}) &= 0 \\ \partial_t(\rho \mathbf{v}) + div_{\mathbf{x}}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p &= div_{\mathbf{x}} \mathbb{S} \\ \partial_t(E) + div_{\mathbf{x}}(E\mathbf{v} + p\mathbf{v}) &= div_{\mathbf{x}} \mathbb{S}\mathbf{v} + div_{\mathbf{x}}(\kappa \nabla_{\mathbf{x}} T) \\ p &= \rho R T \quad \text{ideal gas law} \end{aligned}$$

**Why is there no mass diffusion?**

The most direct answer: All mass transport is encoded in $u$.

To see why that is the case, we turn to the derivation of the flow equations.

# Derivation of flow equations

- Euler: Fixed volume in space. Main principle: Conservation.
- NS: As above+ smoothly deformable mass element; add stress forces. Main principle: Force balance.

Navier-Stokes is Lagrangian.

- Requires mass element smoothness! True along e.g. stagnation line?
- Viscosity appears due to molecular diffusion, and yet mass is constant within an element.

# Stress tensor in Eulerian frame (Newtonian fluid)

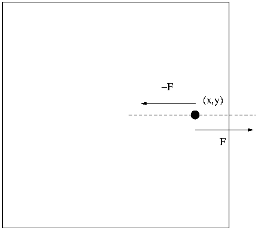



Figure: Viscous force: $\mu u_y$

x-momentum

- x boundary: $u_x, v_y$
- y boundary: $u_y$, $v_x$
- Only $\partial_n u$ can affect $\rho u$.

# Local non-conservation of the NSE.

- In NS, velocity gradients acts as diffusion between fluid volumes.
- Differences in velocity are evened out. Does not inherently conserve momentum!
- Convective velocity is induced between fluid volumes to conserve momentum.

Similarly for heat diffusion. Non-conservative w.r.t internal energy induces micro-advection.

Scrap remaining stress tensor and heat conductive term!

# Diffusion seems to be a more fundamental principle

Random molecular movements give rise to:

- Stresses: Diffusion $\Rightarrow$ Viscosity $\Rightarrow$ stress force
- Heat conduction.
- But somehow, the mass transfer is *purely* advective (through $\mathbf{v}$).

# Eulerian view


Figure: Fluid volumes.

Particle system conserves mass, momentum, kinetic energy, angular momentum and has uniform centre-of-mass movement (no external forces).

# How to regularise the Euler equations?

- Stay in an Eulerian frame.
- Define the meaning of continuum variables (from particle system).
- Determine what properties a 5 eqn system should conserve. (And what not...)
- Use conservation as fundamental principle.
- Model Convective and diffusive term.
  - Convection: same as for the Euler equations.
  - Diffusion: When a particle moves it transports (and conserves) its mass, momentum and kinetic energy. (Globally, $\rho$, $\rho u$, $E$.)

# An Eulerian flow model

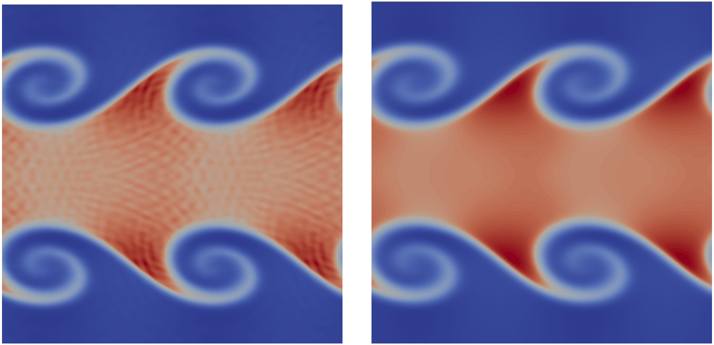
$$
\begin{aligned}
\partial_t \rho + div_{\mathbf{x}}(\rho \mathbf{v}) &= \nabla_{\mathbf{x}} \cdot (\nu \nabla_{\mathbf{x}} \rho) \\
\partial_t(\rho \mathbf{v}) + div_{\mathbf{x}}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p &= \nabla_{\mathbf{x}} \cdot (\nu \nabla_{\mathbf{x}} \rho \mathbf{v})) \\
\partial_t(E) + div_{\mathbf{x}}(E\mathbf{v} + p\mathbf{v}) &= \nabla_{\mathbf{x}} \cdot (\nu \nabla_{\mathbf{x}} E) \\
p &= \rho R T \quad \text{ideal gas law}
\end{aligned}
$$

$\nu \sim \frac{\mu}{\rho}$ since $\mu$ has been extensively measured.

Note the symmetry: All variables are transported by the true mean velocity and random motions are modelled as diffusion.

- Galilean and rotationally invariant.
- The listed problems of NS are all resolved.
- (Yes, they can be derived from Boltzmann.)

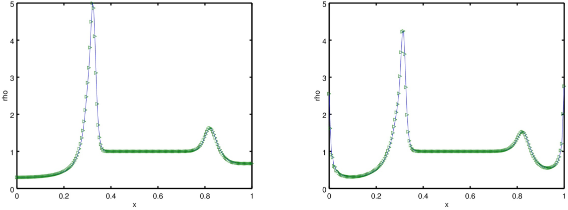
# Kelvin-Helmholtz instability



(a) Navier-Stokes (b) Eulerian

Figure: $\rho$ at $T = 2$ with $256^2$ grid points. $(\nu = \mu/\rho)$

# Blast wave bouncing on walls




(a) Eulerian (b) Navier-Stokes

Figure: $\rho$ at $t = 0.01$. 200 grid points.

# Conclusions

- I propose a modified NS system based on conservation and diffusion.
- It has "nicer" properties than NS.
- In many cases it produces solutions that are almost identical to NS.

Thank you for your attention!