Global Weak Entropy Solutions to Quasilinear Wave Equations of Klein–Gordon and Sine–Gordon Type

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Abstract. We establish the existence of global Lipschitz continuous weak entropy solutions to the Cauchy problem for a class of quasilinear wave equations with an external positional force. We prove the consistency and the convergence of uniformly bounded finite–difference fractional step approximations. Therefore the uniform bound is shown to hold for globally Lipschitz continuous external forces.

Key words and phrases. Klein–Gordon and Sine–Gordon equations, hyperbolic nonhomogeneous systems, shock waves, fractional step approximation, compensated compactness.

1. Introduction

In this paper we establish the existence of global Lipschitz continuous solutions to the Cauchy problem for the one–dimensional quasilinear wave equation

$$\partial_t^2 w - \partial_x \sigma(\partial_x w) + f(w) = 0 \tag{1.1}$$

for all $(x,t) \in \mathbb{R} \times (0,\infty)$, with initial conditions

$$w(x,0) = w_0(x), \quad \partial_t w(x,0) = w_1(x), \tag{1.2}$$

for all $x \in \mathbb{R}$. Here f is a smooth function with $f(0) = 0$ and σ is a given smooth function such that $\sigma'(u) \geq \gamma > 0$ ($\gamma > 0$) and $u\sigma''(u) > 0$ for $u \neq 0$;

w_0 and w_1 are bounded functions with compact support, w_0 is also Lipschitz continuous.

This equation models a vibrating string with an elastic external positional force and can also be deduced (at a very formal level) by applying the principle of the "stationary action" from the Lagrangian density given by

$$\mathcal{L}_1(w_t, w_x, w) = \frac{1}{2} w_t^2 - \Sigma(w_x) - F(w)$$

where $\Sigma' = \sigma$ and $F' = f$.

As an example we can consider the quasilinear Klein–Gordon equation

$$\partial_t^2 w - \partial_x \sigma(\partial_x w) + mw = 0 \quad (m \in \mathbb{R}) \tag{1.3}$$

and the quasilinear Sine–Gordon equation

$$\partial_t^2 w - \partial_x \sigma(\partial_x w) + \sin w = 0 \tag{1.4}$$

Let us notice that the semilinear versions of the equations (1.3), (1.4) exhibit linear dispersive waves [Wh], although this behaviour has not yet been analyzed in detail in the present case.

The Cauchy problem (1.1)-(1.2) will be considered in the following equivalent formulation. Denote by

$$u = \partial_x w \quad , \quad v = \partial_t w \ . \tag{1.5}$$

Then the functions (u, v, w) satisfy the Cauchy problem

$$\begin{aligned} &\partial_t u - \partial_x v = 0 \ , \\ &\partial_t v - \partial_x \sigma(u) = -f(w) \ , \end{aligned} \tag{1.6}$$

$$\partial_t w = v \ , \tag{1.7}$$

for all $(x, t) \in \mathbb{R} \times (0, \infty)$, with initial conditions

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x) \tag{1.8}$$

for all $x \in \mathbb{R}$. Here $u_0 := \partial_x w_0$ and $v_0 := w_1$.

Observe that (1.6)-(1.7) is a nonhomogeneous system of conservation laws. The local existence theorem for this class of problems was first proved in [DH], for initial data having small total variation and in the class of weak entropy BV solutions, by using a simple fractional step version of the Glimm's scheme [Gl]. Global solutions were then obtained under some complete dissipative assumptions on the source term ([DH], Theorem 2) which are not verified in the present case. Other results of convergence of the fractional step version of the Glimm's scheme have been obtained in [YW], for a nonhomogeneous system of isentropic gas dynamics ($\gamma = 1$).

For 2×2 strictly hyperbolic systems, with genuinely nonlinear fields, a quite general theory for L^∞ solutions was developed by Di Perna [DiP]. He used the theory of invariant domains [CCS] to obtain the L^∞ bounds of the parabolic approximations and the method of compensated compactness [Ta] to show the existence of global weak entropy solutions. Moreover, for (1.6), the existence of solutions in L^∞ was also established under some assumptions on the term f, which could be depending on u, v but not on w, and σ having at most one inflection point in the strain hardening case [DiP]. Again the case $f(w) \not\equiv 0$ cannot be directly handled by using this approach.

Here we deal with (1.6)-(1.7) by devising a finite difference approximation of the solutions, which is a modified (implicit) version of the fractional step Lax–Friedrichs and Godunov schemes, in the spirit of [DCL], [MN 1,2]. In section 2 we give some definitions and describe our scheme. The consistency and the convergence of uniformly bounded approximations are shown in Section 3 and in Section 4, respectively. Finally, in Section 5, under a sublinear assumption for the function f, we obtain the L^∞ bounds (locally in time) for the approximating solutions and then the existence of global weak entropy solutions, in particular for the equations (1.1) and (1.2).

2. The approximating solutions

First we define the notion of weak entropy solutions for the problem (1.6)-(1.8). Set $U = (u, v)$, $\mathcal{A}(U) = (-v, -\sigma(u))$, $\mathcal{B}(w) = (0, -f(w))$ and

$U_0 = (u_0, v_0)$. The system (1.6) is now denoted by

$$\partial_t U + \partial_x \mathcal{A}(U) = \mathcal{B}(w) \; . \tag{2.1}$$

Definition 2.1. *The functions (U, w) are a weak solution to (1.6)-(1.8) if, for all smooth functions $\phi \in \left(C_0^\infty(\mathbb{R} \times [0, \infty))\right)^2$ and $\chi \in C_0^\infty(\mathbb{R} \times [0, \infty))$ one has*

$$\begin{aligned} &\iint\limits_{\{t\geq 0\}} \{U \cdot \phi_t + \mathcal{A}(U) \cdot \phi_x + \mathcal{B}(w) \cdot \phi\} \,\mathrm{d}\,x \;\mathrm{d}\,t \\ &\qquad + \int_{\{t=0\}} U_0(x) \cdot \phi(x, 0) \,\mathrm{d}\,x = 0 \; , \end{aligned} \tag{2.2}$$

$$\iint\limits_{\{t\geq 0\}} \{w \chi_t + v \chi\} \,\mathrm{d}\,x \;\mathrm{d}\,t + \int_{\{t=0\}} w_0(x) \chi(x, 0) \,\mathrm{d}\,x = 0 \; . \tag{2.3}$$

Notice that, as a consequence of this definition, since U is locally bounded, then w is a locally Lipschitz continuous function.

