CONSERVATION LAWS WITH A RANDOM SOURCE

Helge Holden, Nils Henrik Risebro

Abstract. We study the scalar conservation law with a noisy non-linear source, viz., $u_t + f(u)_x = h(u, x, t) + g(u)W(t)$, where $W(t)$ is white noise in the time variable, and we analyse the Cauchy problem for this equation where the initial data are assumed to be deterministic. A method is proposed to construct approximate weak solutions, and we then show that this yields a convergent sequence. This sequence converges to a (pathwise) solution of the Cauchy problem. The equation can be considered as a model of deterministic driven phase transitions with a random perturbation in a system of two constituents. Finally we show some numerical results motivated by two-phase flow in porous media.

0. Introduction. There has recently been a considerable increase in the interest in stochastic partial differential equations where one either modifies well-known deterministic partial differential equations by stochastic perturbations or consider random initial data, or both. The motivation for this comes from physical considerations as well as purely mathematical interest.

An important example is the Burgers' equation in various stochastic settings. In fact, Burgers studied heuristically the equation

$$u_t + uu_x = \epsilon u_{xx}, \quad u(x, 0) = u_0(x) \tag{0.1}$$

with white noise initial data as a model of turbulence [5]. Burgers equation (0.1) has over the years been applied to many physical problems. We here mention shortly large scale structures of the universe, [29], dynamics of growing interfaces (adding a force term in (0.1)), [18], [26], transmission of neural signals in squid, [27], chemical oscillations [23], and acoustic turbulence, [10]. We refer to [9] for further discussion of physical applications. The analysis of the stochastic Burgers equation ($\epsilon > 0$) is based on the Forsyth–Florin–Hopf–Cole transformation

$$u = -2\epsilon\frac{\theta_x}{\theta} \tag{0.2}$$

that linearizes (0.1) into the heat equation

$$\theta_t = \theta_{xx}. \tag{0.3}$$

Solving the heat equation explicitly and transforming back, one in fact obtains an explicit solution of Burgers equation (0.1).

Sinai [31] has studied the inviscid Burgers equation (i.e., $\epsilon = 0$) with Brownian motion initial data (see also She, Aurell, and Frisch [30] for numerical results), proving rather detailed results for the solution. Furthermore Sinai has studied asymptotic properties of the the viscous Burgers equation with a random force and/or random initial data, [32], proving that for large t the solution is largely independent of the initial data. Asymptotic properties of Burgers equation with random initial data are also discussed in [4]. For other related results, see [19, 3, 7, 21].

In addition, Holden, Lindstrøm, Øksendal, Ubøe, and Zhang [13,14] and Bertini, Cancrini, and Jona-Lasinio [2] analyze the viscous Burgers equation with a white noise force. Nakazawa [28] studies the inviscid Burgers equation with a random force as a limit $\epsilon \to 0$ in the viscous Burgers' equation. Bulinskii

1991 *Mathematics Subject Classification.* 35L65.

Key words and phrases. Conservation laws, stochastic partial differential equations, phase transitions.

The research has been supported by VISTA (a research cooperation between The Norwegian Academy of Science and Letters and Den norske stats oljeselskap, Statoil), and NAVF (the Norwegian Research Council for Science and the Humanities).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

and Molchanov [4] discuss the asymptotics of Burgers' equation with random initial data. Holden and Risebro [15, 16] analyze a general scalar conservation law in one-dimension with a random flux function and random Riemann initial data.

In an approach closer to the present, Kunita [22] discusses parabolic first order stochastic partial differential equations of the form

$$u_t = F_0(x, t, u, \nabla u) + F(x, t, u, \nabla u) \circ dW \tag{0.4}$$

where u is a vector valued function in several space dimensions. Using stochastic characteristics, Kunita proves existence and uniqueness of a local solution for a class of functions F_0, F and a class of initial data. For semi-linear problems existence of global solutions is established. We prove existence of global solutions for quasilinear scalar equations on the line.

In this paper we take certain aspects of the above analysis further. We study a general conservation law with a random source, viz.

$$u_t + f(u)_x = h(x, t, u) + g(u)W(t), \quad u(x, 0) = u_0(x) \tag{0.5}$$

where f is a smooth function with a finite number of inflection points, and h and g are Lipschitz functions, g with bounded support, and W is white noise in the time variable. Our methods are substantially different from the methods previously used. We will not use any analogue of the transformation (0.2), which is only known in the case of Burgers equations, i.e. with a quadratic f. Our analysis is based on an operator splitting technique. This technique was introduced by Godunov in gas dynamics [8], and has been extended by several authors (see e.g. Crandall and Majda [6], and Holden and Risebro [17]) as a method to analyze multidimensional problems.

The operator splitting approach has also been studied by Bensoussan *et al* [1] for a class of partial differential equations having a stochastic source term. In this work, convergence of the operator splitting method was established for the case where the "deterministic operator" has certain continuity properties, these properties do not hold for (0.5).

Here we use the method to split off the effect of the source term.

To describe our method in more detail, let $u(x, t) = S(t)u_0(x)$ denote the solution of

$$u_t + f(u)_x = 0, \tag{0.6}$$

advancing the initial data a time period t. Let furthermore $u(x, t) = R(t, s : x)u(x, s)$ denote the solution of

$$u_t = h(x, t, u) + g(u)W(t), \tag{0.7}$$

when we take the initial data at time s is advanced to time t. Let $\Delta t > 0$, and $t = n\Delta t$. Then we prove that in a certain sense

$$u(x, t) \approx \prod_{k=1}^{n} [S(\Delta t)R(k\Delta t, (k-1)\Delta t : x)]u_0(x) \tag{0.8}$$

For $t \in \langle n\Delta t, (n+1)\Delta t \rangle$ we use R to determine the solution, using as initial data the above approximation (0.8) at $t = n\Delta t$.

