Degrees of freedom and bounded projections

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Ciarlet's definition of a finite element space

A finite element space is defined by specifying

- the mesh
- the shape functions
- the degrees of freedom (DOFs)

Reference: P. Ciarlet, The finite element method for elliptic problems, 1978

Example: C^0 spaces with respect to a simplicial triangulation

- $\mathcal{T}$ simplicial triangulation of $\Omega \subset \mathbb{R}^n$
- $\mathcal{P}_r, r \geq 1$
- DOFs
$$\int_f u \cdot \eta \, dx_f, \quad \eta \in \mathcal{P}_{r-1-\dim f}(f),$$
where f runs over all subsimplexes of $\mathcal{T}$.

This defines a space V_h of piecewise polynomials of total degree $\leq r$, which are globally continuous, and therefore

$$V_h \subset H^1(\Omega).$$

The DOFs leads to a basis for V_h, referred to as the dual basis.

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The DOFs leads to a basis for V_h, referred to as the dual basis.

Furthermore, the DOFs implicitly defines a canonical projection onto V_h.

The de Rham complex in three dimensions

$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

Finite element discretizations will typically utilize a corresponding subcomplex of the form:

$$\begin{array}{ccccccccc}
\mathbb{R} \hookrightarrow & H^1(\Omega) & \xrightarrow{\text{grad}} & H(\text{curl}, \Omega) & \xrightarrow{\text{curl}} & H(\text{div}, \Omega) & \xrightarrow{\text{div}} & L^2(\Omega) & \to 0 \\
& \Big\downarrow \cup & & \Big\downarrow \cup & & \Big\downarrow \cup & & \Big\downarrow \cup & \\
\mathbb{R} \hookrightarrow & H^1_h & \xrightarrow{\text{grad}} & H_h(\text{curl}) & \xrightarrow{\text{curl}} & H_h(\text{div}) & \xrightarrow{\text{div}} & L^2_h & \to 0.
\end{array}$$

Lowest order spaces, simplicial mesh

$$
\begin{aligned}
H_h^1 : \mathcal{P}_1, &\quad DOFs = \text{vertex values} \\
H_h(\text{curl}) : \text{rigid motions}, &\quad DOFs = \text{tangential components on edges} \\
H_h(\text{div}) : \vec{a} + bx, &\quad DOFs = \text{ normal components on faces} \\
L_h^2 : \text{constants}, &\quad DOFs = \text{values on each simplex}
\end{aligned}
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$$\begin{aligned} H_h^1 : \mathcal{P}_1, &\quad DOFs = \text{vertex values} \\ H_h(\text{curl}) : \text{rigid motions}, &\quad DOFs = \text{tangential components on edges} \\ H_h(\text{div}) : \vec{a} + bx, &\quad DOFs = \text{ normal components on faces} \\ L_h^2 : \text{constants}, &\quad DOFs = \text{values on each simplex} \end{aligned}$$

Furthermore, the following diagram commutes

$$\begin{array}{ccccccccc} \mathbb{R} \hookrightarrow H^1(\Omega) & \xrightarrow{\text{grad}} & H(\text{curl}, \Omega) & \xrightarrow{\text{curl}} & H(\text{div}, \Omega) & \xrightarrow{\text{div}} & L^2(\Omega) & \rightarrow & 0 \\ \Big\downarrow \mathcal{I}_h & & \Big\downarrow \mathcal{I}_h & & \Big\downarrow \mathcal{I}_h & & \Big\downarrow \mathcal{I}_h & & \\ \mathbb{R} \hookrightarrow \quad H_h^1 & \xrightarrow{\text{grad}} & H_h(\text{curl}) & \xrightarrow{\text{curl}} & H_h(\text{div}) & \xrightarrow{\text{div}} & L_h^2 & \rightarrow & 0, \end{array}$$

where the operators $\mathcal{I}_h$ are the corresponding canonical projections.

Differential forms, $\Omega \subset \mathbb{R}^n$

$$0 \to H\Lambda^0 \xrightarrow{d} H\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n \to 0$$

where

$$H\Lambda^k = H\Lambda^k(\Omega) = \{\, \omega \in L^2\Lambda^k(\Omega) \,|\, d\omega \in L^2\Lambda^{k+1}(\Omega) \,\}.$$

For approximations we consider the set up:

$$\begin{array}{ccccccc}
H\Lambda^0(\Omega) & \xrightarrow{d} & H\Lambda^1(\Omega) & \xrightarrow{d} & \cdots & \xrightarrow{d} & H\Lambda^n(\Omega) \\
\downarrow \mathcal{I}_h^0 & & \downarrow \mathcal{I}_h^1 & & & & \downarrow \mathcal{I}_h^n \\
\Lambda_h^0 & \xrightarrow{d} & \Lambda_h^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Lambda_h^n
\end{array}$$

where $\Lambda_h^k \subset H\Lambda^k(\Omega)$.