An entropy–entropy flux pair (η, q) for (2.1) is a couple of smooth functions of U such that

$$\nabla q = \nabla \eta \cdot \nabla \mathcal{A}^T \; ,$$

where ∇ denotes the gradient respect to U. A classical example of strictly convex entropy function is the "mechanical energy":

$$\eta^* = \frac{1}{2} v^2 + \int_0^u \sigma(s) \,\mathrm{d}\,s \; .$$

In analogy with [La2] we give the following definition.

Definition 2.2. *The functions $U = (u, v)$ and w are a weak entropy solution to the problem (1.6)-(1.8) if and only if*

i) (U, w) is a weak solution;

ii) for any convex entropy function η one has

$$\partial_t \eta + \partial_x q \leq (\nabla \eta) \cdot \mathcal{B}(w) \qquad \text{in } \mathcal{D}' \; . \tag{2.4}$$

We recall also that the characteristic velocities for (1.6)-(1.7) are given by

$$\lambda_1 = -\sqrt{\sigma'(u)}\ , \quad \lambda_2 = \sqrt{\sigma'(u)}\ , \quad \lambda_0 = 0\ . \tag{2.5}$$

The right (respectively left) eigenvectors r_i (respectively l_i) corresponding to eigenvalues λ_i $(i = 1, 2)$, can be taken in the form:

$$r_1 = (1, \sqrt{\sigma'(u)})^T, \quad r_2 = (1, -\sqrt{\sigma'(u)})^T\ ;$$

$$l_1 = (\sqrt{\sigma'(u)}, 1), \quad l_2 = (-\sqrt{\sigma'(u)}, 1)\ .$$

The field associated to the eigenvalue λ_i is said to be genuinely nonlinear if

$$\nabla\lambda_i \cdot r_i \neq 0, \qquad (i = 1, 2).$$

In the present case we have

$$\nabla\lambda_i \cdot r_i = (-1)^i \frac{\sigma''(u)}{2\sqrt{\sigma'(u)}}$$

and then both the fields are not genuinely nonlinear for $u = 0$.

The Riemann invariants associated to λ_1 and λ_2 are given respectively by

$$\eta = v - g(u),\ \ \xi = v + g(u), \tag{2.6}$$

where $g(u) = \int_0^u \sqrt{\sigma'(s)}ds$.

Then, in the Riemann invariants coordinates the system (1.6)-(1.7) can be written (formally)

$$\begin{cases} \partial_t \xi + \lambda_1 \partial_x \xi = -f(w)\ , \\ \partial_t \eta + \lambda_2 \partial_x \eta = -f(w)\ , \\ \partial_t w = \frac{\xi+\eta}{2}\ . \end{cases} \tag{2.7}$$

Let us recall now some backgrounds about admissibility of solutions to the Riemann problem for the homogeneous system

$$\text{(2.8)} \qquad \begin{cases} \partial_t u - \partial_x v = 0 \ , \\ \partial_t v - \partial_x \sigma(u) = 0 \ . \end{cases}$$

We have that across any discontinuity curve $(c(t), t)$, a weak solution (u, v) to (2.8) satisfies the Rankine-Hugoniot condition

$$\frac{v_- - v_+}{u_+ - u_-} = \frac{\sigma(u_-) - \sigma(u_+)}{v_+ - v_-} = \dot{c},$$

where $(u_\pm, v_\pm) = (u, v)(c(t) \pm 0, t)$ and $\dot{c}$ is the speed of the discontinuity. For any (u_0, v_0) let the *shock set* through (u_0, v_0) be the set of points (u, v) satisfying the Rankine-Hugoniot condition

$$\frac{v_0 - v}{u - u_0} = \frac{\sigma(u_0) - \sigma(u)}{v - v_0} = \dot{c}(u_0, v_0; u, v).$$

In [Li1,2], T.P. Liu proposed a pointwise entropy condition which is described as follows: across every discontinuity the following condition is satisfied.

$$\text{(2.9)} \qquad \dot{c}(u_-, v_-; u_+, v_+) \leq \dot{c}(u_-, v_-; u, v) \ ,$$

for all (u, v) on the shock set through (u_-, v_-) and (u_+, v_+). This condition extends the celebrated Lax shock inequalities (see [La2]) to systems which are not genuinely nonlinear. Moreover, applying the results in [Li1,2], it easy to show that the Riemann problem for system (2.8) has a unique solution in the class of piecewise self–similar solutions composed by constant states connected by shocks, rarefaction waves and contact discontinuities for arbitrarily large initial data. The invariant regions for this system are of the form

$$\{(u, v); |\xi| \leq N, |\eta| \leq N\},$$

for any $N \geq 0$, where ξ and η are the Riemann invariants.

Moreover, by using the results in [CL], it is possible to show that these solutions are also weak entropy solutions in the sense of Definition 2.2

(with $\mathcal{B} \equiv 0$). Let us point out that condition (2.9) is equivalent to the entropy inequalities (2.4), and then to the Lax shock inequalities, only in the genuinely nonlinear case, but in general it is strictly stronger, see again [CL].

Let us describe now our finite difference scheme which combines the Lax–Friedrichs scheme [La1] with a suitable version of the fractional step method. A similar construction can be handled by using the Godunov scheme.

Consider a partition of $\mathbb{R} \times (0, \infty)$ into horizontal layers. For any $k \in \mathbb{N}$ let

$$S_k = \{(x,t) \,|\, kh \leq t < (k+1)h\} ,$$

where h is the time mesh–length. Fix $k \geq 0$ and set $I_k = \{i \,|\, i+k \text{ is even}\}$. Then define, for any $i \in I_k$,

$$Q_{i,k} = \{(x,t) \in S_k \,|\, (i-1)l < x < (i+1)l\} \ ,$$

where l is the space mesh–length.

Moreover we shall denote by $V^h = \left(U^h, w^h\right) = (u^h, v^h, w^h)$ the approximate solutions we construct by using this partition.