We have previously analyzed the situation where the flux function f or the Riemann initial data are random [15, 16].

We consider the equation (0.5) as a model of phase transitions for flow in porous media: Assume that we have a system consisting of two phases, denoted by A and B, with a possibility of a phase transition, or chemical reaction, between them. Let u and $v(= 1 - u)$ denote the respective saturations of A and B, i.e., the fraction of the available pore volume occupied by each of the phases. Conservation of mass is then described by (0.5), where the right hand side describes the reaction or phase transition between A and B. In many applications the chemical rules governing such phase transitions cannot be worked out from first principles, or the conditions under which the reaction takes place are largely unknown, so that it is natural to model the reaction terms with a noisy component.

1. Construction of approximate solutions. For simplicity we will in the following assume that the deterministic drift term h is zero, and then towards the end of this section briefly discuss the case with a non-zero drift term. Consider therefore the following conservation law with a random source term

$$\begin{aligned} u_t + f(u)_x &= g(u)W(t) \\ u(x,0) &= u_0(x). \end{aligned} \tag{1.1}$$

Here the unknown u is a scalar stochastic variable, $f(u)$ and $g(u)$ are uniformly Lipschitz continuous functions, and g has bounded support, $W(t)$ is white noise in the time variable t. The deterministic initial function $u_0(x)$ is assumed to be in $BV \cap L_1$, and in addition we assume that it has a finite number of extrema. The equation (1.1) is to be interpreted in the distributional sense, i.e., a solution $u(x,t;\omega)$ is to satisfy

$$\int_{-\infty}^{\infty}\int_0^{\infty}[u\phi_t + f(u)\phi_x]dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)dx = -\int_{-\infty}^{\infty}\int_0^{\infty}\phi(x,t)g(u)dB(t)dx \tag{1.2}$$

for all test functions ϕ in C_0^1. Here $B(t)$ is Brownian motion in time, and the stochastic integral is to be interpreted in the Ito sense.

We will now explain a method for construction of approximate solutions to (1.1) or (1.2), and then show that this method almost surely yields a compact sequence. Then we justify the term 'approximate solution' by showing that this sequence almost surely tends to a weak solution ω-wise. Let $R(t,s)$ formally denote the operator which takes a number $\bar{u}$ to the solution of the ordinary stochastic differential equation

$$du(\tau) = g(u)dB(\tau), \quad u|_{t=s} = \bar{u}, \tag{1.3}$$

i.e., $u(t) = R(t,s)\bar{u} = \bar{u} + \int_s^t g(u(\tau))dB(\tau)$. We denote by $S(t-s)$ the operator which takes an initial function $u(x,s)$ at time s to the weak entropy solution of the conservation law

$$u_t + f(u)_x = 0, \quad u|_{t=s} = u(x,s) \tag{1.4}$$

at time t, i.e. $u(x,t) = S(t-s)u(x,s)$. Note that $S(t)$ maps L_1 into BV. We now define the sequence of approximate solutions. Let Δt be some small positive number, and divide the $x-t$ plane into strips $[n\Delta t, (n+1)\Delta t]$, $n = 0,1,2,\ldots$. Let the function $u_{\Delta t}(x,t)$ be defined by

$$u_{\Delta t}(x,t) = \begin{cases} u^n(x) & \text{for } t = n\Delta t, \\ R(t, n\Delta t)u^n(x) & \text{for } t \in \langle n\Delta t, (n+1)\Delta t\rangle, \end{cases} \tag{1.5}$$

where the sequence $u^n(x)$ is defined by

$$\begin{aligned} u^0(x) &= u_0(x) \\ u^{n+1}(x) &= S(\Delta t)R\left((n+1)\Delta t, n\Delta t\right)u^n(x). \end{aligned} \tag{1.6}$$

Lemma 1.1. *There is a constant M_1 independent of n, Δt and ω such that*

$$||u_{\Delta t}||_\infty \le M_1. \tag{1.7}$$

Proof. From general theory of scalar conservation laws we know that $S(t)$ obeys the maximum principle:

$$||S(t)v||_\infty \le ||v||_\infty \tag{1.8}$$

for any v in $L_1 \cap L_\infty$. Let u_-^n denote $\lim_{t\nearrow n\Delta t}[u_{\Delta t}(x,t)]$. Then (1.8) implies that $||u^{n+1}||_\infty \le ||u_-^{n+1}||_\infty$. Since g has compact support, i.e. $\text{supp}(g) \subset [-M,M]$ for some number M, $R(t,s)$ will take $[-M,M]$ into itself and be the identity outside this interval. Therefore $||u^{n+1}||_\infty \le \max\{M, ||u^n||_\infty\}$. □

Lemma 1.2. *Assume that u_0 is of bounded variation and is such that it has finitely many maxima and minima and has bounded support. Then there is a constant M_2 independent of Δt and t such that*

$$T.V._x\left[u_{\Delta t}(x,t)\right] \le M_2 \tag{1.9}$$

Proof. Assume that u^n has bounded support and finitely many extrema and is of bounded variation. It is therefore monotone on a finite number of intervals; N_n. Furthermore $T.V._x(u^n) \le 2M_1N_n$. Now u_-^{n+1} will almost surely also be monotone on the same intervals, and with the same bound, M, on each interval. This is so by the uniqueness of $R(t,s)u$, i.e., if $a < b$ then $R(t,s)a < R(t,s)b$ almost surely. From general theory for scalar conservation laws, u^{n+1} will be monotone on N_{n+1} intervals, where $N_{n+1} \le N_n$. Thus $T.V.u^{n+1} \le 2M_1N_{n+1} \le 2M_1N_0$. □