The spaces $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$

The space $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ consists of all k forms u such that

$$u|_T \in \mathcal{P}_r\Lambda^k(T), \quad T \in \mathcal{T}_h, \qquad \text{and } [\operatorname{tr} u]_f = 0 \quad \forall f \in \Delta_{n-1}(\mathcal{T}_h).$$

The spaces $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ are defined similarly with $u|_T \in \mathcal{P}_r\Lambda^k(T)$ replaced by $u|_T \in \mathcal{P}_r^-\Lambda^k(T)$. Here $\mathcal{P}_r^-\Lambda^k$ consists of all

$$u \in \mathcal{P}_r\Lambda^k \quad \text{such that } u \lrcorner x \in \mathcal{P}_r\Lambda^{k-1}.$$

Degrees of freedom

All the spaces Λ_h^k above have DOFs defined with respect to the subsimplexes of of $\mathcal{T}_h$ of the form

$$\int_f \operatorname{tr}_f u \wedge \eta, \quad \eta \in \mathcal{P}'(f,k,r), \quad f \in \Delta(\mathcal{T}_h), \dim f \geq k,$$

where $\mathcal{P}'(f,k,r) \subset \Lambda^{dimf-k}(f)$.

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where $\mathcal{P}'(f,k,r) \subset \Lambda^{dimf-k}(f)$.
If

$$\Lambda_h^k = \mathcal{P}_r^- \Lambda^k(\mathcal{T}_h) \quad \text{then } \mathcal{P}'(f,k,r) = \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f-k}(f),$$

while if

$$\Lambda_h^k = \mathcal{P}_r \Lambda^k(\mathcal{T}_h) \quad \text{then } \mathcal{P}'(f,k,r) = \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f-k}(f).$$

Furthermore, the corresponding canonical projections commute with the exterior derivative.

Approximation of Hilbert complexes

Framework:

$$\begin{array}{ccccccc} V^0 & \xrightarrow{d_0} & V^1 & \xrightarrow{d_1} & \cdots & \xrightarrow{d_{n-1}} & V^n \\ \Big\downarrow \pi_h & & \Big\downarrow \pi_h & & & & \Big\downarrow \pi_h \\ V_h^0 & \xrightarrow{d_0} & V_h^1 & \xrightarrow{d_1} & \cdots & \xrightarrow{d_{n-1}} & V_h^n \end{array}$$

Stability of discrete problems *iff* $\pi_h^k : V^k \to V_h^k$ are *uniformly bounded* in $\mathcal{L}(V^k, V^k)$, and *commutes with d*, i.e.,

$$d_k \circ \pi_h^k = \pi_h^{k+1} \circ d_k$$

Furthermore,

$$c_{p,h} \leq c_p \|\pi\|_{\mathcal{L}(V^k, V^k)}.$$

A problem: The canonical projections are in general *not bounded.*

Construction of bounded cochain projections by smoothing

Consider operators of the form

$$Q^k_{\epsilon,h} = \mathcal{I}^k_h \circ R^k_{\epsilon,h},$$

where $R^k_h = R^k_{\epsilon,h}$ is a proper smoothing operator which commutes with the exterior derivative d.

An operator of the form Q^k_h can be made bounded on $L^2\Lambda^k(\Omega)$, and will commute with d. However, in general it is *not a projection* onto the finite element space Λ^k_h.

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The so called *smoothed projections* are of the form

$$\pi^k_h = (Q^k_{\epsilon,h}|_{\Lambda_h})^{-1} \circ Q^k_{\epsilon,h},$$

for ϵ sufficiently small, but not too small. (cf. Schöberl 2007, Christiansen 2007, Arnold–Falk–W 2006).

These constructions give bounded, but *nonlocal* projections.

The Clément operator

The Clément operator $\mathcal{I}_h : L^2 \to \mathcal{P}_r\Lambda^0(\mathcal{T}_h)$ is defined by

$$\int_f \mathcal{I}_h u \cdot \eta \, dx_f = \int_f P_f u \cdot \eta \, dx_f, \quad \eta \in \mathring{\mathcal{P}}_r(f), \quad f \in \Delta(\mathcal{T}_h),$$

where P_f is the local $L^2(\Omega_f)$ projection onto $\mathcal{P}_r$.