We shall also require the following condition: for any $T > 0$ there exists a positive constant C_T such that

$$\sup |\lambda_i(V^h)| \leq \frac{l}{2h} \leq C_T \quad , \quad i = 1, 2 \ . \tag{2.10}$$

Let us start the construction of V^h setting for any $i \in I_{-1}$

$$U^{i,0} = \frac{1}{2l} \int_{(i-1)l}^{(i+1)l} U^0(x)\,\mathrm{d}\,x \ , \tag{2.11}$$

where $U^0 = (u^0, v^0)$, and for any $k > 0$ and any $i \in I_{k-1}$

$$U^{i,k} = \frac{1}{2l} \int_{(i-1)l}^{(i+1)l} U^h(x, kh-0)\,\mathrm{d}\,x \ . \tag{2.12}$$

Then, for any $k \geq 0$ and $i \in I_k$, let $\tilde{U}^h = (\tilde{u}^h, \tilde{v}^h)$ be the solution of the Riemann problem for the homogeneous system (2.8), for $(x,t) \in Q_{i,k}$, with the following initial condition at $t = kh$

$$\tilde{U}^h(x, kh+0) = \begin{cases} U^{i-1,k}, & (i-1)l < x < il \ , \\ U^{i+1,k}, & il < x < (i-1)l \ . \end{cases} \tag{2.13}$$

Thanks to (2.10) the elementary waves do not interact on S_k. At this point we define the fractional step part of our scheme, which allows us to treat this special class of nonhomogeneous terms. Therefore we define on S_k

$$w^h(x,t) = \int_{-\infty}^{x} \tilde{u}^h(y,t)\,\mathrm{d}\,y \ , \tag{2.14}$$

$$u^h(x,t) = \tilde{u}^h(x,t) \ , \tag{2.15}$$

$$v^h(x,t) = \tilde{v}^h(x,t) - \int_{kh}^{t} f\left(w^h(x,s)\right)\,\mathrm{d}\,s \ . \tag{2.16}$$

3. Convergence of the scheme

In this section we first establish the compactness properties of our scheme to use the convergence result of Di Perna [DiP], which, in our case, can be stated in the following way.

Theorem 3.1. *(Compactness framework) Let* $\{V^h\} = \{(U^h, w^h)\}$ *be a sequence of approximate solutions to (1.6)-(1.7) which satisfy the following assumptions:*

(H_1) for any $T > 0$ *there exists a constant* $C_T > 0$ *such that*

$$|V^h(x,t)| \leq C_T$$

for a.e. $(x,t) \in \mathbb{R} \times (0,T)$*;*

(H_2) for any entropy–entropy flux pair (η, q) *for (1.6) the sequence of distributions*

$$\Lambda^h := \partial_t \eta(U^h) + \partial_x q(U^h)$$

is relatively compact in $H^{-1}_{\mathrm{loc}}(\Omega)$ *for any bounded open set* $\Omega \subseteq \mathbb{R} \times (0,\infty)$*.*

Then there is a converging subsequence $\{V^{h_\nu}\}$ *and a locally bounded limit function* $V = (U, w)$ *such that*

i)

$$U^{h_\nu} \longrightarrow U$$

in $L^p_{\mathrm{loc}}(\mathbb{R} \times (0,\infty))$ *for any* $p \in [1,\infty)$*, as* $h_\nu \to 0$*;*

ii)

$$w^{h_\nu} \longrightarrow w$$

in L^∞ weak-$*$, as $h_\nu \to 0$.

Moreover, if the following assumption is also verified

(H_3) for any $T > 0$, there is a constant $D_T > 0$ such that

$$|\nabla_{x,t} w^h| \leq D_T$$

for a.e. $(x,t) \in \mathbb{R} \times (0,T)$;

then, by passing to another subsequence, still denoted by h_ν, we have

$$w^{h_\nu} \rightrightarrows w$$

uniformly on compact subset of $\mathbb{R} \times (0,\infty)$, as $h_\nu \to 0$, and w is a locally Lipschitz continuous in $\mathbb{R} \times (0,\infty)$.

The proof of this statement follows closely the Di Perna's argument and is omitted.

In Section 5 under an additional sublinear growth assumption on f we show that our approximation satisfy the assumption (H_1). Here we show that (H_2) holds true in the general case for the approximation given by the scheme (2.14)-(2.16).

Theorem 3.2. *Assume that the initial data u_0, v_0, w_0 are bounded measurable functions with compact support and $u_0 = \partial_x w_0$. Let $\{V^h\} = \{(U^h, w^h)\}$ be a locally uniformly bounded approximating sequence given by the scheme (2.14)-(2.16).*

Then the sequence of measures

$$\Lambda^h = \partial_t \eta(U^h) + \partial_x q(U^h)$$

is a relatively compact subset of H^{-1}_{loc}, for all pair (η, q).

Proof. Let Ω be a bounded open set contained in $\mathbb{R} \times [0,\infty)$ and let $T > 0$ be such that $\Omega \subseteq \mathbb{R} \times (0,T)$. Let N be an integer which satisfies $(N+1)h \geq T > Nh$. For any function $\phi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} \int\!\!\int \big(\eta(U^h)\phi_t + q(U^h)\phi_x\big)\, \mathrm{d}x\, \mathrm{d}t \qquad (3.1)\\ = \langle M^h + \Sigma^h + R^h + L^h, \phi\rangle, \end{aligned}$$

where

(3.2)
$$\langle M^h, \phi\rangle = \int \left[\phi(x,T)\eta\left(\tilde{U}^h(x,T-0)\right) - \phi(x,0)\eta\left(\tilde{U}^h(x,0)\right)\right] \mathrm{d}\,x \ ;$$
(3.3)
$$\langle \Sigma^h, \phi\rangle = \int_0^T \sum_c \{\dot{c}[\tilde{\eta}]_c - [\tilde{q}]_c\}\phi\left(c(t),t\right) \mathrm{d}\,t \ ;$$
(3.4)
$$\langle R^h, \phi\rangle = \int\int \left(\eta(U^h) - \eta(\tilde{U}^h)\right)\phi_t + \left(q(U^h) - q(\tilde{U}^h)\right)\phi_x \,\mathrm{d}\,x\,\mathrm{d}\,t \ ;$$
(3.5)
$$\langle L^h, \phi\rangle = \sum_{k=1}^{N}\sum_{i\in I_{k-1}} \int_{(i-1)l}^{(i+1)l} \left(\eta(\tilde{U}^h(x,kh-0)) - \eta(U^h(x,kh+0))\right)\phi(x,kh)\,\mathrm{d}\,x \ .$$

We denote here by $c = c(t)$ any shock curve of the solution $\tilde{U}^h$ of the homogeneous problem (2.11) in S_k $(k \geq 0)$ and by $\dot{c}$ the shock speed given by the Rankine-Hugoniot formula; $[\tilde{\eta}]_c$ and $[\tilde{q}]_c$ denote the jump along $x = c(t)$ of the functions $\eta(\tilde{U}^h)$ and $q(\tilde{U}^h)$ respectively. It will be also convenient to introduce, for any piecewise smooth function $U = U(x,t)$, the notations $U_\pm^k = U(x, kh \pm 0)$ and $[U]_k = (U_-^k - U_+^k)$.

