AN OPERATOR SPLITTING METHOD FOR CONVECTION-DIFFUSION EQUATIONS

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Abstract. We present a semi-discrete method for constructing approximate solutions to the m-dimensional ($m \geq 1$) convection-diffusion equation $u_t + \nabla \cdot \mathbf{f}(u) = \varepsilon \Delta u$. The method is based on the use of operator splitting to isolate the convection part and the diffusion part of the equation. In the case $m > 1$, dimensional splitting is used to reduce the m-dimensional convection problem to a series of one-dimensional problems. We show that the method produces a compact sequence of approximate solutions. Finally, a fully discrete method is analyzed, and demonstrated in the case of one and two space dimensions.

1. Introduction. In this paper we present an operator splitting method for the scalar convection-diffusion equation

$$u_t + \sum_{i=1}^{m} f_i(u)_{x_i} = \varepsilon \sum_{i=1}^{m} u_{x_i x_i}, \qquad u(x_1, \dots, x_m, 0) = u_0(x_1, \dots, x_m). \tag{1.1}$$

For brevity of notation, we shall write this equation as

$$u_t + \nabla \cdot \mathbf{f}(u) = \varepsilon \Delta u, \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}),$$

where $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{f}(u) = (f_1(u), \dots, f_m(u))$, $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_m)$, and $\Delta = \sum_{i=1}^{m} \partial^2/\partial x_i^2$.

Equations such as (1.1) arise in a variety of applications, ranging from models of turbulence [1], via traffic flow [17], to two phase flow in porous media [19]. Equation (1.1) can also be viewed as a model problem for a system of convection-diffusion equations, such as three phase flow in porous media [23], or the Navier-Stokes equations.

Of particular importance is the case where convection dominates diffusion, i.e., ε is small compared with other scales in (1.1). This is often the case in models of two phase flow in oil reservoirs. Accurate numerical simulation of such models are consequently often complicated by both unphysical oscillations and numerical diffusion. The operator splitting approach presented here is especially well suited to the case where $\varepsilon \ll 1$.

The quasi-linear parabolic equation (1.1) was first properly analyzed by Oleinik in [18], where she proved existence of a unique classical solution, and also that weak solutions to (1.1) coincide with classical solutions.

Operator splitting, or fractional steps, methods have been extensively used in connection with conservation laws, starting with Godunov [10], who used this method to study gas dynamics. Operator splitting (dimensional splitting) for a scalar conservation law in several space dimensions was studied by Crandall and Majda in [2], where they analyzed both semi-discrete and discrete methods, and showed the convergence of the operator splitting for several numerical schemes. Holden and Risebro [12] studied dimensional splitting coupled with front tracking, and convergence rates for dimensional splitting methods were obtained in [22] and [14].

If $\varepsilon \ll 1$, then (1.1) is "almost hyperbolic", and it is natural to exploit this when constructing numerical methods. This approach has indeed been taken by several authors, we only mention Douglas and Russel [5], [21], [8], Espedal and Ewing [6], [7], and more recently Dahle [4]. In [21] a characteristic element method is used to solve the hyperbolic part of (1.1). In [5], error estimates are obtained for a linear version of (1.1).

The splitting method analyzed in this paper can be summarized as follows: Let $v(\mathbf{x}, t) = \mathcal{S}(t) v_0(\mathbf{x})$ be the unique entropy satisfying solution to

$$v_t + \nabla \cdot \mathbf{f}(v) = 0, \qquad v(\mathbf{x}, 0) = v_0(\mathbf{x}),$$

and let $w(\mathbf{x}, t) = \mathcal{H}(t) w_0(\mathbf{x})$ be the solution of

$$w_t = \varepsilon \Delta w, \qquad w(\mathbf{x}, 0) = w_0(\mathbf{x}).$$

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Operator splitting is based on the following approximation

$$u(\mathbf{x}, n\Delta t) \approx [\mathcal{H}(\Delta t)\mathcal{S}(\Delta t)]^n \, u_0(\mathbf{x}). \tag{1.2}$$

Here we study the convergence properties as the time step $\Delta t \to 0$. In applications, the exact solution operators $\mathcal{S}(t)$ and $\mathcal{H}(t)$ are replaced by approximations. We use one-dimensional front tracking as defined by Dafermos [3], see also [11], as an approximate solution operator for the hyperbolic part if $m = 1$. If $m > 1$, we use the dimensional splitting method described in [12] as an approximate solution operator for the hyperbolic part. In both one and several space dimensions a finite element method is used as an approximate operator for the diffusion part. We also establish convergence of the approximate solutions in the fully discrete case.

The rest of this paper is organized as follows: In section 2 we obtain compactness of the sequence of approximate solutions generated by (1.2), and show that any convergent subsequence converges to the unique solution of (1.1). In section 3, we show the same if the exact solution operators are replaced by the approximations mentioned above. In section 4 we study applications of the method in one and two space dimensions, and present tentative convergence rates based on the numerical examples.

2. The semi-discrete method. In this section we shall describe the operator splitting of (1.1), that is, we will obtain the solution of (1.1) through a composition of the solution operator to a hyperbolic equation and to a parabolic equation.

We study (1.1) for $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$, where $m \geq 1$. Let therefore $\mathbf{f}(u) = (f_1(u), \dots, f_m(u))$, where u is a scalar. Furthermore, let $\mathcal{S}(t)$ be the operator which takes an initial function $v_0(\mathbf{x})$ to the entropy weak solution at time t of

$$v_t + \nabla \cdot \mathbf{f}(v) = v_t + \sum_{i=1}^{m} f_i(v)_{x_i} = 0, \qquad v(\mathbf{x}, 0) = v_0(\mathbf{x}), \tag{2.1}$$

that is, we write the (weak) solution $v(\mathbf{x}, t)$ as $\mathcal{S}(t)v_0(\mathbf{x})$. We shall also need the solution operator taking the initial data $v_0(\mathbf{x})$ to the entropy weak solution at time t of the one dimensional conservation law

$$v_t + f_i(v)_{x_i} = 0, \qquad v(\mathbf{x}, 0) = v_0(\mathbf{x}). \tag{2.2}$$

This operator we denote $\mathcal{S}^{f_i}(t)$. Note that in (2.2), x_j, $j \neq i$, act only as parameters. Following Kružkov [15], we know that (2.1) has a unique entropy weak solution if $\mathbf{f}$ is Lipschitz continuous, and the initial data are in $L_\infty \bigcap B.V.$. Here, $B.V.$ denotes the space of functions with bounded total variation. Similarly, let $\mathcal{H}(t)$ be the solution operator (at time t) to the parabolic equation

$$w_t = \varepsilon \Delta w = \varepsilon \sum_{i=1}^{m} w_{x_i x_i}, \qquad w(\mathbf{x}, 0) = w_0(\mathbf{x}). \tag{2.3}$$

Now fix $T > 0$ and $\Delta t > 0$, and let N be such that $N\Delta t = T$. Let $u^0(\mathbf{x}) = u_0(\mathbf{x})$, and define $\{u^n(\mathbf{x})\}_{n=1}^N$ inductively by

$$u^{n+1}(\mathbf{x}) = \left[\mathcal{H}(\Delta t)\mathcal{S}^{f_m}(\Delta t) \dots \mathcal{S}^{f_1}(\Delta t)\right] u^n(\mathbf{x}), \tag{2.4}$$

for $n = 0, \dots, N-1$. Observe that by the results in [2], $\mathcal{S}(\Delta t) \approx \left[\mathcal{S}^{f_m}(\Delta t) \dots \mathcal{S}^{f_1}(\Delta t)\right]$.