Lemma 1.3. *There is a constant M_3, independent of Δt, and t_1 and t_2, $0 \le t_1 \le t_2$, such that*

$$\begin{aligned}\int_{\mathbb{R}} \left|u_{\Delta t}(x,t_1) - u_{\Delta t}(x,t_2)\right| \, dx \le \\ M_3\left|t_1 - t_2\right| + O(\Delta t) + I(\lceil t_1/\Delta t\rceil \Delta t, \lfloor t_2/\Delta t\rfloor \Delta t) + \\ \int_{\mathbb{R}} \left|\int_{t_1}^{\lceil t_1/\Delta t\rceil \Delta t} g\left(u_{\Delta t}(x,s)\right) \, dB(s)\right| \, dx + \int_{\mathbb{R}} \left|\int_{\lfloor t_2/\Delta t\rfloor \Delta t}^{t_2} g\left(u_{\Delta t}(x,s)\right) \, dB(s)\right| \, dx,\end{aligned} \tag{1.10}$$

where

$$I(n,m) = \sum_{i=n}^{m} \int_{\mathbb{R}} \left|\int_{i\Delta t}^{(i+1)\Delta t} g\left(u_{\Delta t}(x,s)\right) \, dB(s)\right| \, dx, \tag{1.11}$$

where $\lfloor t\rfloor$ is the greatest integer smaller or equal to t, and similarly $\lceil s\rceil$ is the smallest integer greater or equal to s.

Remark: Note that as both Δt and $|t_1 - t_2|$ tend to zero, all terms on the right hand side of (1.10) tend to zero almost surely.

Proof. The left hand side of (1.10) is less than or equal to

$$\begin{aligned}\int_{\mathbb{R}} \left|\int_{t_1}^{\lceil t_1/\Delta t\rceil \Delta t} g\left(u_{\Delta t}(x,s)\right) \, dB(s)\right| \, dx \\ + \int_{\mathbb{R}} \left|\int_{\lfloor t_2/\Delta t\rfloor \Delta t}^{t_2} g\left(u_{\Delta t}(x,s)\right) \, dB(s)\right| \, dx + \left|\left|u^n - u^m\right|\right|_1\end{aligned} \tag{1.12}$$

where $m = \lceil t_1/\Delta t\rceil \Delta t$ and $n = \lfloor t_2/\Delta t\rfloor \Delta t$. Thus we must show

$$\left|\left|u^m - u^n\right|\right|_1 \le M(t_2 - t_1) + I(n,m) + O(\Delta t). \tag{1.13}$$

We will show (1.13) for $m = n+1$, the general case will then follow by induction.

$$\begin{aligned}\left|\left|u^{n+1} - u^n\right|\right|_1 &\le \left|\left|u^{n+1} - u_-^{n+1}\right|\right|_1 + \left|\left|u_-^{n+1} - u^n\right|\right|_1 \\ &\le \Delta t M + \left|\left|R((n+1)\Delta t, n\Delta t)\, u^n - u^n\right|\right|_1 \\ &= M\Delta t + I(n, n+1)\end{aligned} \tag{1.14}$$

The middle inequality comes from general theory of hyperbolic conservation laws: If $v(x,t)$ is a weak solution of

$$u_t + f(u)_x = 0 \qquad v(x,0) = v_0(x), \tag{1.15}$$

then we always have

$$\left|\left|v(\cdot,t_1) - v(\cdot,t_2)\right|\right|_{L_1(\mathbb{R},dx)} \le O(|t_1 - t_2|). \tag{1.16}$$

Consequently the lemma holds. □

Lemma 1.4. *Let $\{\Delta t_j\}$ be some sequence with $\Delta t_j \to 0$ as $j \to \infty$. Then as $j \to \infty$, almost surely a subsequence of $\{u_{\Delta t_j}\}$ converges in $L_1(\mathbb{R} \times [0,\infty\rangle, dx\,dt)$, uniformly in $L_1(\mathbb{R}, dx)$ for each t.*

Proof. To simplify the notation let u_j denote $u_{\Delta t_j}$. Lemmas 1 and 2 allows us to define a convergent subsequence for each t_i, almost surely, in some dense countable set $\{t_i\}$ in $[0,T]$. This is standard procedure for conservation laws and is a direct consequence of Helly's theorem, see e.g. [33] for a detailed treatment. Assume that t is not in $\{t_i\}$. Let $\{t_k\}$ be some sequence in $\{t_i\}$ tending to t as $k \to \infty$. Then we have

$$\begin{aligned} ||u_n(\cdot,t) - u_m(\cdot,t)||_1 \le {} & ||u_n(\cdot,t) - u_n(\cdot,t_k)||_1 + \\ & ||u_n(\cdot,t_k) - u_m(\cdot,t_k)||_1 + ||u_m(\cdot,t_k) - u_m(\cdot,t)||_1. \end{aligned} \tag{1.17}$$

Choosing m and n large, i.e. choosing Δt small, we ensure that the middle term is small. Since the right hand side of (1.10) vanishes as Δt and $|t_1 - t_2|$ become small, we can choose both m, n, and k large, to make the first and last terms on the right hand side of (1.17) small. Thus we find that this sequence also converges for t not in the set $\{t_i\}$. We denote the limit by $u(x,t)$. □