Also, for simplicity, in this proof we shall omit the index h of the approximate solutions U^h, $\tilde{U}^h$.

Then we take a strictly convex entropy function η and its associated entropy flux q. It follows that

$$\sum_{k\geq 1}\int \left[\eta(\tilde{U})\right]_k \mathrm{d}\,x + \int_0^T \sum_c \{\dot{c}[\tilde{\eta}]_c - [\tilde{q}]_c\}\,\mathrm{d}\,t \leq C \ . \tag{3.6}$$

Note that, since the entropy inequality

$$\dot{c}[\tilde{\eta}]_c - [\tilde{q}]_c \geq 0$$

is satisfied along the shock waves, we have

$$0 \leq \int_0^T \sum_c \{\dot{c}[\tilde{\eta}]_c - [\tilde{q}]_c\}\,\mathrm{d}\,t \leq C \ . \tag{3.7}$$

On the other hand we have, by the Taylor expansion of the function η,

$$
\begin{aligned}
\sum_{k\geq 1}\int \left[\eta(\tilde U)\right]_k \,\mathrm{d}\,x &= \sum_{k=1}^{N}\sum_{i\in I_{k-1}}\int_{(i-1)l}^{(i+1)l}\left(\eta(\tilde U^k_-)-\eta(\tilde U^k_+)\right)\,\mathrm{d}\,x\\
&= \sum_{k=1}^{N}\sum_{i\in I_{k-1}}\int_{(i-1)l}^{(i+1)l}\nabla\eta\left(U^k_+\right)\cdot(\tilde U^k_- - U^k_+)\,\mathrm{d}\,x\\
&+ \sum_{k=1}^{N}\sum_{i\in I_{k-1}}\int_{(i-1)l}^{(i+1)l}\bigg(\int_0^1(1-\theta)(\tilde U^k_- - U^k_+)\\
&\nabla^2\eta\left(U^k_+ + \theta(\tilde U^k_- - U^k_+)\right)(\tilde U^k_- - U^k_+)^T\,\mathrm{d}\,\theta\bigg)\,\mathrm{d}\,x\\
&= I_1 + I_2\ .
\end{aligned}
\tag{3.8}
$$

Therefore, from the boundedness of V^h, we have

$$
\begin{aligned}
|I_1| &\leq \sum_{k=1}^{N}\sum_{i\in I_{k-1}}\left|\int_{(i-1)l}^{(i+1)l}\eta_v\left(U^k_+\right)\left(\int_{(k-1)h}^{kh}f(w^h)\,\mathrm{d}\,s\right)\,\mathrm{d}\,x\right|\\
&\leq \int_0^T\int_{-L}^{L}|\eta_v||f|\,\mathrm{d}\,x\,\mathrm{d}\,t \leq C\ .
\end{aligned}
\tag{3.9}
$$

Moreover the entropy function η is strictly convex, namely there exists a constant $\nu > 0$ such that, for any vector $U\in\mathbb{R}^2$

$$
U\nabla^2\eta U^T \geq \nu|U|^2\ .
$$

Then by (3.6), (3.8), we have the energy inequality

$$
\sum_{k=1}^{N}\sum_{i\in I_{k-1}}\int_{(i-1)l}^{(i+1)l}|\tilde U^k_- - U^k_+|^2\,\mathrm{d}\,x \leq \frac{2}{\nu}|I_2| \leq C\ . \tag{3.10}
$$

Let us consider now the identity (3.1) for a general (not necessarily strictly convex) entropy function η. Set $\phi^{ik} = \phi(il, kh)$ and denote by

$$
\langle L_1^h,\phi\rangle = \sum_{k=1}^{N}\sum_{i\in I_{k-1}}\phi^{ik}\int_{(i-1)l}^{(i+1)l}\left(\eta(U^k_-)-\eta(\tilde U^k_+)\right)\,\mathrm{d}\,x\ ,
$$

$$
\langle L_2^h,\phi\rangle = \sum_{k=1}^{N}\sum_{i\in I_{k-1}}\int_{(i-1)l}^{(i+1)l}\left(\eta(\tilde U^k_-)-\eta(U^k_-)\right)\phi(x,kh)\,\mathrm{d}\,x\ ,
$$

$$
\langle L_3^h,\phi\rangle = \sum_{k=1}^{N}\sum_{i\in I_{k-1}}\int_{(i-1)l}^{(i+1)l}\left(\eta(U^k_-)-\eta(\tilde U^k_+)\right)(\phi(x,kh)-\phi^{ik})\,\mathrm{d}\,x\ .
$$

Then, for any entropy pair (η, q) we derive, from (3.7),(3.8) and (3.10),

$$|\langle M^h, \phi\rangle| \leq C||\phi||_{C_0(\Omega)} \ ,$$
$$|\langle \Sigma^h, \phi\rangle| \leq C||\phi||_{C_0(\Omega)} \ ,$$
$$|\langle L_1^h, \phi\rangle| \leq C||\phi||_{C_0(\Omega)} \ ,$$

The contribution of the term L_2^h includes the effects of the external force f

$$\begin{aligned}|\langle L_2^h, \phi\rangle| &= \left|\sum_{k=1}^{N} \sum_{i\in I_{k-1}} \int_{(i-1)l}^{(i+1)l} \left(\eta(\tilde{U}_-^k) - \eta(U_-^k)\right) \phi(x, kh) \,\mathrm{d}\, x\right| \\ &\leq \|\phi\|_{C_o(\Omega)} \sum_{k=1}^{N} \sum_{i\in I_{k-1}} \int_{(i-1)l}^{(i+1)l} \left| \left(\int_0^1 \eta_v \left(U_-^k + \theta(\tilde{U}_-^k - U_-^k)\right) \,\mathrm{d}\, \theta\right) \right. \\ &\left. \left(\int_{(k-1)h}^{kh} f(w^h) \,\mathrm{d}\, s\right)\right| \mathrm{d}\, x \leq C\|\phi\|_{C_0(\Omega)} \ .\end{aligned}$$

Hence

$$\left\|(M^h + \Sigma^h + L_1^h + L_2^h)\right\|_{\mathcal{M}} \leq C \ ,$$

where $\mathcal{M} = C_0^*$ is the space of bounded measures on Ω.