We will now show that the sequence $\{u^n\}$ is compact in L_1^{loc}. To accomplish this we use a technique introduced by Oleinik [18]. This approach is now standard in the theory of conservation laws, and involves proving a priori uniform L_∞, $B.V.$, and $L_1(\mathbf{x})$ Lipschitz in time, bounds on $\{u^n\}$.

To simplify the notation we define

$$\mathcal{S}^{\mathbf{f}}(t) = \mathcal{S}^{f_m}(t) \dots \mathcal{S}^{f_1}(t).$$

We also note that if $w_0(\mathbf{x})$ is bounded, then the solution of (2.3) is given by convolution with the "heat kernel" $K(\mathbf{x}, t)$, where

$$K(\mathbf{x}, t) = \frac{1}{(4\pi\varepsilon t)^{m/2}} \exp\left[\frac{-|\mathbf{x}|^2}{4\varepsilon t}\right] = k(x_1, t) \dots k(x_m, t)$$

with $k(x, t)$ given by

$$k(x, t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \exp\left[\frac{-x^2}{4\varepsilon t}\right],$$

i.e., we can write

$$\mathcal{H}(t)w_0(\mathbf{x}) = \int_{\mathbb{R}^m} K(\mathbf{x} - \mathbf{y}, t)\, w_0(\mathbf{y})\, d\mathbf{y}. \tag{2.5}$$

Lemma 2.1. *For $n = 0, \ldots, N$,*

$$||u^n||_\infty \le ||u_0||_\infty. \tag{2.6}$$

Proof. This follows from the fact that both $\mathcal{S}^{\mathbf{f}}(t)$ and $\mathcal{H}(t)$ obey a maximum principle similar to (2.6). $\square$

Lemma 2.2. *For $n = 0, \ldots, N$,*

$$T.V.(u^n) \le T.V.(u_0). \tag{2.7}$$

Proof. Recall that for a function $g(x)$ its total variation is defined by

$$T.V.(g) = \limsup_{h\to 0} \int_{\mathbb{R}} \frac{|g(x+h) - g(x)|}{h}\, dx,$$

and for a function $g(\mathbf{x})$ of several variables

$$T.V.(g) = \sum_{i=1}^{m} \int_{\mathbb{R}^{m-1}} T.V._{x_i}(g)\, d^{m-1}x,$$

where $T.V._{x_i}(g)$ denotes the total variation of g with respect to the ith variable. We have that $T.V.\left(\mathcal{S}^{f_i}(t)v_0\right) \le T.V.(v_0)$ and that

$$||\mathcal{S}^{f_i}(t)v_0 - \mathcal{S}^{f_i}(t)\hat{v}_0||_1 \le ||v_0 - \hat{v}_0||_1.$$

These two facts imply, see [12], that

$$T.V.\left(\mathcal{S}^{\mathbf{f}}(t)v_0\right) \le T.V.(v_0).$$

The lemma will follow by induction if we can show that

$$T.V.(\mathcal{H}(t)w_0) \le T.V.(w_0). \tag{2.8}$$

Let $\mathbf{e}_i$ denote the unit vector in the ith direction. We calculate

$$\begin{aligned}\limsup_{h\to 0} \int_{\mathbb{R}} \frac{|w(\mathbf{x} + h\mathbf{e}_i, t) - w(\mathbf{x}, t)|}{h}\, dx_i &= \limsup_{h\to 0} \int_{\mathbb{R}} \int_{\mathbb{R}^m} K(\mathbf{y}, t) \frac{|w_0(\mathbf{x} + h\mathbf{e}_i - \mathbf{y}) - w_0(\mathbf{x} - \mathbf{y})|}{h}\, d\mathbf{y}\, dx_i \\ &\le \int_{\mathbb{R}^m} K(\mathbf{y}, t) T.V._{x_i}(w_0)\; d\mathbf{y} = T.V._{x_i}(w_0).\end{aligned}$$

Hence, (2.8) holds. $\square$

Lemma 2.3. *There is a constant C, independent of k, l and Δt, such that*

$$||u^k - u^l||_1 \le C\sqrt{(k-l)\Delta t} \tag{2.9}$$

for all $0 \le l \le k \le N$.

Proof. We begin by showing that the splitting solutions are weakly Lipschitz continuous in the time variable, that is,

$$\left|\int_{\mathbb{R}^m} \phi(\mathbf{x})\big(u^k(\mathbf{x}) - u^l(\mathbf{x})\big)\, d\mathbf{x}\right| \le C_1(k-l)\Delta t\big(||\phi||_\infty + \max_i ||\phi_{x_i}||_\infty\big), \qquad \forall \phi \in C_0^1. \tag{2.10}$$

Using finite speed of propagation, we have that $\mathcal{S}^{f_i}(t)$ (and thus $\mathcal{S}^{\mathbf{f}}(t)$) obey a stronger condition than (2.10), namely

$$\int_{\mathbb{R}} \left|\mathcal{S}^{f_i}(t + \Delta t)v_0(\mathbf{x}) - \mathcal{S}^{f_i}(t)v_0(\mathbf{x})\right|\, dx_i \le C_2\Delta t.$$

From this it follows that

$$\left|\int_{\mathbb{R}^m} \phi(\mathbf{x})\left(\mathcal{S}^{\mathbf{f}}(t + \Delta t)v_0(\mathbf{x}) - \mathcal{S}^{\mathbf{f}}(t)v_0(\mathbf{x})\right)\, d\mathbf{x}\right| \le C_3\Delta t||\phi||_\infty. \tag{2.11}$$

Using the differential equation for $w(\mathbf{x},t)=\mathcal{H}(t)w_0(\mathbf{x})$ and integration by parts, we get the bound

$$\begin{aligned}\left|\int_{\mathbb{R}^m}\phi(\mathbf{x})\big(\mathcal{H}(t+\Delta t)w_0(\mathbf{x})-\mathcal{H}(t)w_0(\mathbf{x})\big)\,d\mathbf{x}\right| &= \left|\sum_{i=1}^{m}\int_{\mathbb{R}^m}\phi(\mathbf{x})\left(\int_t^{t+\Delta t}\varepsilon w(\mathbf{x},\tau)_{x_ix_i}\,d\tau\right)\,d\mathbf{x}\right| \\ &\le C_4\Delta t\max_i\|\phi_{x_i}\|_\infty,\end{aligned}\tag{2.12}$$

where we mention that C_4 depends on $T.V.(w_0)$. Using (2.11) and (2.12), we now readily compute

$$\begin{aligned}\left|\int_{\mathbb{R}^m}\phi(\mathbf{x})\big(u^k(\mathbf{x})-u^l(\mathbf{x})\big)\,d\mathbf{x}\right| &\le \sum_{i=l}^{k-1}\left|\int_{\mathbb{R}^m}\phi(\mathbf{x})\big(u^{i+1}(\mathbf{x})-u^i(\mathbf{x})\big)\,d\mathbf{x}\right| \\ &\le C_1(k-l)\Delta t\big(\|\phi\|_\infty+\max_i\|\phi_{x_i}\|_\infty\big).\end{aligned}$$

