CONSERVATION LAWS WITH VANISHING NONLINEAR DIFFUSION AND DISPERSION

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Abstract. We study the limiting behavior of the solutions to a class of conservation laws with vanishing nonlinear diffusion and dispersion terms. We prove the convergence to the entropy solution of the first order problem under a condition on the relative size of the diffusion and the dispersion terms. This study is closely motivated by the pseudo-viscosity approximation introduced by Von Neumann in the 50's.

Key words and phrases: Nonlinear dispersive waves, Korteweg-de Vries equation, pseudo-viscosity, Burgers equation, shock waves, measure-valued solutions.

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1. Introduction

This paper is concerned with the convergence of smooth solutions $u = u^{\varepsilon,\delta}$ to the initial value problem for the nonlinear dispersive equation

$$\partial_t u + \partial_x f(u) = \varepsilon\, \partial_x \beta(\partial_x u) - \delta\, \partial_x^3 u, \quad (x,t) \in \mathbb{R} \times (0,\infty), \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0^{\varepsilon,\delta}(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where the parameters $\varepsilon > 0$ and $\delta > 0$ will converge to zero and $u_0^{\varepsilon,\delta}$ is an approximation of a given initial condition $u_0 : \mathbb{R} \to \mathbb{R}$. The flux $f = f(u)$ and the (degenerate) viscosity $\beta = \beta(\lambda)$ are given smooth functions satisfying certain assumptions to be listed shortly. When $\varepsilon = 0$, the equation (1.1) reduces to the generalized Korteweg-de Vries (KdV) equation, the original one corresponding to the special flux $f(u) = u^2/2$. On the other hand, when $\delta = 0$, we recover a nonlinear degenerate parabolic equation. The first case was extensively studied from mathematical and numerical standpoints; see for instance [13, 14, 6] and the references therein (see also Section 2 for some background). For $\delta = 0$ and under suitable assumptions, this equation was treated in [22] as a simplified model of the pseudo viscosity approximation proposed by von Neumann and Richtmyer [24] and then studied in [26, 25] for numerical purposes.

In this paper we prove, under an assumption on the relative size of the parameters ε and δ, that the sequence $\{u^{\varepsilon,\delta}\}$ converges to an entropy solution u of the first order hyperbolic conservation law

$$\partial_t u + \partial_x f(u) = 0 \tag{1.3}$$

as ε and $\delta \to 0$. Throughtout the paper, we assume the following conditions on f and β:

(A_1) There are two constants $C_1 > 0$ and $m > 1$ such that

$$|f'(u)| \le C_1\,(1 + |u|^{m-1}) \qquad \text{for all } u \in \mathbb{R}.$$

(A_2) The function $\beta = \beta(\lambda)$ is non-decreasing and satisfies $\beta(0) = 0$.

Observe that, from (A_2), we can deduce $\beta(\lambda)\lambda \ge 0$ for all $\lambda \in \mathbb{R}$. The following assumption will also be in use:

(B_1) There exist constants C_2, C_3, $N > 0$, and $r \ge 1$ such that

$$C_2\,|\lambda|^{3r} \le \beta(\lambda)\lambda \le C_3\,|\lambda|^{3r} \qquad \text{for all } |\lambda| \ge N.$$

The function β, therefore, is at least quadratic and could vanish on a bounded interval.

Two other assumptions on the function β will be required for certain results below:

(B_2) There are constants $C_4 > 0$ and $N > 0$ such that

$$\beta(\lambda)\lambda \ge C_4\,|\lambda|^3 \qquad \text{for all } |\lambda| \ge N.$$

(B_3) There are constants $C_5 > 0$ and $r \ge 1$ such that

$$\beta(\lambda)\lambda \ge C_5\,|\lambda|^{3r} \qquad \text{for all } \lambda \in \mathbb{R}.$$