Therefore, as noticed by Di Perna [DiP],

$$M^h + \Sigma^h + L_1^h + L_2^h$$

is in a compact subset of $W^{-1,q_1}(\Omega)$, for $1 < q_1 < 2$, as a consequence of the Murat's lemma [Mu]. Next we have, for $\frac{1}{2} < \alpha < 1$,

$$\begin{aligned}|\langle L_3^h, \phi\rangle| &\leq \sum_{k=1}^{N} \sum_{i\in I_{k-1}} \int_{(i-1)l}^{(i+1)l} \left|\eta(U_-^k) - \eta(\tilde{U}_+^k)\right| \left|\phi(x, kh) - \phi^{ik}\right| \,\mathrm{d}\, x \\ &\leq h^\alpha \|\phi\|_{C_0^\alpha} \sum_k \left(\sum_i \int_{(i-1)l}^{(i+1)l} |\eta(U_-^k) - \eta(\tilde{U}_+^k)|^2 \,\mathrm{d}\, x\right)^{1/2} \\ &\leq C\, h^{\alpha-\frac{1}{2}} \|\nabla\eta\|_{L^\infty} \|\phi\|_{C_0^\alpha} \left(\sum_{k=1}^{N} \sum_{i\in I_{k-1}} \int_{(i-1)l}^{(i+1)l} |U_-^k - \tilde{U}_+^k|^2 \,\mathrm{d}\, x\right)^{1/2} \\ &\leq C\, h^{\alpha-\frac{1}{2}} \|\phi\|_{C_0^\alpha} \ .\end{aligned}$$

Then, by using the Sobolev embedding theorem

$$\|\phi\|_{C_0^\alpha} \leq C\|\phi\|_{W_0^{1,q}}$$

for all $q > \frac{2}{1-\alpha}$.

The previous inequality leads to

$$\|L_3^h\|_{W^{-1,q_2}} \leq C\, h^{\alpha-\frac{1}{2}}$$

for all $1 < q_2 < \frac{2}{1+\alpha}$.

Then

$$M^h + \Sigma^h + L^h$$

lies in a compact subset of $W^{-1,q_0}(\Omega)$, where $1 < q_0 = \min(q_1, q_2) < \frac{2}{1+\alpha}$.

Furthermore, from the boundedness assumptions on the approximating scheme, we have that

$$M^h + \Sigma^h + L^h = \Lambda^h - R^h$$

is bounded in $W^{-1,r}(\Omega)(r > 1)$. Therefore, again from the Murat's lemma, it follows that

$$M^h + L^h + \Sigma^h$$

lies in a compact subset of $H^{-1}_{\text{loc}}(\Omega)$. Moreover, since

$$\begin{aligned}\left|\langle R^h, \phi\rangle\right| &\leq \sum_k \iint_{S_k} \left|\int_{kh}^{(k+1)h} f(w^h)\,\mathrm{d}\,s\right| (|\nabla\eta|\,|\phi_t| + |\nabla q|\,|\phi_x|)\;\mathrm{d}\,x\,\mathrm{d}\,t \\ &\leq C\,h\|\phi\|_{H_0^1(\Omega)}\ ,\end{aligned}$$

it follows that R^h is in a compact subset of $H^{-1}_{\text{loc}}(\Omega)$. Then the conclusion follows. □

4. Consistency of the scheme

To prove the consistency of our scheme we need the next elementary but crucial result which combines an original argument due to Makino and Takeno [MT, Proposition 3] and our energy estimate (3.10).

Lemma 4.1. *Under the assumptions of Theorem 3.1, there exists a positive constant C such that*

$$\sum_{k=1}^{N} \sum_{i\in I_{k-1}} \int_{(i-1)l}^{(i+1)l} \int_{(k-1)h}^{kh} |\tilde{U}^h(x, kh+0) - \tilde{U}^h(x,t)|^2 \,\mathrm{d}\,x \,\mathrm{d}\,t \le Ch \; .$$

Proof. From Proposition 3 in [MT], we have that there exists a constant $C > 0$ such that, for all $k \ge 0$ and for $t \in ((k-1)h, kh)$,

$$\int_{(i-1)l}^{(i+1)l} \left|\tilde{U}^h(x,t) - \tilde{U}^h(x, kh-0)\right|^2 \,\mathrm{d}\,x$$
$$\le C \int_{(i-1)l}^{(i+1)l} \left|\tilde{U}^h(x, kh-0) - \frac{1}{2l}\int_{(i-1)l}^{(i+1)l} \tilde{U}^h(y, kh-0)\,\mathrm{d}\,y\right|^2 \,\mathrm{d}\,x \; .$$

Then

$$\int_{(i-1)l}^{(i+1)l} |\tilde{u}^h(x,t) - \tilde{u}^{hk}_{+}|^2 \,\mathrm{d}\,x$$
$$\le 2\int_{(i-1)l}^{(i+1)l} |\tilde{u}^h(x,t) - \tilde{u}^{hk}_{-}|^2 + |\tilde{u}^{hk}_{+} - \tilde{u}^{hk}_{-}|^2 \,\mathrm{d}\,x$$
$$\le C\int_{(i-1)l}^{(i+1)l} |\tilde{U}^{hk}_{+} - \tilde{U}^{hk}_{-}|^2 \,\mathrm{d}\,x \; .$$