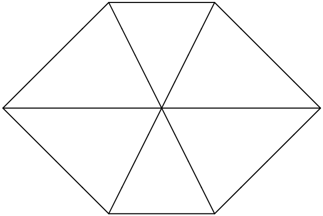
This operator is bounded even in L^2, and it has "optimal" approximation properties, but it is *not a projection*. Furthermore, it is *not obvious* how to extend the construction into a cochain projection.

Macroelements

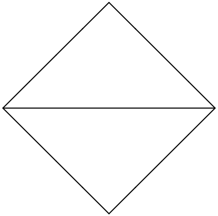
For each $f \in \Delta(\mathcal{T}_h)$ Ω_f is given by

$$\Omega_f = \bigcup \{\, T \mid T \in \mathcal{T}_h,\, f \in \Delta(T)\,\}.$$

Vertex macroelement, $n = 2$.



Edge macroelement, $n = 2$.



The modified Clement operator onto $\mathcal{P}_r\Lambda^0(\mathcal{T}_h) \subset H^1$

The operator π_h is constructed by a recursive procedure. In particular,

$$\pi_{0,h} u = \sum_{z \in \Delta_0(\mathcal{T}_h)} E_z(P_z u)(z),$$

where P_z is the local H^1 projection onto $\mathcal{P}_r\Lambda^0(\mathcal{T}_{z,h})$.

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For $1 \le m \le n$ we define $\pi_{m,h}$ by

$$\pi_{m,h} u = \pi_{m-1,h} u + \sum_{f \in \Delta_m(\mathcal{T}_h)} E_f \operatorname{tr}_f P_f (u - \pi_{m-1,h} u)$$

For $\dim f \ge 1$ the operators P_f are local H^1 projections onto to the space

$$\check{\mathcal{P}}_r\Lambda^0(\mathcal{T}_{f,h}) = \{ u \in \mathcal{P}_r\Lambda^0(\mathcal{T}_{f,h}) \,|\, \operatorname{tr}_f u \in \mathring{\mathcal{P}}_r(f) \},$$

and $E_f : \mathring{\mathcal{P}}_r(f) \to P_r\Lambda^0(\mathcal{T}_{f,h})$ is the discrete harmonic extension. This will lead a local projection $\pi_h = \pi_{n,h}$ which is bounded in H^1.

The simplest example

Consider the (modified) Clement projection onto the piecewise linear space $\mathcal{P}_1\Lambda^0(\mathcal{T}_h)$. The operator π_h^0 has the form

$$(\pi_h^0 u)(x) = \sum_{z\in\Delta_0(\mathcal{T}_h)} (P_z u)(z)\lambda_z(x)$$

Here the projections P_z are local H^1 projections with respect to macroelement Ω_z.
More preciesely,

$$P_z u = \int_\Omega u \cdot \mathrm{vol}_{\Omega_z}\, dx + Q_z u.$$

where $Q_z u \in \mathcal{P}_1\Lambda^0(\mathcal{T}_{z,h})$ has mean value zero on Ω_z, and satisfies

$$\langle \mathrm{grad}\, Q_z u, \mathrm{grad}\, v\rangle_{\Omega_z} = \langle \mathrm{grad}\, u, \mathrm{grad}\, v\rangle_{\Omega_z},$$

for all $v \in \mathcal{P}_1\Lambda^0(\mathcal{T}_{z,h})$ with mean value zero. Here $\mathrm{vol}_{\Omega_z} = |\Omega_z|^{-1}\kappa_{\Omega_z}$.

Commuting projections

To obtain commuting projections we need to define π_h^1 into the space $\mathcal{P}_1^- \Lambda^1(\mathcal{T}_h)$ such that

$$\operatorname{grad} \pi_h^0 u = \pi_h^1 \operatorname{grad} u.$$

In particular, we have to express

$$\operatorname{grad} \pi_h^0 u = \operatorname{grad} \sum_{z \in \Delta_0(\mathcal{T}_h)} (\int_\Omega u \cdot \mathrm{vol}_{\Omega_z} \, dx + (Q_z u)(z)) \lambda_z,$$

in terms of $\operatorname{grad} u$.

The appearance of the δ operator

So consider the operator

$$(M_h u)(x) = \sum_{z \in \Delta_0(\mathcal{T}_h)} (\int_\Omega u \cdot \mathrm{vol}_{\Omega_z}\, dx) \lambda_z(x).$$

We need to express grad $M_h u$ in terms of grad u.