Finally, we claim that the estimate (2.9) follows from an approximation argument. To see this, let $\omega_h(\mathbf{x})$ be a standard m-dimensional C_0^1-mollifier with smoothing radius h. Let $d(\mathbf{x})=u^k(\mathbf{x})-u^l(\mathbf{x})$, and define $\beta(\mathbf{x})=\operatorname{sign}(d(\mathbf{x}))$ for $|\mathbf{x}|\le r-h$ and $\beta(\mathbf{x})=0$ for $|\mathbf{x}|>r-h$, where $r>h$. Moreover, define $\beta^h=\omega_h*\beta$ and note that $\beta^h\in C^1$ with support in $\{|\mathbf{x}|\le r\}$. By choosing $\phi=\beta^h$ in (2.10) and recalling several elementary properties of the mollifier function, e.g. $|\left(\beta^h\right)_{x_i}|\le \text{Const.}/h$, it follows that

$$\begin{aligned}\left|\int_{|\mathbf{x}|\le r}\left|u^k(\mathbf{x})-u^l(\mathbf{x})\right|\,d\mathbf{x}\right| &\le \int_{|\mathbf{x}|\le r}\left||d(\mathbf{x})|-\beta^h(\mathbf{x})d(\mathbf{x})\right|\,d\mathbf{x}+\int_{|\mathbf{x}|\le r}\left|\beta^h(\mathbf{x})d(\mathbf{x})\right|\,d\mathbf{x} \\ &\le C_5h+C_6\left((k-l)\,\Delta t\right)/h.\end{aligned}$$

Choosing $h=\sqrt{(k-l)\Delta t}$ and letting $r\to\infty$, we obtain (2.10). This concludes the proof of the lemma. □

In order to investigate the convergence of a sequence of approximate solutions, we need to work with functions that are not only defined for each $t=n\Delta t$, but in the entire interval $[0,T]$. To do this we define

$$\alpha_{j,n}=n+\frac{j}{m+1}$$

for $j=0,\dots,m+1$, and

$$u^{i,n}(\mathbf{x})=\mathcal{S}^{f_i}(\Delta t)\dots\mathcal{S}^{f_1}(\Delta t)u^n(\mathbf{x})$$

for $i=1,\dots,m$. Let now the sequence $\{u_{\Delta t}\}$ be defined by

$$u_{\Delta t}(\mathbf{x},t)=\begin{cases}\mathcal{S}^{f_i}\left((m+1)\left(t-\alpha_{i-1,n}\right)\right)u^{i-1,n}(\mathbf{x}), & \text{for } t\in[\alpha_{i-1,n}\Delta t,\alpha_{i,n}\Delta t\rangle,\\ \mathcal{H}\left((m+1)\left(t-\alpha_{m,n}\right)\right)u^{m,n}(\mathbf{x}), & \text{for } t\in[\alpha_{m,n}\Delta t,\alpha_{m+1,n}\Delta t\rangle.\end{cases}$$

This method of extending $\{u^n\}$ to a function defined for all t was first used by Crandall and Majda in [2]. Regarding $\{u_{\Delta t}\}$ we have the following lemma.

Lemma 2.4. *Let $\{\Delta t\}$ be some sequence tending to zero. Then there exists a subsequence $\{\Delta t_j\}$ and a function u such that as $j\to\infty$, $u_{\Delta t_j}\to u$ in $L_1^{\text{loc}}(\mathbb{R}^m\times[0,T])$.*

Proof. By Lemma 2.1 and 2.2 the sequence $\{u_{\Delta t}(\cdot,t)\}$ is uniformly bounded and has uniformly bounded total variation. Using Helly's theorem we can therefore conclude that a subsequence $\left\{u_{\Delta t_j}(\cdot,t)\right\}$ converges in L_1 on bounded boxes $[-r,r]^m$ for each fixed t. Since r is arbitrary, we can apply this argument a countable number of times to conclude that there is a further subsequence, again denoted by $\left\{u_{\Delta t_j}\right\}$, such that $\left\{u_{\Delta t_j}(\cdot,t)\right\}$ converges in L_1^{loc} for each fixed t. By a diagonalization we can have such a convergence for a dense countable subset $\{t_l\}$ in $[0,T]$. Now, for some $t\notin\{t_l\}$, choose $\{t_k\}\subset\{t_l\}$ such that $t_k\to t$ as $k\to\infty$. We compute

$$\begin{aligned}\int_{[-r,r]^m}\left|u_{\Delta t_i}(\mathbf{x},t)-u_{\Delta t_j}(\mathbf{x},t)\right|\,d\mathbf{x} &\le \int_{[-r,r]^m}\left|u_{\Delta t_i}(\mathbf{x},t)-u_{\Delta t_i}(\mathbf{x},t_k)\right|\,d\mathbf{x} \\ &\quad+\int_{[-r,r]^m}\left|u_{\Delta t_i}(\mathbf{x},t_k)-u_{\Delta t_j}(\mathbf{x},t_k)\right|\,d\mathbf{x} \\ &\quad+\int_{[-r,r]^m}\left|u_{\Delta t_j}(\mathbf{x},t_k)-u_{\Delta t_j}(\mathbf{x},t)\right|\,d\mathbf{x} \\ &=\mathcal{O}\left(\sqrt{|t-t_k|}\right)+\int_{[-r,r]^m}\left|u_{\Delta t_i}(\mathbf{x},t_k)-u_{\Delta t_j}(\mathbf{x},t_k)\right|\,d\mathbf{x},\end{aligned}$$

which vanishes as i, j and k become large. Thus $\left\{u_{\Delta t_j}(\cdot,t)\right\}$ is a Cauchy sequence in L_1^{loc} for all $t\in[0,T]$, and therefore converges. A closer inspection shows that this convergence is uniform in t for $t\in[0,T]$. We denote the limit by u. This concludes the proof of the lemma. □

Next, we justify the term "approximate solution" by showing:

Theorem 2.5. *Assume that u_0 is in $L_\infty \cap B.V.$, and that $\mathbf{f}(u)$ is Lipschitz continuous. Then, for any sequence $\{\Delta t\}$ tending to zero, there exists a subsequence $\{\Delta t_j\}$ such that the corresponding subsequence $\{u_{\Delta t_j}\}$ converges to a solution of the initial value problem*

$$u_t + \nabla \cdot \mathbf{f}(u) = \varepsilon \Delta u, \quad u(\mathbf{x}, 0) = u_0(\mathbf{x}).$$

Remark. With a solution of (1.1) we understand a function $u(\mathbf{x}, t)$ which is twice continuously differentiable in $\mathbf{x}$ and once in t such that the differential equation is satisfied in the classical sense for $t > 0$. Moreover, the initial data should be assumed in the weak sense. It is well-known that such a solution exists, cf. [18], [16], [15].

Proof. From Lemma 2.4 we have convergence of the sequence $\{u_j\}$ $(= \{u_{\Delta t_j}\})$ to u. We continue by showing that u is indeed a weak solution to (1.1). To this end define the functional

$$\mathcal{L}_\phi(u) = \int_{\mathbb{R}^m} \int_0^T (u\phi_t + \mathbf{f}(u) \cdot \nabla\phi + \varepsilon u \Delta\phi) \, dt \, d\mathbf{x} + \int_{\mathbb{R}^m} u_0(\mathbf{x})\phi(\mathbf{x}, 0) \, d\mathbf{x},$$

for $\phi \in C_0^2(\mathbb{R}^m \times [0, T])$. If $\mathcal{L}_\phi(u) = 0$ for all such ϕ, then u is a weak solution. But we have more, for following [18], we have that a weak solution to (1.1) is, in fact, a classical solution possessing the necessary smoothness for $t > 0$. Therefore it suffices to show that $\mathcal{L}_\phi(u) = 0$, where $u = \lim_j u_j$, for a proper set of test functions ϕ.