So the conclusion for $\tilde{u}^h$ follows by the energy inequality (3.10). In the same way it is easy to seen that

$$\int_{(i-1)l}^{(i+1)l} |\tilde{v}^h(x,t) - \tilde{v}^{hk}_{+}|^2 \,\mathrm{d}\,x$$
$$\le C_1\int_{(i-1)l}^{(i+1)l} |\tilde{U}^{hk}_{+} - \tilde{U}^{hk}_{-}|^2 \,\mathrm{d}\,x$$
$$+ C_2 \int_{(i-1)l}^{(i+1)l} \left(\frac{1}{2l}\int_{(i-1)l}^{(i+1)l}\int_{(k-1)h}^{hk} f(w^h)\,\mathrm{d}\,s\,\mathrm{d}\,y\right)^2 \,\mathrm{d}\,x.$$

Therefore the conclusion follows from (3.10) and the boundedness of w^h.□

Theorem 4.2. *Assume that the sequence given by the scheme (2.14)–(2.16) verifies the hypothesis (H_1), (H_2) of Theorem 3.1. Then (H_3) is also verified and the limit functions (U, w) are a weak entropy solution to the Cauchy problem (1.6)-(1.7)-(1.8).*

Proof. Consistency with initial data follows easily by arguing as in [DCL]. Then, take any smooth function $\phi \in C_0^\infty(\mathbb{R} \times (0,\infty))$. We have

$$\int\int (u^h\phi_t - v^h\phi_x)\,\mathrm{d}\,x\,\mathrm{d}\,t = \int\int (\tilde{u}^h\phi_t - \tilde{v}^h\phi_x)\,\mathrm{d}\,x\,\mathrm{d}\,t$$
$$+\sum_{k=1}^{N}\iint_{S_{k-1}}\left(\int_{(k-1)h}^{t} f(w^h)\,\mathrm{d}\,s\right)\phi_x\,\mathrm{d}\,x\,\mathrm{d}\,t = A_1^h + A_2^h\ .$$

From the assumption (H_1) there exists a constant $C_T \geq 0$ such that

$$\sup_x \sup_{1\leq k\leq N}\left|\int_{(k-1)h}^{kh} f(w^h)\,\mathrm{d}\,s\right| \leq C_T h\ ,$$

and then

$$|A_2^h| \leq C\,h\ .$$

On the other hand, thanks to the energy estimate (3.9), we have

$$\begin{aligned}
|A_1^h| &= \left|\sum_{k=1}^{N}\int \phi(x,kh)\left(\tilde{u}^h(x,kh+0) - \tilde{u}^h(x,kh-0)\right)\,\mathrm{d}\,x\right| \\
&\leq \left|\sum_{k=1}^{N}\sum_i \int_{(i-1)l}^{(i+1)l}\left(\phi(x,kh) - \phi^{ik}\right)\left(\tilde{u}^h(x,kh+0) - \tilde{u}^h(x,kh-0)\right)\mathrm{d}\,x\right| \\
&+\left|\sum_{k=1}^{N}\sum_i \phi^{ik}\int_{(i-1)l}^{(i+1)l}\left[\tilde{u}^h(x,kh-0) - \frac{1}{2l}\int_{(i-1)l}^{(i+1)l}\tilde{u}^h(y,kh-0)\,\mathrm{d}\,y\right]\mathrm{d}\,x\right| \\
&\leq C\sqrt{l}\|\phi\|_{C_0^1}\left\{\sum_{\substack{i,k\\ |il|\leq L}}\int_{(i-1)l}^{(i+1)l}|\tilde{u}^h(x,kh-0) - \tilde{u}^h(x,kh+0)|^2\,\mathrm{d}\,x\right\}^{\frac{1}{2}} \\
&\leq C\sqrt{h}\ .
\end{aligned}$$

Consider now the second equation in (1.6). For any smooth function

$\phi \in C_0^\infty(\mathbb{R}\times(0,\infty))$ we have

$$
\begin{aligned}
\int\!\!\int & (v^h\phi_t - \sigma(u^h)\phi_x - f(w^h)\phi)\,\mathrm{d}\,x\,\mathrm{d}\,t \\
&= \int\!\!\int (\tilde{v}^h\phi_t - \sigma(\tilde{u}^h)\phi_x - f(w^h)\phi)\,\mathrm{d}\,x\,\mathrm{d}\,t \\
&+ \sum_{k=1}^{N}\iint_{S_{k-1}}\left(\int_{(k-1)h}^{t} f(w^h)\,\mathrm{d}\,s\right)\phi_t\,\mathrm{d}\,x\,\mathrm{d}\,t \\
&= \sum_{k=1}^{N}\int \phi(x,kh)\left(\tilde{v}^h(x,kh-0)-\tilde{v}^h(x,kh+0)\right)\,\mathrm{d}\,x \\
&- \int\!\!\int f(w^h)\phi\,\mathrm{d}\,x\,\mathrm{d}\,t + \sum_{k=1}^{N}\iint_{S_{k-1}}\left(\int_{(k-1)h}^{t} f(w^h)\,\mathrm{d}\,s\right)\phi_t\,\mathrm{d}\,x\,\mathrm{d}\,t \\
&= B_1^h + B_2^h\ .
\end{aligned}
$$

Again from the assumption (H_1) we have

$$|B_2^h| \le C\,h\ .$$

Furthermore we have

$$
\begin{aligned}
|B_1^h| &= \Bigg|\sum_{k=1}^{N}\int \phi(x,kh)\left(\tilde{v}^h(x,kh-0)-\tilde{v}^h(x,kh+0)\right)\,\mathrm{d}\,x \\
&- \int\!\!\int f(w^h)\phi\ \mathrm{d}\,x\ \mathrm{d}\,t\Bigg| \\
&\le \left|\sum_{k=1}^{N}\sum_{i\in I_{k-1}}\int_{(i-1)l}^{(i+1)l}\left(\phi(x,kh)-\phi^{ik}\right)\left(\tilde{v}^h(x,kh+0)-\tilde{v}^h(x,kh-0)\right)\,\mathrm{d}\,x\right| \\
&+ \Bigg|\sum_{k=1}^{N}\sum_{i\in I_{k-1}}\phi^{ik}\int_{(i-1)l}^{(i+1)l}\Bigg[\tilde{v}^h(x,kh-0)-\frac{1}{2l}\int_{(i-1)l}^{(i+1)l}\tilde{v}^h(y,kh-0)\,\mathrm{d}\,y \\
&+ \frac{1}{2l}\int_{(i-1)l}^{(i+1)l}\int_{(k-1)h}^{kh} f(w^h)\,\mathrm{d}\,y\ \mathrm{d}\,t\Bigg] - \int\!\!\int f(w^h)\phi\ \mathrm{d}\,x\ \mathrm{d}\,t\Bigg| \\
&\le C\sqrt{h} + \left|\sum_{k,i}\int_{(i-1)l}^{(i+1)l}\int_{(k-1)h}^{kh} f(w^h)(\phi^{ik}-\phi(x,t))\,\mathrm{d}\,x\ \mathrm{d}\,t\right| \le C\sqrt{h}\ .
\end{aligned}
$$