Observe that both (B_2) and (B_3) are refinements to the lower bound estimate in (B_1), and that (B_2) is actually a consequence of (B_3). For our main theorems, we shall assume either (B_1), or (B_1) and (B_3). The condition (B_2) will be relevant only in the course of the discussion of the proofs. One typical example of a function β satisfying (A_2) and (B_1) (and also (B_2), (B_3)) is given by

$$\beta(\lambda) = |\lambda|^{3r-2}\lambda \quad (r \ge 1). \tag{1.4}$$

Recall that, for the Cauchy problem for (1.3), existence and uniqueness of an entropy solution were first proved by Kruzkov [16] (Section 2 below). The *vanishing viscosity limit*, which corresponds to $\delta = 0$ and $\varepsilon \to 0$, was studied by several authors in the special case of a linear (and therefore non-degenerate) diffusion term $\beta(\lambda) = \lambda$; this activity started with Oleinik's [23] and Kruzkov's [16] works. For more general viscosity coefficients and under the assumptions (A_2)–(B_1), the convergence was proved in [22].

The *vanishing dispersion limit* ($\varepsilon = 0$ and $\delta \to 0$) has also been widely investigated, see Lax-Levermore [20], [7, 30, 31], and the survey paper by Lax [19]. The main source of difficulty lies in the highly oscillating behavior of the solutions $u^{0,\delta}$, which do not converge in a strong topology.

The complete equation (1.1) arises, at least for the linear diffusion $\beta(\lambda) = \lambda$ (the so called Korteweg-de Vries-Burgers equation), as a model of the nonlinear propagation of dispersive and dissipative waves in many different physical systems. For related studies, let us refer to [1, 2, 3, 10, 11]. Travelling waves were studied in [2] and [9] and (partial) analytical results can be found in [15] and [6]. For a study of numerical approximations with similar features as (1.1), see [4].

The vanishing dispersive and diffusion limit (both ε and δ tend to zero) was first studied by Schonbek [27]. She assumed $\beta(\lambda) = \lambda$ and used the compensated compactness method introduced by Tartar [29]. In particular, she proved that the sequence $\{u^{\varepsilon,\delta}\}$ converges to a weak solution to (1.3) (but not necessarily the unique entropy one) under the assumption that either $\delta = 0(\varepsilon^2)$ for $f(u) = u^2/2$, or $\delta = 0(\varepsilon^3)$ for arbitrary subquadratic flux-functions f (i.e. take $m = 2$ in (A_1)). Our inequalities on ε and δ requires that viscosity dominates dispersion, which is expected since $\{u^{\varepsilon,\delta}\}$ are known not to converge to a weak solution of (1.3) (a fortiori to the entropy one) when dispersion effects are dominant.

Here we consider the general nonlinear equation (1.1) and we show that, under the assumption (B_1) and if $\delta \leq C\,\varepsilon^{\frac{5-m}{3-m}}$ $(m < 3)$, the solutions $u^{\varepsilon,\delta}$ converge to the entropy weak solution of (1.3) (Theorem 4.1). Under the assumptions (B_1) and (B_3) and when $\delta \leq C\,\varepsilon^{\frac{5-m}{r(5-m)-1}}$ $(m < 5 - 1/r,\ r \geq 1)$, the same result holds true (Theorem 4.2). Furthermore, our Theorem 4.3 below improves upon Schonbek's result in [27] in the case that $\beta(\lambda) = \lambda$ and f is subquadratic.

The main tool necessary to deal with these singular limits is the concept of entropy measure-valued solution to the equation (1.3). Measure-valued solutions were introduced by DiPerna [7] (see Section 2) to represent the weak-$\star$ limits of the solutions of (1.1), According to DiPerna's theory, the strong convergence of an approximate sequence follows once one obtains

(i) uniform bounds in $L^\infty(0,T;L^q(\mathbb{R}))$ for all $T > 0$ with $q > m$ (where m is given by (A_1));

(ii) weak consistency of the sequence with the entropy inequalities and the strong consistency with the initial data (Cf. the conditions (2.5) and (2.6) of Section 2).