Let now

$$v_j^{i,n}(t) = \mathcal{S}^{f_i}(t) u_j^{i-1,n}.$$

We also define a new test function $\tilde{\phi}$ by

$$\tilde{\phi}(\mathbf{x}, t) = \phi\left(\mathbf{x}, \frac{t}{m+1}\right).$$

Then we have

$$\begin{aligned}
(2.13) \quad & \int_{\mathbb{R}^m} \int_{\alpha_{i-1,n}\Delta t_j}^{\alpha_{i,n}\Delta t_j} \left(\left(\frac{1}{m+1}\right) u_j \phi_t + f_i(u_j) \phi_{x_i} \right) dt \, d\mathbf{x} = \\
& \frac{1}{m+1} \int_{\mathbb{R}^m} \int_0^{\Delta t_j} \left(v_j^{i,n}(\mathbf{x}, \tau) \tilde{\phi}_\tau(\mathbf{x}, \tau + (m+1)\alpha_{i-1,n}\Delta t_j) + f_i(v_j^{i,n}(\mathbf{x}, \tau)) \tilde{\phi}_{x_i}(\mathbf{x}, \tau + (m+1)\alpha_{i-1,n}\Delta t_j) \right) d\tau \, d\mathbf{x} \\
& = \frac{1}{m+1} \int_{\mathbb{R}^m} \left(u_j(\mathbf{x}, \alpha_{i,n}\Delta t) \phi(\mathbf{x}, \alpha_{i,n}\Delta t_j) - u_j(\mathbf{x}, \alpha_{i-1,n}\Delta t) \phi(\mathbf{x}, \alpha_{i-1,n}\Delta t_j) \right) d\mathbf{x}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(2.14) \quad & \int_{\mathbb{R}^m} \int_{\alpha_{m,n}\Delta t_j}^{\alpha_{m+1,n}\Delta t_j} \left(\left(\frac{1}{m+1}\right) u_j \phi_t + \varepsilon u_j \Delta\phi \right) dt \, d\mathbf{x} = \\
& \frac{1}{m+1} \int_{\mathbb{R}^m} \left(u_j(\mathbf{x}, \alpha_{m+1,n}\Delta t_j) \phi(\mathbf{x}, \alpha_{m+1,n}\Delta t_j) - u_j(\mathbf{x}, \alpha_{m,n}\Delta t_j) \phi(\mathbf{x}, \alpha_{m,n}\Delta t_j) \right) d\mathbf{x}.
\end{aligned}$$

If we sum (2.13) over $i = 1, \dots, m$ and add the result to (2.14), and finally sum over $n = 0, \dots, N-1$, the resulting equality takes the form

$$\mathcal{L}_\phi(u_j) = I_1 + I_2,$$

where

$$I_1 = \varepsilon \sum_{n=0}^{N-1} \int_{\mathbb{R}^m} \left(\int_{n\Delta t_j}^{(n+1)\Delta t_j} u_j \Delta\phi \, dt - (m+1) \int_{\alpha_{m,n}\Delta t_j}^{\alpha_{m+1,n}\Delta t_j} u_j \Delta\phi dt \right) d\mathbf{x},$$

and

$$I_2 = \sum_{n=0}^{N-1} \int_{\mathbb{R}^m} \sum_{i=1}^m \left(\int_{n\Delta t_j}^{(n+1)\Delta t_j} f_i(u_j) \phi_{x_i} \, dt - (m+1) \int_{\alpha_{i-1,n}\Delta t_j}^{\alpha_{i,n}\Delta t_j} f_i(u_j) \phi_{x_i} \, dt \right) d\mathbf{x}.$$

We now write I_1 as $I_1 = a + b$, where a and b are given by

$$\begin{aligned}
a &= \varepsilon \sum_{n=0}^{N-1} \int_{\mathbb{R}^m} \left(\int_{n\Delta t_j}^{(n+1)\Delta t_j} \Delta\phi \, dt - (m+1) \int_{\alpha_{m,n}\Delta t_j}^{\alpha_{m+1,n}\Delta t_j} \Delta\phi \, dt \right) u_j(\mathbf{x}, n\Delta t_j) \, d\mathbf{x}, \\
b &= \varepsilon \sum_{n=0}^{N-1} \int_{\mathbb{R}^m} \left(\int_{n\Delta t_j}^{(n+1)\Delta t_j} (u_j(t) - u_j(n\Delta t_j)) \Delta\phi \, dt - (m+1) \int_{\alpha_{m,n}\Delta t_j}^{\alpha_{m+1,n}\Delta t_j} (u_j(t) - u_j(n\Delta t_j)) \Delta\phi \, dt \right) d\mathbf{x}.
\end{aligned}$$

Assuming for the moment that $\phi \in C_0^3(\mathbb{R}^m \times [0,T])$, we may write

$$\Delta\phi(\mathbf{x},t) = \Delta\phi(\mathbf{x}, n\Delta t_j) + \mathcal{O}(t - n\Delta t_j)$$

for $t > n\Delta t_j$. Using this it is not difficult to see that $|a| = \mathcal{O}(\Delta t_j)$, and using the L_1 continuity in time (Lemma 2.3), we obtain that $|b| = \mathcal{O}(\sqrt{\Delta t_j})$. Hence, $|I_1| = \mathcal{O}(\sqrt{\Delta t_j})$. Similarly, $|I_2| = \mathcal{O}(\sqrt{\Delta t_j})$, and consequently,

$$|\mathcal{L}_\phi(u_j)| = \mathcal{O}(\sqrt{\Delta t_j}).$$

Letting $j \to \infty$, we have that $\mathcal{L}_\phi(u) = 0$ for all $\phi \in C_0^3(\mathbb{R}^m \times [0,T])$. A straightforward approximation argument shows this also to hold for all suitable ϕ. This concludes the proof of Theorem 2.5. $\square$

Remark. Since the solution of (1.1) is unique, it follows that the whole sequence $\{u_{\Delta t}\}$ converges.

3. A discrete method. In this section we present a numerical method implementing the ideas of the operator splitting described earlier. Here the solution operator for the hyperbolic part of the equation is replaced by a solution generated by front tracking, and the solution operator for the diffusion part is replaced by a finite element method.

Front tracking is inherently a numerical method for one-dimensional conservation laws, in several dimensions we use the dimensional splitting described in [12] as the approximation to $\mathcal{S}$. Let $\mathcal{S}_\eta^{\mathbf{f}}(t)$ denote the approximate solution operator associated with (2.1), i.e.,

$$\mathcal{S}_\eta^{\mathbf{f}}(t)v_0(\mathbf{x})$$

is the result of using front tracking, possibly coupled with dimensional splitting, for a single timestep of length t, on the piecewise constant function $v_0(\mathbf{x})$. We have that $\mathcal{S}_\eta^{\mathbf{f}}(t)v_0(\mathbf{x})$ is a piecewise constant function of $\mathbf{x}$ (on a fixed grid if $m > 1$). Here η is a parameter measuring the discretization used in front tracking. If $m = 1$, then $\eta = \delta$ with δ denoting the distance between the interpolation points used when approximating the flux function f with a piecewise linear function. If $m > 1$, then $\eta = (\delta, \Delta\mathbf{x}) = (\delta, \Delta x_1, \dots, \Delta x_m)$, where Δx_i denotes the grid spacing in the ith direction. For a detailed description of the front tracking method for scalar conservation laws in one dimension, we refer the reader to [11], and for a description of the dimensional splitting method to [12].