Lemma 1.5. *The limit $u(x,t;\omega)$ is a weak solution to (1.1) for each ω.*

Proof. We have to show that u satisfies (1.2). Let $\tilde{u}$ denote the solution of the conservation law without the source term, and with initial value $R((n+1)\Delta t, n\Delta t)\, u^n(x)$, i.e.,

$$\tilde{u}(x,t) = S(t - n\Delta t) R((n+1)\Delta t, n\Delta t)\, u^n(x), \qquad t \ge n\Delta t. \tag{1.18}$$

We then know that $\tilde{u}$ satisfies the following Lipschitz inequality

$$\begin{aligned} \int_{\mathbb{R}} |\tilde{u}(x,t) - u^{n+1}(x)| dx &= \left|\left| S(t - n\Delta t) u_-^{n+1} - S(\Delta t) u_-^{n+1} \right|\right|_1 \\ &\le M((n+1)\Delta t - t) \end{aligned} \tag{1.19}$$

for some constant M. Furthermore, since $\tilde{u}$ is a weak solution of the conservation law we have

$$\begin{aligned} \int_{\mathbb{R}} \int_{n\Delta t}^{(n+1)\Delta t} [\tilde{u}\phi_t + f(\tilde{u})\phi_x] dx dt - \int_R u^{n+1} \phi\, dx + & \\ \int_{\mathbb{R}} R((n+1)\Delta t, n\Delta t)\, u^n(x) \phi\, dx &= 0. \end{aligned} \tag{1.20}$$

We can approximate the first integral in (1.20) by

$$\int_{\mathbb{R}} \int_{n\Delta t}^{(n+1)\Delta t} [u^{n+1}\phi_t + f\left(u^{n+1}\right)\phi_x] dx dt \tag{1.21}$$

making an error only of order $(\Delta t)^2$. This is due to the Lipschitz property (1.19). Inserting (1.3) and (1.21) in (1.20), and summing for all n less than or equal to N, such that $N\Delta t = T$ and $\mathrm{supp}(\phi)$ is contained in $t < T$, we find

$$\begin{aligned} \sum_{n=0}^{N} \int_{\mathbb{R}} \int_{n\Delta t}^{(n+1)\Delta t} [u_{\Delta t}(x,(n+1)\Delta t)\phi_t + f\left(u_{\Delta t}(x,(n+1)\Delta t)\right)\phi_x] dx dt + & \\ \int_0^T \int_{\mathbb{R}} g\left(u_{\Delta t}(x,t)\right) \phi\, dx dB(t) + \int_{\mathbb{R}} u_0 \phi(x,0) dx &= T \cdot O(\Delta t). \end{aligned} \tag{1.22}$$

The bounded convergence theorem implies that the first term of (1.22) will converge to

$$\int_0^\infty \int_{\mathbb{R}} [u\phi_t + f(u)\phi_x]\, dx dt. \tag{1.23}$$

Since $\int_{\mathbb{R}} u_{\Delta t}\, dx$ is bounded as a function of t, and tends to $\int_{\mathbb{R}} u\, dx$ for almost all t, we have that $\int_{\mathbb{R}} g\left(u_{\Delta t}\right)\, dx$ will tend to $\int_{\mathbb{R}} g\left(u\right)\, dx$ in $L_2[0,T]$. By the Ito isometry, this implies that, as $\Delta t \to 0$, the stochastic integral in (1.22) will converge to

$$\int_0^\infty \int_{\mathbb{R}} g\left(u(x,t)\right) \phi \, dx \, dB(t), \tag{1.24}$$

thereby making u a weak solution. □

Although we have set $h = 0$ in the proof of Lemmas 1.1 — 1.3, it is not difficult to extend the proofs given to the case where h is a function of u, x and t which is uniformly Lipschitz in all its variables, and has bounded support in u. Indeed the proofs of Lemma 1.1 and of Lemma 1.3 carry over literally to cover this case, and Lemma 1.2 can be extended to this case via a Gronwall type inequality. Having shown these three lemmas existence of a weak solution limit will follow as above. Therefore we have proved the following theorem:

Theorem. *Let $f(u)$ and $g(u)$ be real Lipschitz functions, g with bounded support, and let $h(x,t,u)$ be a Lipschitz function which has bounded support in u. Let $u_0(x)$ be a (deterministic) essentially bounded function with bounded support which has finitely many extrema. Then, almost surely, there exists a weak solution to the stochastic conservation law:*

$$\begin{gathered} u_t + f(u)_x = h(x,t,u) + g(u)W(t) \\ u(x,0;\omega) = u_0(x), \end{gathered} \tag{1.25}$$

where $W(t)$ is the white noise process, and the right hand side is to be interpreted in the Ito sense.

2. Numerical examples. We now present an example where our ideas are applied to a problem motivated by oil recovery processes. We assume that we have a one dimensional oil reservoir where a slug of gas is located in the middle of the reservoir. This slug is surrounded by oil. For some reason one wants to extract this gas. This is done by injecting oil at the left end of the reservoir and extracting oil, and later on gas, at the right end. We assume that the oil has some natural tendency to change into gas, the rate of this reaction is proportional to the product of the oil and gas. The reaction rate is however, uncertain, and we model this by adding a noise term. The resulting equation is solved by an operator splitting method similar to the one used in the previous section.