Take now any convex entropy function η and let q be the correspondent entropy–flux. For any nonnegative function $\phi \in C_0^\infty(\mathbb{R}\times(0,\infty))$ we have

$$
\begin{aligned}
I^h &= \int\int \eta(U^h)\phi_t + q(U^h)\phi_x - \eta_v(U^h)f(w^h)\phi \ \mathrm{d}x\ \mathrm{d}t \\
&= \int\int \eta(\tilde U^h)\phi_t + q(\tilde U^h)\phi_x - \eta_v(\tilde U^h)f(w^h)\phi \ \mathrm{d}x\ \mathrm{d}t \\
&+ \sum_{k=1}^{N} \iint_{S_{k-1}} \Big\{ [\eta(U^h) - \eta(\tilde U^h)]\phi_t + [q(U^h) - q(\tilde U^h)]\phi_x \\
&- [\eta_v(U^h) - \eta_v(\tilde U^h)] f(w^h)\phi \Big\}\,\mathrm{d}x\ \mathrm{d}t \\
&= C_1^h + C_2^h\ .
\end{aligned}
$$

Since the functions η, q, η_v are smooth there exists a constant C such that

$$
|C_2^h| \le C\,h\ . \tag{4.1}
$$

Consider now the term C_1^h. We have

$$
\begin{aligned}
C_1^h &= \sum_{k=1}^{N} \int \phi(x,kh)\left(\eta(\tilde U_-^{hk}) - \eta(\tilde U_+^{hk})\right)\,\mathrm{d}x \\
&- \int\int \eta_v(\tilde U^h)f(w^h)\phi\ \mathrm{d}x\ \mathrm{d}t \\
&= \sum_{k,i} \int_{(i-1)l}^{(i+1)l} \left(\phi(x,kh) - \phi^{ik}\right)\left(\eta(\tilde U_-^{hk}) - \eta(\tilde U_+^{hk})\right)\,\mathrm{d}x \\
&+ \sum_{k,i} \phi^{ik} \int_{(i-1)l}^{(i+1)l} \left[\eta(\tilde U_-^{hk}) - \eta(\tilde U_+^{hk})\right]\,\mathrm{d}x \\
&- \int\int \eta_v(\tilde U^h)f(w^h)\phi\ \mathrm{d}x\ \mathrm{d}t
\end{aligned}
$$

$$= \sum_{k,i} \int_{(i-1)l}^{(i+1)l} \left(\phi(x,kh) - \phi^{ik}\right) \left(\eta(\tilde U_-^{hk}) - \eta(\tilde U_+^{hk})\right) \mathrm{d}\,x$$
$$+ \sum_{k,i} \phi^{ik} \int_{(i-1)l}^{(i+1)l} \eta_u(\tilde U_+^{hk}) \left[\tilde u^h(x,kh-0) - \frac{1}{2l}\int_{(i-1)l}^{(i+1)l} \tilde u^h(y,kh-0)\,\mathrm{d}\,y\right] \mathrm{d}\,x$$
$$+ \sum_{k,i} \phi^{ik} \int_{(i-1)l}^{(i+1)l} \eta_v(\tilde U_+^{hk}) \Big[\tilde v^h(x,kh-0) - \frac{1}{2l}\int_{(i-1)l}^{(i+1)l} \tilde v^h(y,kh-0)\,\mathrm{d}\,y$$
$$+ \frac{1}{2l}\int_{(i-1)l}^{(i+1)l}\int_{(k-1)h}^{kh} f(w^h)\,\mathrm{d}\,y\ \mathrm{d}\,t\Big]\,\mathrm{d}\,x$$
$$+ \sum_{k,i} \phi^{ik} \int_{(i-1)l}^{(i+1)l} Q_\eta(\tilde U_-^{hk}, \tilde U_+^{hk})\,\mathrm{d}\,x$$
$$- \int\int \eta_v(\tilde U^h) f(w^h)\phi\ \ \mathrm{d}\,x\ \mathrm{d}\,t$$
$$= C_{11}^h + C_{12}^h + C_{13}^h + C_{14}^h + C_{15}^h\ .$$

Here, for any $a, b \in \mathbb{R}^2$, we set

$$Q_\eta(a,b) = \int_0^1 (1-\theta)(a-b)\nabla^2\eta\left(b + \theta(a-b)\right)(a-b)^T\,\mathrm{d}\,\theta\ .$$

So, from the convexity of η we have that

$$C_{14}^h \geq 0\ . \tag{4.2}$$

As previously it is easy to see that

$$|C_{11}^h| \leq C\sqrt{h}\ ; \tag{4.3}$$

$$C_{12}^h = 0\ ; \tag{4.4}$$

$$|C_{13}^h + C_{15}^h| = \left|\sum_{k=1}^N \sum_{i\in I_{k-1}} \int_{(i-1)l}^{(i+1)l}\int_{(k-1)h}^{kh} f(w^h)\left(\eta_v(\tilde U_+^{hk})\phi^{ik} - \eta_v(\tilde U^h)\phi\right)\,\mathrm{d}\,x\ \mathrm{d}\,t\right|$$
$$\leq \left|\sum_{k=1}^N \sum_{i\in I_{k-1}} \int_{(i-1)l}^{(i+1)l}\int_{(k-1)h}^{kh} f(w^h)\eta_v(\tilde U_+^{hk})(\phi^{ik} - \phi)\,\mathrm{d}\,x\ \mathrm{d}\,t\right|$$
$$+ \left|\sum_{k=1}^N \sum_{i\in I_{k-1}} \int_{(i-1)l}^{(i+1)l}\int_{(k-1)h}^{kh} f(w^h)\phi\left(\eta_v(\tilde U_+^{hk}) - \eta_v(\tilde U^h)\right)\,\mathrm{d}\,x\ \mathrm{d}\,t\right|$$
$$= D_1^h + D_2^h\ .$$

It is easily seen that

$$(4.5) \qquad |D_1^h| \leq C\,h\ .$$

Furthermore by using the Lemma 4.1 we have immediately that
(4.6)

$$|D_2^h| \leq C \sum_{k=1}^{N} \sum_{i\in I_{k-1}} \left(\int_{(i-1)l}^{(i+1)l} \int_{(k-1)h}^{kh} |\phi|\, |\tilde{U}^h(x,kh+0) - \tilde{U}^h(x,t)|^2 \,\mathrm{d}\,x\ \mathrm{d}\,t \right)^{\frac{1}{2}} \leq C\sqrt{h}\ .$$

Then, by summing up (4.1)-(4.6), we have

$$I^h \geq -C\sqrt{h}$$

and in the limit we obtain the entropy inequality (2.4).