We shall use a finite element method for the solution of (2.3), with elements determined by the discontinuities in $\mathcal{S}_\eta^{\mathbf{f}}(t)v_0(\mathbf{x})$. Let $\mathcal{H}_{\Delta\mathbf{x}}(t)$ denote the operator which takes an initial function

$$w_0(\mathbf{x}) = \sum_{i=1}^{M} a_i\varphi_i(\mathbf{x})$$

to the approximate solution of (2.3) obtained by the element method using basis functions $\varphi_i(\mathbf{x})$, $i = 1,\dots,M$. We assume that these basis functions are associated with $\Delta\mathbf{x}$ such that $M \to \infty$ as $\Delta\mathbf{x} \to 0$. The approximate solution is then written as

$$\mathcal{H}_{\Delta\mathbf{x}}(t)w_0(\mathbf{x}) = \sum_{i=1}^{M} a_i(t)\varphi_i(\mathbf{x}),$$

where $a_i(t)$ is the solution of the following system of ordinary differential equations:

$$\sum_{i=1}^{M} \frac{da_i(t)}{dt}(\varphi_i, \varphi_j) + \varepsilon \sum_{i=1}^{M} (\nabla\varphi_i, \nabla\varphi_j)\, a_i(t) = 0, \qquad j = 1,\dots,M, \tag{3.0}$$

where $(\cdot,\cdot)$ denotes inner product in L_2. For a description of finite element methods for problems such as (2.3), see [13]. Standard theory for parabolic equations, see e.g. [20, Ch. 10], says that $\mathcal{H}_{\Delta\mathbf{x}}(\cdot)w_0$ is a continuous function of t taking values in L_2. If we use a suitable numerical scheme when integrating (3.0), and if w_0 is bounded, then also $\mathcal{H}_{\Delta\mathbf{x}}(t)w_0$ will be a bounded function for all t, independently of M. We restrict ourselves to some bounded (rectangular) domain Ω in $\mathbb{R}^m$, therefore $\mathcal{H}_{\Delta\mathbf{x}}(\cdot)w_0$ is a continuous function of t taking values in L_1, and we have the following estimate:

$$||\mathcal{H}_{\Delta\mathbf{x}}(t)w_0 - w_0||_1 \le h(t), \tag{3.1}$$

where $h(t)$ is some continuous function with $h(0) = 0$.

As mentioned above, we shall restrict ourselves to a bounded rectangular domain Ω in $\mathbb{R}^m$, $x_i \in [a_i, b_i]$. To do this, we ought to impose boundary values, but for ease of presentation we shall make these as simple as possible. If $m = 1$, we shall require the solution to satisfy

$$u(a,t) = u_a, \quad u(b,t) = u_b, \tag{3.2a}$$

where u_a and u_b are constants. If $m > 1$, we require the solution to satisfy

$$u|_{\partial\Omega} = c, \tag{3.2b}$$

where c is some constant, and we observe that there is no loss of generality in choosing $c = 0$.

Now we can describe the numerical approximation. Let δ and Δx_i, $i = 1, \ldots, m$, be small numbers, and let $u_\eta^0(\mathbf{x})$ be a function which is constant in each grid block,

$$\{\mathbf{x} = (x_1, \ldots, x_m) : a_i + n_i \Delta x_i \le x_i < a_i + (n_i + 1)\, \Delta x_i,\ i = 1, \ldots, m\}$$

where $n_i = 0, \ldots, M_i$, and $a_i + M_i \Delta x_i = b_i$. We choose this function such that

$$\lim_{\eta \to 0} ||u_\eta^0 - u_0||_1 = 0. \tag{3.3}$$

For $t = n\Delta t$, $n = 0, \ldots, N$, we define $u_\eta^n(\mathbf{x})$ as

$$u_\eta^n(\mathbf{x}) = \left[\mathcal{H}_{\Delta\mathbf{x}}(\Delta t)\mathcal{S}_\eta^{\mathbf{f}}(\Delta t)\right]^n u_\eta^0(\mathbf{x}). \tag{3.4}$$

Here, the case $m = 1$ needs special mention. In this case $\mathcal{S}_\eta^{\mathbf{f}}(\Delta t)$ does not give a result that is constant on a fixed grid. Therefore, to ensure convergence of $\mathcal{H}_{\Delta x}$, we have to add grid points whenever the spacing between two discontinuities become larger than $C\Delta x$ for some fixed constant C. In the computations presented in section 4, we use $C = 1$.

By mimicking the proofs of Lemma 2.1 to Lemma 2.3, and using the remarks above, it is straightforward to prove:

Lemma 3.1. *Let $u_\eta^n(\mathbf{x})$ be defined by (3.4), then we have*

$$\begin{aligned} ||u_\eta^n||_\infty &\le M, \\ T.V.\left(u_\eta^n\right) &\le M, \\ ||u_\eta^k - u_\eta^l||_1 &\le M\tilde{h}(|(k-l)\Delta t|), \end{aligned} \tag{3.5}$$

where M is some number independent of n, l, k, η and Δt, and $\tilde{h}(t)$ is a continuous function with $\tilde{h}(0) = 0$.

An immediate consequence of this lemma is that our numerical method, (3.4), converges as η and Δt tend to zero. Furthermore, we can also mimic the arguments used in proving Theorem 2.5 to show:

Theorem 3.2. *Assume that u_0 is in $L_\infty \cap B.V.$, and that $\mathbf{f}(u)$ is Lipschitz continuous. Let $u_{\eta,\Delta t}(\mathbf{x}, t)$ be given by (3.4) for $t = n\Delta t$, and the analog of $u_{\Delta t}(\mathbf{x}, t)$ (the semi-discrete method) for $t \ne n\Delta t$. Then*

$$u(\mathbf{x}, t) = \lim_{\eta, \Delta t \to 0} u_{\eta,\Delta t}(\mathbf{x}, t)$$

is the unique solution to the initial value problem

$$u_t + \nabla \cdot \mathbf{f}(u) = \varepsilon \Delta u, \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}).$$

Remark. In the implementation presented in section 4, we have used Strang splitting

$$u_\eta^n(\mathbf{x}) = \left[\mathcal{H}_{\Delta\mathbf{x}}(\Delta t/2)\,\mathcal{S}_\eta^{\mathbf{f}}(\Delta t)\mathcal{H}_{\Delta\mathbf{x}}(\Delta t/2)\right]^n u_\eta^0(\mathbf{x})$$

instead of (3.4). It is clear that convergence and Theorem 3.2 hold for this construction as well. Furthermore, it should be noted that if $\mathcal{H}_{\Delta x}(t)$ is chosen properly, our discrete method (3.4) is unconditionally stable in the sense that the time step is not limited by the space discretization.

4. Some numerical experiments. In this section we shall present an implementation of the operator splitting algorithm presented in the last section in the case of one and two space dimensions. In particular, we will try to numerically determine a convergence rate of this method. To do this we shall measure the *error* at a fixed time T (in the experiments below we use $T = 1$). Since the convergence results above were formulated in L_1, we measure the error in a relative L_1 - norm, i.e.,

$$E = \frac{||u_\eta^N - u_*(\cdot, T)||_1}{||u_*(\cdot, T)||_1},$$

where u_* is some reference solution. If we have no exact solution available, then we have used a solution calculated with a very fine discretization. We shall in the following adopt the notation $\mathbf{x} = (x, y)$, $\mathbf{f} = (f, g)$, and $\Delta\mathbf{x} = (\Delta x, \Delta y)$.