We let u denote the gas saturation, i.e. the fraction of the available pore volume occupied by gas. We assume that only gas and oil are present so that the oil saturation is given by $1-u$. As flux function f we use for simplicity

$$f(u) = \frac{u^2}{u^2 + (1-u)^2}, \tag{2.1}$$

this type of flux function has the correct s-shape and is often used for model studies. We set the reaction terms to be

$$h(x,t,u) = h(u) = k_1 u(1-u), \qquad \text{and} \qquad g(u) = k_2 u(1-u)\left(u - 1/2\right). \tag{2.2}$$

The initial saturation is given by

$$u_0(x) = \begin{cases} 0 & \text{for } x \le -\frac{1}{2} \\ 2x+1 & \text{for } -\frac{1}{2} < x \le 0 \\ -2x+1 & \text{for } 0 < 0 \le \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < x, \end{cases} \tag{2.3}$$

cf. figure 2.1.

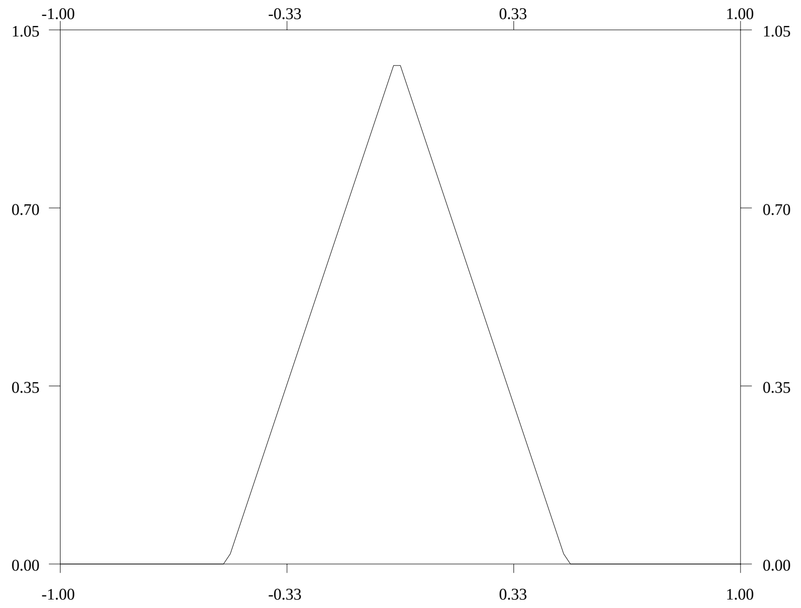

Figure 2.1. The initial function u_0.

In order to apply the operator splitting technique we need numerical methods for the conservation law. We have used two methods: the Lax–Friedrichs method and a front tracking method. A description of these methods would be outside the scope of the present work, we refer the reader instead to [24] for information on difference methods in general and Lax–Friedrichs method in particular, and to [25, 12, 11] for a description of the front tracking method. We also need to integrate the stochastic differential equation, we have used the Euler $\sqrt{\Delta t}$ order method, [20].

In figure 2.2 we show the solution where we have no phase transitions, i.e. both k_1 and k_2 are zero. Figure 2.2a shows the solution in the $x-t$ plane, the thin lines shown can be thought of as characteristics, although they are only approximations to these. For a complete discussion of the front tracking technique applied to scalar conservation laws we again refer the reader to [12, 11]. Figure 2.2b shows the solution at $t = 0.4$.

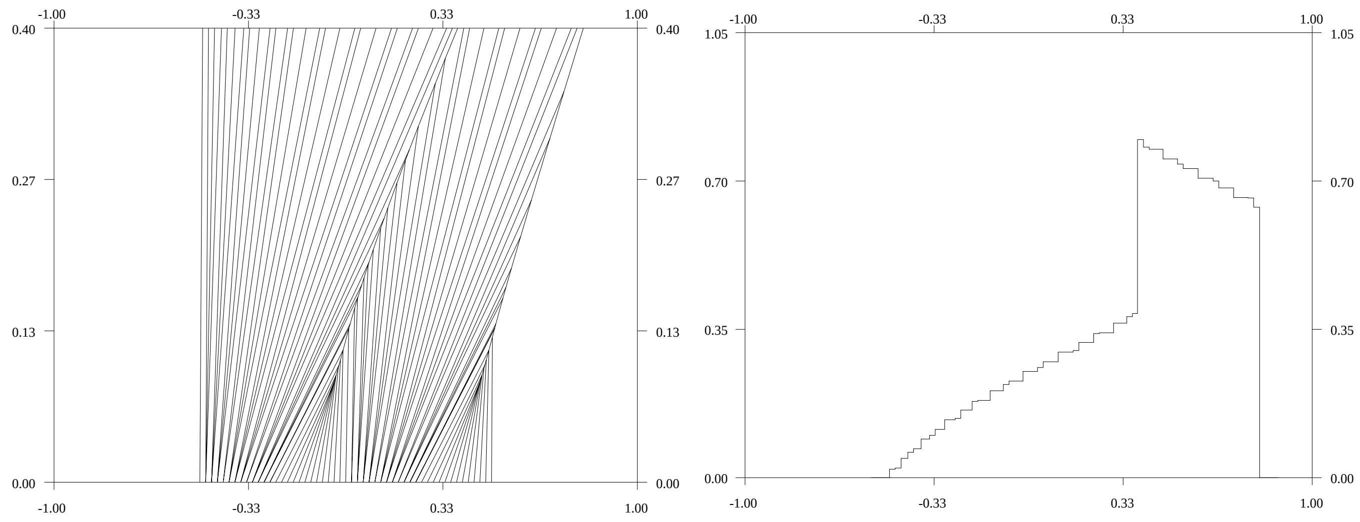

Figure 2.2a (left) and figure 2.2b (right).