Such a framework can be used to establish as well the convergence of difference schemes as shown in [5]. Observe however that, because of the oscillatory behavior of $u^{\varepsilon,\delta}$, no maximum principle is available for the equation (1.1) and the L^q (uniform) estimates are more natural. Therefore we actually use the L^q version of DiPerna's result derived by Szepessy in [28].

One basic estimate that does not require any restriction on the parameters ε and δ is the so-called energy estimate based on the simplest quadratic function u^2 (Lemma 3.1). The derivation of the necessary L^q estimates with larger exponents $(q > 2)$ is based on the nonlinearity of the viscosity coefficient β and requires suitable restriction on ε and δ.

The problem studied in this paper is motivated by the numerical method proposed by Von Neumann and Richtmyer [24] (see also [26]) and based on the pseudo-viscosity approximation $\beta(\lambda) = |\lambda|\lambda$. An important advantage of the nonlinear (quadratic) diffusion term $\beta(\lambda)$ is that it adds sharply localized viscosity near shocks and a small quantity (possibly zero if β vanishes for $\lambda < 0$) elsewhere. The pseudo-viscosity idea was proposed by Von Neumann to stabilize an unstable scheme such as the Lax-Wendroff scheme [21, 18, 8]: a minimal amount of numerical viscosity is added in order to prevent both the formation of unphysical (entropy violating) shocks and the generation of highly oscillatory approximate solutions.

2. Preliminaries

This section contains short background material on Young measures, entropy measure valued (mv) solutions (Section 2.1), and dispersive equations (Section 2.2).

2.1 Young Measures and Measure-Valued Solutions

Following Schonbek [27], we describe a representation theorem for Young measures associated with a sequence of uniformly bounded functions of L^q where $q \in (1, \infty)$ is fixed in the whole of this subsection. The corresponding setting in L^∞ was first established by Tartar [29].

Lemma 2.1. *Let $\{u_j\}$ be a uniformly bounded sequence in $L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$. Then there exists a subsequence $\{u_{j'}\}$ and a weak-$\star$ measurable, mapping $\nu : \mathbb{R} \times \mathbb{R}_+ \to \mathrm{Prob}\,(\mathbb{R})$ taking its values in the spaces of non-negative measures with total mass one (probability measures) such that, for all functions $g \in \mathcal{C}(\mathbb{R})$ satisfying*

$$g(u) = 0(|u|^r) \quad \text{as } |u| \to \infty \tag{2.1}$$

for some $r \in [0, q)$, the following limit representation holds

$$\lim_{j' \to \infty} \iint_{\mathbb{R} \times \mathbb{R}_+} g(u_{j'}(x,t))\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \iint_{\mathbb{R} \times \mathbb{R}_+} \int_{\mathbb{R}} g(\lambda)\,\mathrm{d}\nu_{(x,t)}(\lambda)\,\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t \tag{2.2}$$

for all $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$.

The measure-valued function $\nu_{(x,t)}$ is a Young measure associated with the sequence $\{u_j\}$. The following result reveals the connection between the structure of ν and the strong convergence.

Lemma 2.2. *Suppose that ν is a Young measure associated with a sequence $\{u_j\}$ that is uniformly bounded in $L^\infty(\mathbb{R}_+;\ L^q(\mathbb{R}))$. For $u \in L^\infty(\mathbb{R}_+;\ L^q(\mathbb{R}))$, the following statements are equivalent:*

(i) $\lim_{j\to\infty} u_j = u$ *in $L^\infty(\mathbb{R}_+; L^r_{\mathrm{loc}}(\mathbb{R}))$ for some $r \in [1, q)$;*
(ii) $\nu_{(x,t)} = \delta_{u(x,t)}$ *for almost every $(x,t) \in \mathbb{R}\times\mathbb{R}_+$.*