Finally let us show that the limit function w verifies the equation (1.7). From the scheme (2.14)–(2.16) we have

$$\partial_t w^h = \int_{-\infty}^{x} \partial_t u^h = \int_{-\infty}^{x} \partial_x \tilde{v}^h = \tilde{v}^h$$

in the weak sense. So the consistency is ensured if we show that

$$(\tilde{v}^h - v^h) \longrightarrow 0$$

for almost every $(x,t) \in \mathbb{R} \times (0,\infty)$.

In fact we have, for any $\phi \in C_0^\infty(\mathbb{R} \times (0,\infty))$

$$\left| \int\!\!\int |\tilde{v}^h - v^h| \phi \ \mathrm{d}\,x \right| = \left| \sum_{k=1}^{N} \iint_{S_k} | \int_{(k-1)h}^{t} f(w^h)\,\mathrm{d}\,\tau | \phi \ \mathrm{d}\,x\ \mathrm{d}\,t \right| \leq C\,h\ .$$

The function w is obviously Lipschitz continuous since, for almost every $(x,t) \in \mathbb{R} \times (0,\infty)$

$$\partial_x w = u$$

and

$$\partial_t w = v\ .$$

Moreover, from the second equation in (1.6) we have

$$\partial_t w - \partial_x \sigma(\partial_x w) + f(w) = 0$$

in the weak sense. The proof is complete. □

5. L^∞ estimates

In this section we establish the L^∞ bounds we need on the approximate solutions given by the scheme (2.14)–(2.16). Let us make the following natural assumption, easily verified for the problems (1.3) and (1.4):

(H_4) there exists a constant $L > 0$ such that

$$|f'(w)| \leq L$$

for all $w \in \mathbb{R}$.

Under this assumption we can state our main result.

Theorem 5.1. *Assume that the initial data u_0, v_0, w_0 are bounded measurable functions with compact support and $u_0 = \partial_x w_0$. Under the assumption (H_4) there exists a global weak entropy solution to the Cauchy problem (1.6)-(1.7)-(1.8).*

The result follows immediately from the Theorems 3.1, 3.2, 4.2 and the following L^∞ bounds on the approximate solutions.

Theorem 5.2. *Under the assumptions of the Theorem 5.1, for any $T > 0$ there exist some constants $M_T, N_T > 0$ and $h_T > 0$ such that*

$$|u^h| \leq M_T,\ |v^h| \leq N_T,\ |w^h| \leq N_T\ , \tag{5.1}$$

for a.e. $(x,t) \in \mathbb{R} \times (0,T)$ and any $h \in (0, h_T)$.

Proof. Since u_0, v_0 and w_0 are function with a compact support, u^h, v^h will also are with a compact support and, from the structure of the scheme, also w^h will be with a compact support.

Consider the discrete Riemann invariant (ξ^h, η^h), respectively $(\tilde{\xi}^h, \tilde{\eta}^h)$, given by (2.6) applied to (u^h, v^h), respectively $(\tilde{u}^h, \tilde{v}^h)$. Therefore, from (2.14)–(2.16), we obtain, for all $(x,t) \in S_k$

$$\begin{cases} \xi^h(x,t) = \tilde{\xi}^h(x,t) - \int_{kh}^t f\left(w^h(x,s)\right) \mathrm{d}\,s\ , \\ \eta^h(x,t) = \tilde{\eta}^h(x,t) - \int_{kh}^t f\left(w^h(x,s)\right) \mathrm{d}\,s\ . \end{cases} \tag{5.2}$$

Notice that, from (2.8) and (2.14), we have for all $(x,t) \in S_k$,

$$w^h(x,t) = w^h(x, kh+0) + \int_{kh}^{t} \tilde{v}^h(x,s)\,\mathrm{d}\,s\ .$$

Also, from (H$_4$), since $f(0) = 0$

$$|f(w)| \le L\,w \tag{5.3}$$

for any $w \in \mathbb{R}$.

Denoting by

$$\begin{aligned} M^k &= \max\left(\sup_x |\xi^h(x, kh+0)|, \sup_x |\eta^h(x, kh+0))\right) \\ &= \max\left(\sup_{S_k} |\tilde{\xi}^h(x,t)|, \sup_{S_k} |\tilde{\eta}^h(x,t)|\right), \end{aligned}$$

$$N^k = \sup_x |w^h(x, kh+0)|\ ,$$

we have, for $(x,t) \in S_k$,

$$\begin{aligned} |\tilde{v}^h| &\le M^k\ , \\ |g(\tilde{u}^h)| &\le M^k\ , \end{aligned}$$

where $g(u) = \int_0^u \sqrt{\sigma'(s)}\,\mathrm{d}\,s$, and

$$|w^h| \le N^k + h\,M^k\ .$$

Therefore, from (5.1) and (5.2), it follows, for $(x,t) \in S_k$,

$$|\xi^h| \le M^k + L\,h(N^k + h\,M^k)$$

and

$$|\eta^h| \le M^k + L\,h(N^k + h\,M^k)\ .$$

Hence

$$M^{k+1} \le (1 + L\,h^2)M^k + L\,h\,N^k$$

and

$$N^{k+1} \leq h\, M^k + N^k \ .$$

Then, by induction, the following inequality holds

$$\begin{aligned} M^k + N^k &\leq (M^0 + N^0)(1 + (L+1)h)^k \\ &\leq (M^0 + N^0) e^{(L+1)T} \ . \end{aligned}$$

The proof is complete. □

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