The appearance of the δ operator

So consider the operator

$$(M_h u)(x) = \sum_{z \in \Delta_0(\mathcal{T}_h)} \left(\int_\Omega u \cdot \mathrm{vol}_{\Omega_z}\, dx\right) \lambda_z(x).$$

We need to express grad $M_h u$ in terms of grad u.
If $f = [x_0, x_1]$ consider grad $M_h(u) \cdot (x_1 - x_0)$ on f.

$$\begin{aligned} \mathrm{tr}_f \operatorname{grad} M_h(u) \cdot (x_1 - x_0) &= \int_\Omega u(\mathrm{vol}_{\Omega_{x_1}} - \mathrm{vol}_{\Omega_{x_0}})\, dx \\ = \int_\Omega u(\delta z^0)_f\, dx &= \int_\Omega u(\operatorname{div} z_f^1)\, dx, \end{aligned}$$

where z_f^1 satisfies div $z_f^1 = (\delta z^0)_f$ and have zero normal components on the boundary of $\Omega_f^e = \Omega_{x_0} \cup \Omega_{x_1}$.

We can therefore conclude that

$$\operatorname{grad} M_h u = \sum_{f \in \Delta_1(\mathcal{T}_h)} \left(\int_{\Omega_f^e} \operatorname{grad} u \cdot z_f^1 \, dx \right) \phi_f,$$

where

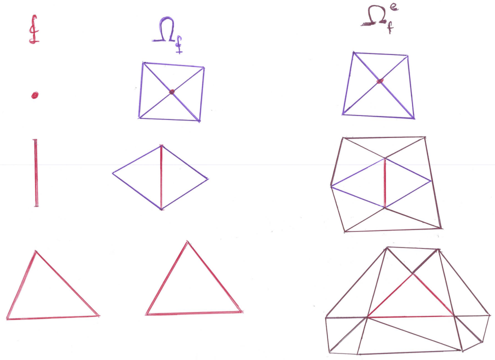
$$\phi_f = \lambda_0 (\operatorname{grad} \lambda_1) - \lambda_1 (\operatorname{grad} \lambda_0), \quad f = [x_0, x_1]$$

and $\lambda_i = \lambda_{x_i}$.

The extended macroelements

$$\Omega_f^e = \bigcup_{g \in \Delta_0(f)} \Omega_g, \quad f \in \Delta(\mathcal{T}_h).$$

If $g \in \Delta(f)$ then $\Omega_f \subset \Omega_g$ and $\Omega_f^e \supset \Omega_g^e$.
In 2D we have:



A double complex

Commuting diagram:

$$\begin{array}{ccc}
\bigoplus_{f\in\Delta_m(\mathcal{T})} \mathring{H}\Lambda^k(\Omega_f^e) & \xrightarrow{d} & \bigoplus_{f\in\Delta_m(\mathcal{T})} \mathring{H}\Lambda^{k+1}(\Omega_f^e) \\
\Big\downarrow \delta & & \Big\downarrow \delta \\
\bigoplus_{f\in\Delta_{m+1}(\mathcal{T})} \mathring{H}\Lambda^k(\Omega_f^e) & \xrightarrow{d} & \bigoplus_{f\in\Delta_{m+1}(\mathcal{T})} \mathring{H}\Lambda^{k+1}(\Omega_f^e)
\end{array}$$

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\end{array}$$

If $f = [x_0, x_1, \ldots x_{m+1}] \in \Delta_{m+1}(\mathcal{T}_h)$ then

$$(\delta u)_f = \sum_{j=0}^{m+1} (-1)^j u_{f_j},$$

where $f_j = [x_0, \ldots, x_{j-1}, \hat{x}_j, x_{j+1}, \ldots x_{m+1}]$. For more details see

The modified Clement operator for k–forms

We recall that for $k = 0$ the operator π^0 is constructed by a recursive procedure of the form

$$\pi^0_m u = \pi^0_{m-1} u + \sum_{f \in \Delta_m(\mathcal{T}_h)} E^0_f \operatorname{tr}_f P^0_f (u - \pi^0_{m-1} u), \quad 0 \le m \le n,$$

where $\pi^0 = \pi^0_n$.

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where $\pi^0 = \pi^0_n$.

For $k \ge 1$ we will utilize a similar construction The projection π^k is defined by the recursion

$$\pi^k_m u = \pi^k_{m-1} u + \sum_{f \in \Delta_m(\mathcal{T}_h)} E^k_f \circ \operatorname{tr}_f \circ P^k_f (u - \pi^k_{m-1} u), \qquad k \le m \le n,$$

where $\pi^k = \pi^k_n$. *To start the iteration* we need to define $\pi^k_{k-1} u$ properly. Here the double complex construction is used.

Main references

Falk–W,
-Local bounded cochain projections, Math Comp 2014

and

Arnold–Falk–W,
-Finite element exterior calculus, homological techniques and applications, Acta Numerica 2006

-Finite element exterior calculus: from Hodge theory to numerical stability, Bulletin AMS 2010