Figure 2.3 depicts the situation with only deterministic phase transfer, i.e., $k_2 = 0$, and $k_1 = 10$. Figures 2.3a and 2.3b are analogous to 2.2a and 2.2b. Note how characteristics now tend to “multiply” due to phase transfer, and how much of the oil has turned into gas at $t = 0.4$.

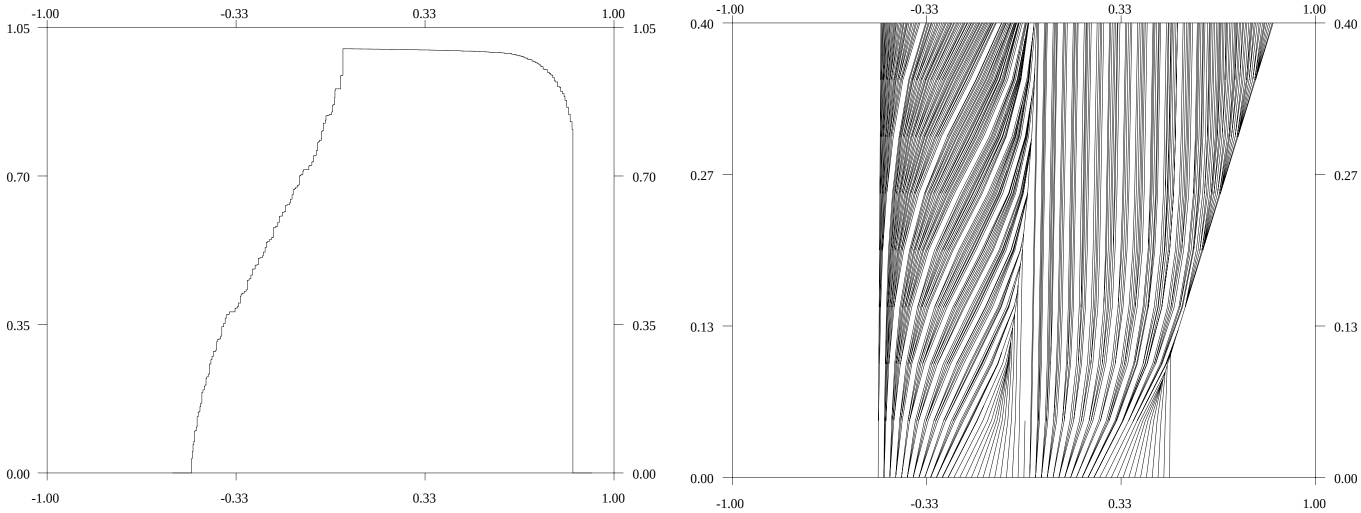

Figure 2.3a (left) and figure 2.3b (right).

In the next set of figures we show a realization of the case where we have only stochastic phase transitions. We see that the characteristics and shock paths now exhibit wiggly behavior as expected. Here $k_2 = 10$. Figure 2.4.

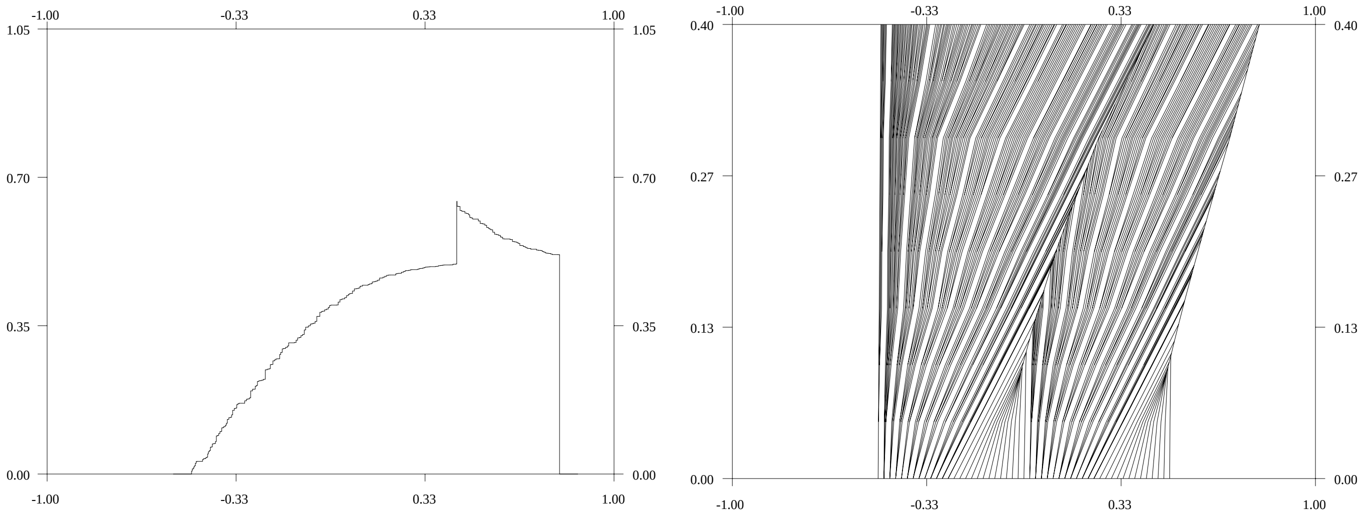

Figure 2.4a (left) and figure 2.4b (right).

Finally we show a realization where we have both deterministic and stochastic transitions, $k_1 = k_2 = 10$. Figure 2.5a-b.