When trying to determine the convergence rate in the one-dimensional case we assume that the error is of the form

$$E \approx C\left[\Delta x^{\alpha} + \Delta t^{\beta}\right],$$

for some constant C independent of Δx and Δt. In our experiments we perform two kinds of computations: One is to test the spatial convergence rate, where we fix a small Δt and observe the convergence order α with respect to Δx. The other is to test the temporal convergence rate with respect to Δt. Here we fix a small Δx and observe β. We use least-squares fit to obtain the exponents α and β.

In the two-dimensional case we shall assume that the error is of the form $E \approx C\Delta x^{\alpha}$ with $\Delta y = \Delta x$, and relate the time step Δt to space discretization Δx through the CFL number; $(\Delta t/\Delta x)\max(||f'||_{\infty}, ||g'||_{\infty})$. The reason for doing so is that we want to explore, in addition to the convergence rate, large time step behavior of the method. Our method is unconditionally stable in the sense that there is no CFL number restriction on the discretization parameters in order to obtain convergence, therefore it is interesting to see whether the error increases if larger time steps are taken. We will present L_1 - errors and convergence rates for three different CFL numbers, namely, 1, 2 and 4.

In one dimension, front tracking for a scalar conservation law has two discretization parameters; the distance between the (linear) interpolation points used when approximating the flux function, and the initial discretization used when approximating the initial function. Furthermore, the operator splitting needs two additional parameters, namely the timestep Δt and the size of the support of the basis functions. In one dimension we have used piecewise linear basis functions of the sort

$$\varphi_j(x) = \begin{cases} 0 & \text{if } x \leq x_{j-1} \\ 1 & \text{if } x = x_j \\ 0 & \text{if } x \geq x_{j+1}. \end{cases}$$

Here, the numbers $\{x_j\}$ are given as the position of the jth front from the left, or, if the spacing between adjacent fronts is greater than Δx, then $x_{j+1} = x_j + \Delta x$, where Δx is the discretization used when approximating the initial function u_0. That is, we use

$$u_{\eta}^{0}(x) = u_0(a + (j + 1/2)\Delta x) \quad \text{for } x \in \langle a + j\Delta x, a + (j+1)\Delta x]. \tag{4.0}$$

We set the distance between the interpolation points in the flux function to 0.1 for all computations in one dimension and 0.01 for the computations in two dimensions. In two dimensions, the approximation, $u_{\eta}^{0}(x, y)$, of the initial data is given by the two-dimensional analoug of (4.0). Furthermore, the finite element method uses a standard uniform triangulation of Ω, and piecewise linear basis functions given by

$$\varphi_j(N_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

where N_i is some node in the triangulation.

When integrating (3.0) numerically, for stability reasons, we have set the matrix $\{(\phi_i, \phi_j)\}$ equal to the identity matrix, a technique known as *lumping*, see [9]. The integration of (3.0) is done by Euler's backward method. This might seem as a restriction on the order of the splitting method, but as the numerical convergence orders obtained here are all well below 1, we believe this not to be so.

Example 1 (one-dimensional test case). Our first example will be the solution of Burgers' equation in one space dimension,

$$u_t + \left(\frac{1}{2}u^2\right)_x = \varepsilon u_{xx}. \tag{4.1}$$

This equation has a time independent solution given by

$$u(x, t) = -\tanh\left(\frac{x}{2\varepsilon}\right), \tag{4.2}$$

so that it is well suited as a test case.

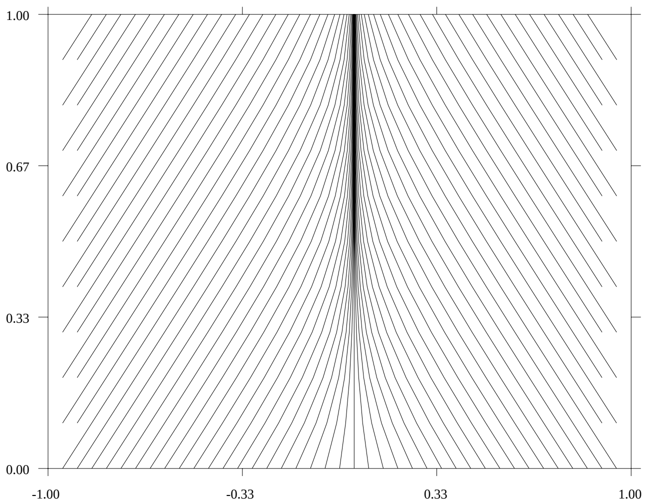

Figure 4.1. Discontinuities in the (x, t) plane.

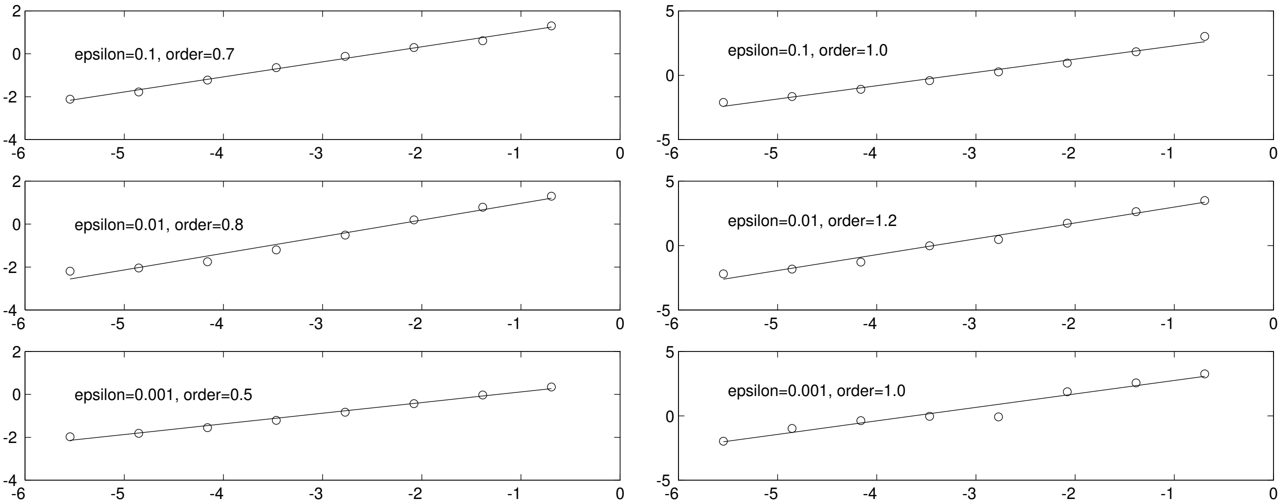

Figure 4.2. Numerical convergence rates in Δt (left) and Δx (right), with u_0 given by (4.2).

In Figure 4.1. we show all discontinuities in the approximated solution in the (x, t) plane for t in the interval $[0, 1]$. This computation was done with $\varepsilon = 0.1$, $\Delta x = 0.05$ and $\Delta t = 0.1$. Note how the discontinuities "converge" towards the steepest place in the exact solution at $x = 0$.

In the left column of Figure 4.2 we show the numerical convergence rates obtained with $\Delta x = 2^{-8}$, with computational domain $[-1, 1]$, and boundary data given by (4.2). These rates were obtained using $\Delta t = 2^{-1}, \ldots, 2^{-8}$. The figure shows the logarithm of the error versus the logarithm of Δt. The straight lines are obtained from standard linear regression. In the right column we show numerical convergence rates in Δx. Here $\Delta t = 2^{-8}$, and $\Delta x = 2^{-1}, \ldots, 2^{-8}$. The three rows of the figure correspond to $\varepsilon = 0.1$, $\varepsilon = 0.01$ and $\varepsilon = 0.001$.