In (ii) above, the notation $\delta_{u(x,t)}$ is used for the Dirac mass defined by

$$\iint_{\mathbb{R}\times\mathbb{R}_+} < \delta_{u(x,t)}, g(\cdot) > \ \phi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \iint_{\mathbb{R}\times\mathbb{R}_+} g(u(x,t))\,\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

for all $g \in \mathcal{C}(\mathbb{R})$ satisfying (2.1) and all $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$. Following DiPerna [7] and [28], we now define the measure-valued solutions to the first order Cauchy problem

$$\partial_t u + \partial_x f(u) = 0, \tag{2.3}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{2.4}$$

Definition 2.1. *Assume that f satisfies the growth condition (2.1) and $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$. A Young measure ν associated with a sequence $\{u_j\}$, which is assumed to be uniformly bounded in $L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$, is called an entropy mv solution to the problem (2.3)-(2.4) if*

$$\partial_t\langle\nu_{(\cdot)}, |\lambda - k|\rangle + \partial_x\langle\nu_{(\cdot)}, \operatorname{sgn}(\lambda-k)(f(\lambda)-f(k))\rangle \le 0 \tag{2.5}$$

in the distributional sense for all $k \in \mathbb{R}$ and, for all interval $I \subseteq \mathbb{R}$,

$$\lim_{T\to 0+}\frac{1}{T}\int_0^T\int_I \langle\nu_{(x,t)}, |\lambda - u_0(x)|\rangle\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.6}$$

Remark. A function $u \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}) \cap L^q(\mathbb{R}))$ is an entropy weak solution to (2.3)-(2.4) in the sense of Kružkov [16] if and only if the Dirac measure $\delta_{u(\cdot)}$ is an entropy mv solution. In the case $p = +\infty$, existence and uniqueness of such solutions were proved in [16]. The following results of entropy mv solutions were proved in [28]: Theorem 2.3 states that entropy mv solutions are actually Kruzkov's solutions. Theorem 2.4 states that the problem has a unique L^q solution.

Theorem 2.3. *Assume that f satisfies* (2.1) *and $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$. Suppose that ν is an entropy mv solution to* (2.3)-(2.4). *Then there exists a function $w \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}) \cap L^q(\mathbb{R}))$ such that*

$$\nu_{(x,t)} = \delta_{w(x,t)} \tag{2.7}$$

for almost every $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

Theorem 2.4. *Assume that f satisfies* (2.1) *and $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$. Then there exists a unique entropy solution $u \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}) \cap L^q(\mathbb{R}))$ to* (2.3)-(2.4) *which, moreover, satisfies*

$$\|u(\cdot,t)\|_{L^r(\mathbb{R})} \leq \|u_0\|_{L^r(\mathbb{R})} \tag{2.8}$$

for almost every $t \in \mathbb{R}_+$ and for all $r \in [1,q]$. Moreover the measure-valued mapping $\nu_{(x,t)} = \delta_{u(x,t)}$ is the unique entropy mv solution of the same problem.

Combining Theorems 2.3 and 2.4 and Lemma 2.2 we obtain our main convergence tool.

Corollary 2.5. *Assume that f satisfies* (2.1) *and $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$. Let $\{u_j\}$ be a sequence of functions that are uniformly bounded in $L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$ for $q \geq 1$, and let ν be a Young measure associated with this sequence. If ν is an entropy mv solution to* (2.3)-(2.4), *then*

$$\lim_{j\to\infty} u_j = u \quad \text{in } L^\infty(\mathbb{R}_+; L^r_{loc}(\mathbb{R})) \tag{2.9}$$

for all $r \in [1,q)$, where $u \in L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$ is the unique entropy solution to (2.3)-(2.4).