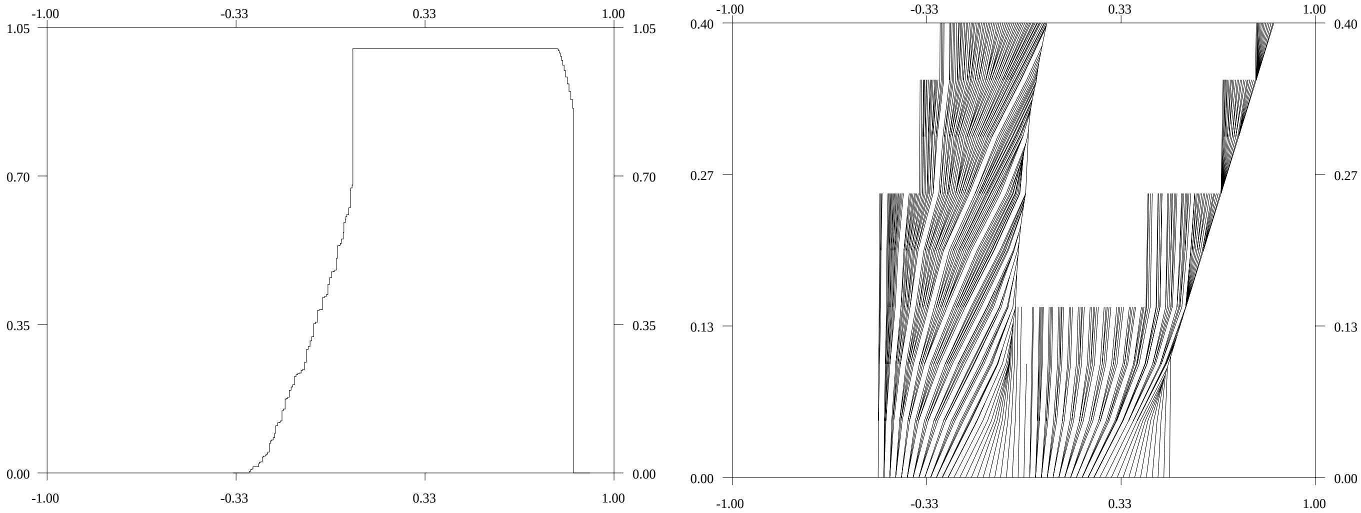

Figure 2.5a (left) and figure 2.5b (right).

It is interesting to investigate the relationship between the expected solution of the conservation law with the noisy source, and the conservation law with the expected noise, i.e., no noise. Since both the conservation law without source and the ordinary stochastic differential equation are nonlinear, it is not to be expected (sic!) that the expected solution is the solution of the deterministic equation with the expected initial data. See Holden, Risebro [15] for a partial discussion of conservation laws with stochastic initial data in connection with oil recovery problems. We will here discuss this important issue only numerically. We used the same equation and initial values as before, but with only stochastic phase transfer, $k_1 = 0$ and $k_2 = 10$. Now we used the Lax-Friedrichs scheme to solve the conservation law, and again the Euler $\sqrt{\Delta t}$ order method to solve the stochastic equation. On the interval $[-1.0, 1.0]$ we used 200 mesh points, i.e., $\Delta x = 0.01$ and $\Delta t = 0.005$. In figure 2.6 we see the solution of the conservation law (without noise) as the thick gray line. The dotted line is the average of 200 realizations, and the thin solid line shows the standard deviation multiplied by 10. Although the average is close to the solution without noise, it is not equal to it, and further numerical computations indicate that it does not converge as the number of realizations increases.

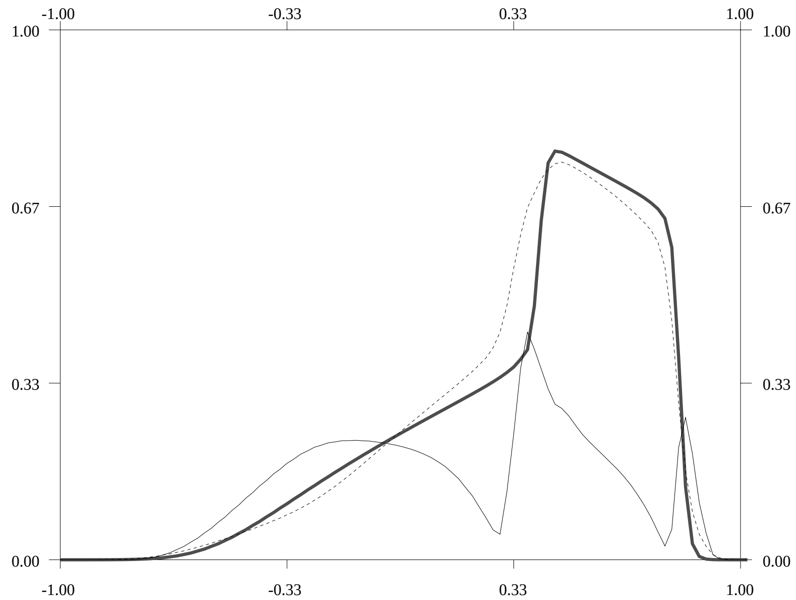

Figure 2.6.