The computations above correspond to the case where the conservation law would have a shock solution, we can do a similar analysis in the case where the conservation law has a rarefaction wave solution. Also in this case, via the Hopf-Cole transform, we have an explicit solution. Now we have initial data

$$u_0(x) = \begin{cases} -1 & \text{for } x \le 0 \\ 1 & \text{for } x > 0, \end{cases} \tag{4.3}$$

and the solution is

$$u(x, t) = \frac{g(-x, t) - g(x, t)}{g(-x, t) + g(x, t)},$$

where g is given by

$$g(x, t) = e^{\frac{t+2x}{4\varepsilon}} \operatorname{erfc}\left(\frac{t + x}{\sqrt{4\varepsilon t}}\right).$$

The computational domain is the interval $[-1, 1]$, and the boundary values are $g(\mp 1, t)$. In Figure 4.3 we show the numerical convergence rates in Δt and Δx for this "rarefaction case". The most notable feature of these computations is the poor convergence rate in Δt. In the "visual norm" the approximate solution is reasonably close to the true solution even for large Δt (see Figure 4.4), so we believe that this poor convergence rate is due to the error made by approximating with a piecewise constant function.

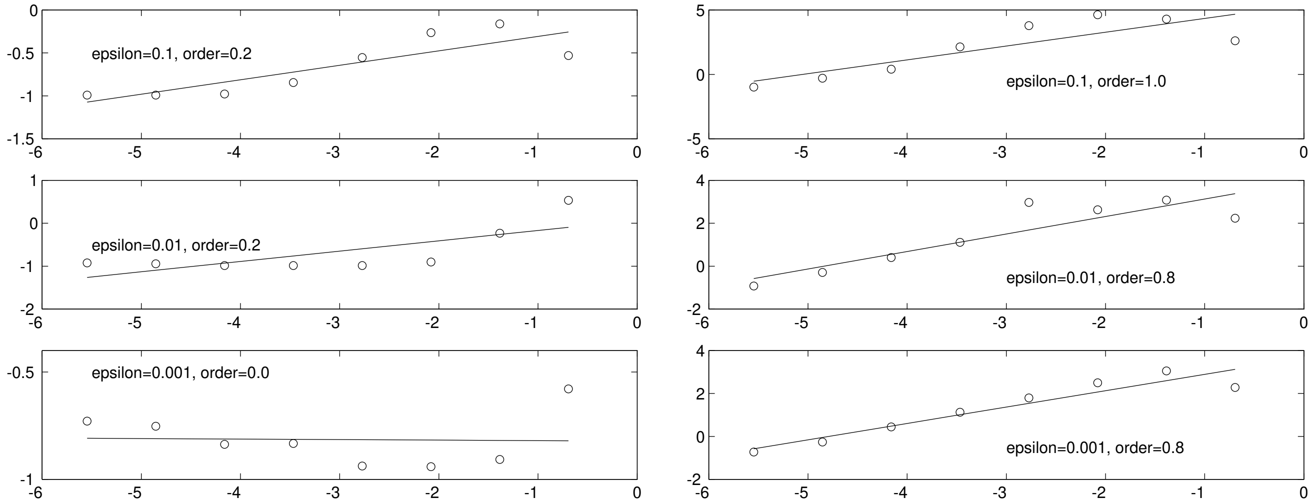

Figure 4.3. Example 1; Numerical convergence rates in Δt (left) and Δx (right), with u_0 given by (4.3)

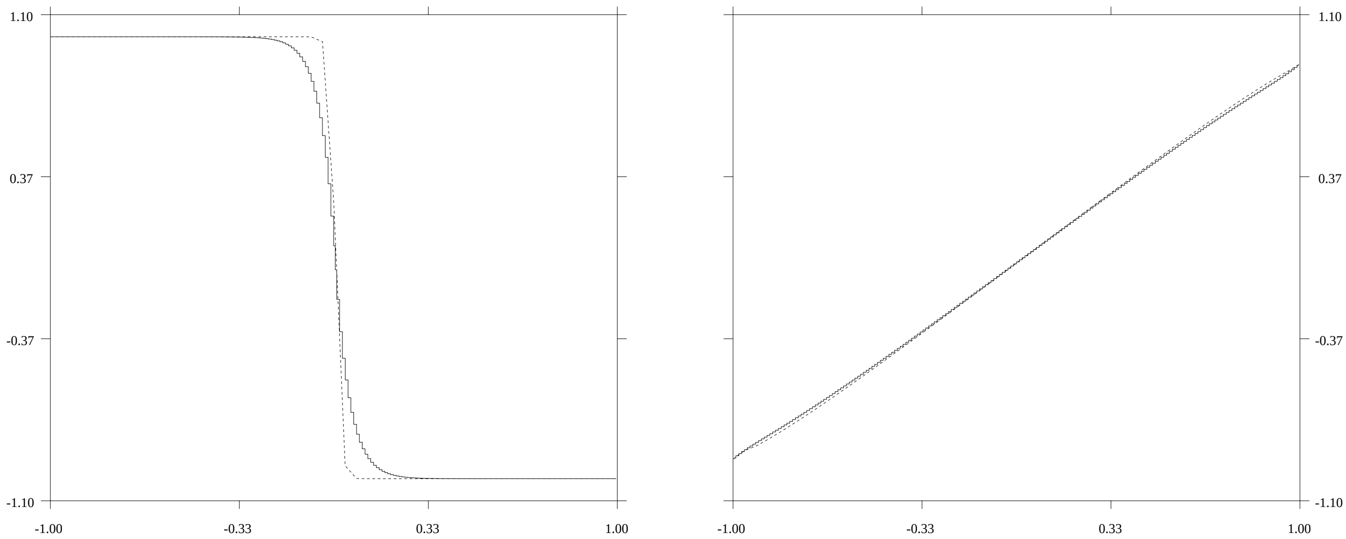

Figure 4.4. Exact solutions (dotted line) versus approximate solutions (piecewise constant) from Example 1. The shock case (left) and the rarefaction case (right) are both calculated with $\Delta x = 0.01$, $\Delta t = 0.5$, and $\varepsilon = 0.01$.

Example 2 (two-dimensional test case). We consider the equation

$$u_t + (u + (u - 0.25)^3)_x - (u + u^2)_y = \varepsilon(u_{xx} + u_{yy}),$$

with initial data given by

$$u_0(x, y) = \begin{cases} 1 & \text{for } (x - 0.25)^2 + (y - 2.25)^2 < 0.5 \\ 0 & \text{otherwise.} \end{cases}$$

The computational domain is $[-2, 5] \times [-2, 5]$, and we calculate approximate solutions for $\Delta x = 7 \cdot 2^{-4}, \dots, 7 \cdot 2^{-8}$. Boundary values are set to zero, which is consistent with the initial data. Figure 4.5 shows "log-log plots" of the error and numerical convergence rates in Δx for $\varepsilon = 0.1$, 0.01 and CFL numbers 1, 2, and 4. See Figure 4.7 for a picture of an approximate solution. We observe that the error and the convergence rates are more or less