References

1. A. Bensoussan, R. Glowinski, A. Rascanu, *Approximation of some stochastic differential equations by the splitting up method*, Appl. Math. Optim. **25** (1992), 81-106.
2. L. Bertini, N. Cancrini, G. Jona-Lasinio, *The stochastic Burgers equation*, Comm. Math. Phys. **165** (1994), 211–232.
3. Z. Brzezniak, M. Capinski, F. Flandoli, *Stochastic partial differential equations and turbulence*, Math. Mod. Meth. Appl. Science **1** (1991), 41-59.
4. A. V. Bulinskii, S. A. Molchanov, *Asymptotical normality of a solution of Burgers' equation with random initial data*, Theory Prob. Appl. **36** (1992), 217–236.
5. J. M. Burgers, *The Nonlinear Diffusion Equation*, Reidel, Dordrecht, 1974.
6. M. Crandall, A. Majda, *The method of fractional steps for conservation laws*, Numer. Math. **34** (1980), 285-314.
7. J.-D. Fournier, U. Frisch, *L'équation de Burgers déterministe et statistique*, J. Méch. Théor. Appl. **2** (1983), 699–750.
8. S. K. Godunov, *Finite difference methods for numerical computations of discontinuous solutions of the equations of fluid dynamics*, Mat. Sbornik. **47** (1959), 271-295.
9. S. Gurbatov, A. Malakov, A. Saichev, *Nonlinear random waves and turbulence in nondispersive media: waves, rays and particles*, Manchester Univ. Press, Manchester, 1991.
10. S. N. Gurbatov, A. I. Saichev, *Degeneracy of one-dimensional acoustic turbulence at large Reynolds numbers*, Sov. Phys. JETP **53** (1981), 347-354.
11. H. Holden, L. Holden, *First order nonlinear scalar hyperbolic conservation laws in one dimension*, Ideas and Methods in Mathematical Analysis, Stochastics, and Applications (S. Albeverio, J. E. Fenstad, H. Holden, T. Lindstrøm, eds.), Cambridge University Press, Cambridge, 1992, pp. 480-510.
12. H. Holden, L. Holden, R. Høegh-Krohn, *A numerical method for first order nonlinear scalar conservation laws in one–dimension*, Comput. Math. Applic. **15** (1988), 595–602.
13. H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe, T.-S. Zhang, *The Burgers equation with a noisy force*, Comm. Part. Diff. Eqn. **19** (1994), 119–141.
14. H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe, T.-S. Zhang, *The stochastic Wick-type Burgers equation*, Proceedings, Edinburgh 1994, London Math. Soc. Lecture Notes (to appear).
15. H. Holden, N. H. Risebro, *Stochastic Properties of the scalar Buckley–Leverett equation*, SIAM J. Math. Anal. **51** (1991), 1472-1488.
16. H. Holden, N. H. Risebro, *A stochastic approach to conservation laws*, Third International Conference on Hyperbolic Problems. Theory, Numerical Methods and Applications, Uppsala, 1990 (B. Engquist, B. Gustafsson, eds.), Studentlitteratur/Chartwell-Bratt, Lund–Bromley, 1991, pp. 575–587.
17. H. Holden, N. H. Risebro, *A fractional steps method for scalar conservation laws without the CFL condition*, Math. Comp. **60** (1993), 221 – 232.
18. M. Kardar, G. Parisi, Y.-C. Zhang, *Dynamic scaling of growing surfaces*, Phys. Rev. Lett. **56** (1986), 889-892.
19. S. Kida, *Asymptotic properties of Burgers turbulence*, J. Fluid. Mech. **93** (1979), 337–377.

20. P.E. Kloeden, E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer, Berlin, 1992.
21. H. Konno, *On stochastic Burgers equation*, J. Phys. Soc. Japan **54** (1985), 4475-4478.
22. H. Kunita, *First order stochastic partial differential equations*, Stochastic Analysis, Taniguchi Symposium, Katata and Kyoto 1982 (K. Ito, ed), North-Holland, Amsterdam, 1984, pp. 249–269.
23. Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, Springer, Berlin, 1984.
24. R. LeVeque, *Numerical Methods for Conservation Laws*, Birkhäuser, Basel, 1991.
25. B.J. Lucier, *A moving mesh numerical method for hyperbolic conservation laws*, Math. Comp. **46** (1986), 59-69.
26. E. Medina, T. Hwa, M. Kardar, T.-C. Zhang, *Burgers equation with correlated noise: Renormalization-group analysis and applications to directed polymers and interface growth*, Phys. Rev. **39A** (1989), 3053-3075.
27. T. Musha, Y. Kosugi, G. Matsumoto, M. Suzuki, *Modulation of the time relation of action potential impulses propagating along the axon*, IEEE Trans. Biomedical Eng. **BME–28** (1981), 616–623.
28. H. Nakazawa, *Stochastic Burgers' equation in the inviscid limit*, Adv. Appl. Math. **3** (1982), 18-42.
29. S.F. Shandarin, Ya.B. Zeldovich, *The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium*, Rev. Mod. Phys. **61** (1989), 185–220.
30. Z.-S. She, E. Aurell, U. Frisch, *The inviscid Burgers equation with initial data of Brownian motion*, Comm. Math. Phys. **148** (1992), 623-641.
31. Ya.G. Sinai, *Statistics of shocks in solutions of inviscid Burgers equation*, Comm. Math. Phys. **148** (1992), 601-621.
32. Ya.G. Sinai, *Two results concerning asymptotic behavior of solutions of the Burgers equation with force*, J. Stat. Phys. **64** (1991), 1-12.
33. J. Smoller, *Shock Waves and Reaction–Diffusion Equations*, Springer, New York, 1983.

Department of Mathematical Sciences
The Norwegian Institute of Technology
University of Trondheim
N–7034 Trondheim, Norway
(Holden)
E-mail address: holden@imf.unit.no

Department of Mathematics
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway
(Risebro)
E-mail address: nilshr@math.uio.no