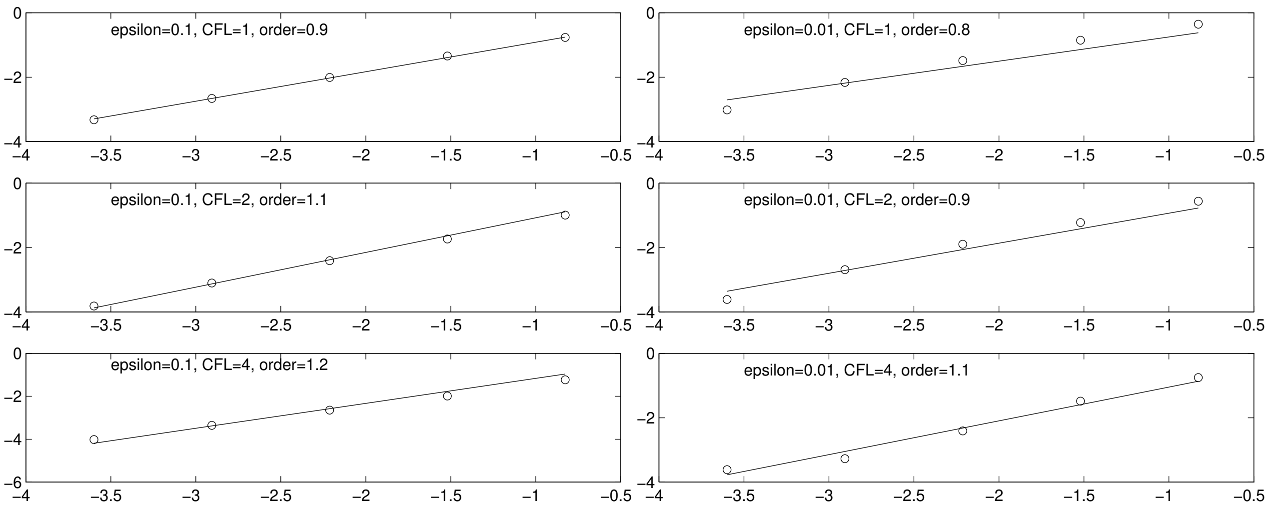

Figure 4.5. Example 2; Numerical convergence rates in Δx for different values of the CFL number.

independent of the choice of the CFL number, and, consequently, large time steps can be taken without loosing accuracy.

Example 3 (two-dimensional test case). We now present an example where we generate approximate solutions to the equation

$$u_t + f(u)_x + g(u)_y = \varepsilon(u_{xx} + u_{yy}),$$

with initial data

$$u_0(x, y) = \begin{cases} 1 & \text{for } x^2 + y^2 < 0.5 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(u) = \frac{u^2}{u^2 + (1-u)^2},$$
$$f(u) = g(u)(1 - 5(1-u)^2).$$

The flux functions f and g are both “S-shaped” with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. This problem is motivated from two-phase flow in porous media with a gravitation pull in the x-direction. The computational domain is $[-3, 3] \times [-3, 3]$, and approximate solutions are computed for $\Delta x = 6 \cdot 2^{-4}, \dots, 6 \cdot 2^{-8}$. Boundary values are again put equal to zero. We use the same values as in the previous example for the viscosity constant ε and the CFL number. The results are presented in Figure 4.6 (see also Figure 4.7), and we observe again that the accuracy is uncorrelated to the CFL number. Finally, let us also mention (without “log-log plots”) that the convergence rate in Δt for both the two-dimensional examples seems to be well below 1, which is consistent with the observations in Example 1.

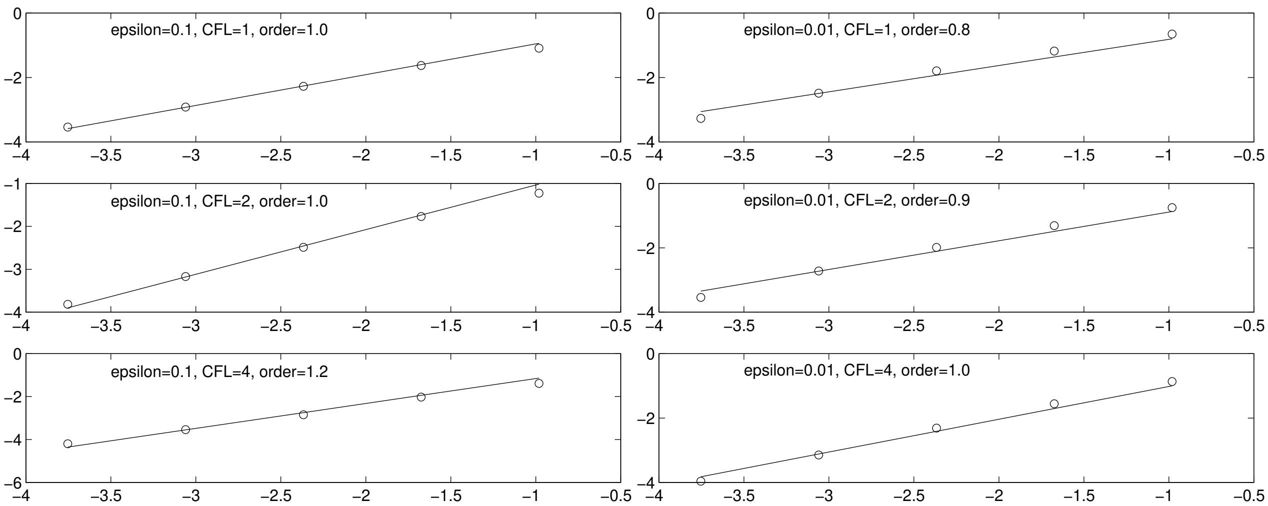

Figure 4.6. Example 3; Numerical convergence rates in Δx for different values of the CFL number.

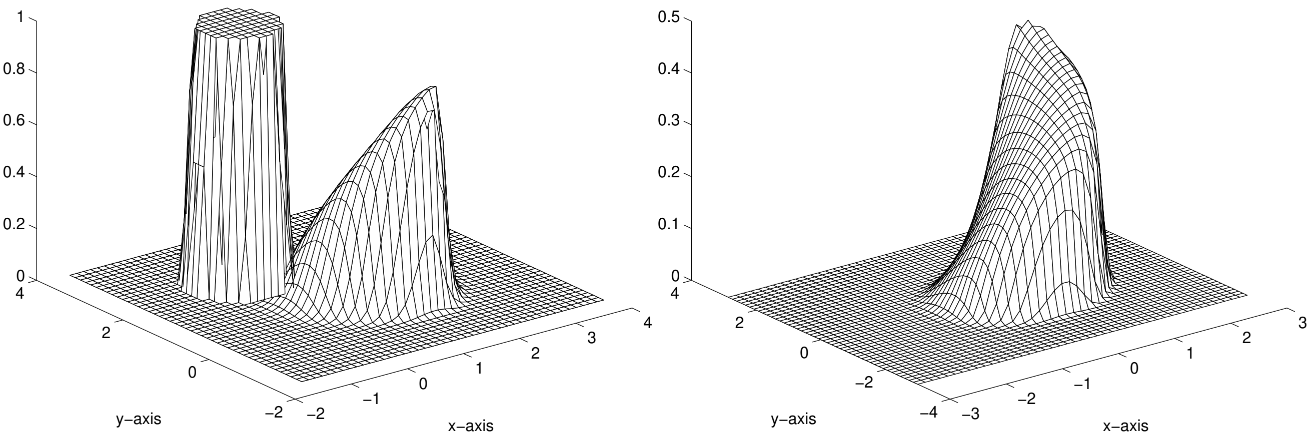

Figure 4.7. Left: Approximate solution and initial data taken from Example 2. Right: Approximate solution from Example 3. The solutions are both computed on a 45×45 grid, with 10 time steps, and $\varepsilon = 0.01$.

In closing, we would like to mention that we plan to continue the study of convergence rates, this time in a more theoretical setting, in a forthcoming paper.

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