2.2 Nonlinear Dispersive Equations.

Consider the fully nonlinear, KdV-type equation in one space variable

$$\partial_t u + F(u, \partial_x u, \partial_x^2 u, \partial_x^3 u) = 0 \tag{2.10}$$

for $(x,t) \in \mathbb{R} \times (0,T)$ together with the Cauchy data

$$u(x,0) = u_0(x) \qquad \text{for } x \in \mathbb{R}. \tag{2.11}$$

In recent years this class of problems has been extensively studied; see for instance [13, 14, 15] and the references cited therein. In particular, under suitable assumptions, these equations enjoy a gain of regularity of the solutions with respect to their initial data (Cf. [6, 12, 17] and the references above). We observe however that a complete theory of global existence remains to be developped. Presenting a complete review is far beyond the aim of this presentation. Here we need only recall a result –that is sufficient to deal with (1.1)– of local existence and uniqueness ot the smooth solutions to the equation (2.10). More details and proofs can be found in the paper by Craig-Kappeler-Strauss [6] (see also [15]).

Using the notation $U = (u_0, u_1, u_2, u_3)$, the assumptions on the smooth function $F = F(U)$ are as follows:

(H_1) There exists $\delta > 0$ such that

$$\partial_{u_3} F(U) \geq \delta > 0 \qquad \text{for all } U \in \mathbb{R}^4;$$

(H_2) $\partial_{u_2} F(U) \leq 0 \qquad$ for all $U \in \mathbb{R}^4$.

The equation (1.1) does satisfy these hypotheses provided $\delta > 0$ and (A_2) holds.

Theorem 2.6 (Uniqueness). *Let $T > 0$ be fixed and assume F satisfies* (H_1)-(H_2). *For any $u_0 \in H^7(\mathbb{R})$ there is at most one solution $u \in L^\infty((0,T); H^7(\mathbb{R}))$ to the problem* (2.10)-(2.11).

Theorem 2.7 (Existence). *Assume F satisfies* (H_1)-(H_2). *Let N be an integer ≥ 7 and let $C_0 > 0$ be a given constant. There exists a time $T > 0$, depending only on C_0, such that for all $u_0 \in H^N(\mathbb{R})$ with $\|u_0\|_{H^7} \leq C_0$, there exists at least one solution $u \in L^\infty((0,T); H^N(\mathbb{R}))$ to the problem* (2.10)-(2.11).

Observe that the setting above based on a Sobolev norm of relatively high order ($H^7(\mathbb{R})$) is the optimal result provided by the current techniques of analysis of dispersive equations. The *global* existence of smooth solutions to (1.1)-(1.2) appears to be an open problem. In the following we tacitly restrict attention to a time $T_* \in (0, \infty]$ chosen such that the problem (1.1)-(1.2) is well-posed in the strip $\mathbb{R} \times (0, T_*)$.

3. A Priori Estimates

In this section we consider a sequence $\{u^{\varepsilon,\delta}\}$ of smooth solutions of (1.1)-(1.2) that vanish at infinity. We assume also that the initial data $\{u_0^{\varepsilon,\delta}\}$ are smooth functions with compact support, and are uniformly bounded in $L^1(\mathbb{R}) \cap L^q(\mathbb{R})$ for a suitable $q > 1$. In what follows, whenever it does not lead to confusion, we omit the indices ε and δ. Similarly all constants are denoted by $C, C_1, \cdots$. We begin with the natural energy estimate based the quadratic function $u^2/2$. (Arbitrary convex functions could not be easily used here because of the dispersive term.)

Lemma 3.1. *For any $T > 0$, it holds that*

$$\int_{\mathbb{R}} u^2(x,T)\,\mathrm{d}x + 2\,\varepsilon \int_0^T \int_{\mathbb{R}} \beta(u_x(x,t))u_x(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}} u_0^2(x)\,\mathrm{d}x. \tag{3.1}$$

Proof. We multiply (1.1) by u and integrate in space. Integrating by parts we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\int_{\mathbb{R}} u^2\,\mathrm{d}x + \int_{\mathbb{R}} u f'(u)u_x\,\mathrm{d}x = -\varepsilon \int_{\mathbb{R}} \beta(u_x)u_x\,\mathrm{d}x - \delta \int_{\mathbb{R}} u\,u_{xxx}\,\mathrm{d}x. \tag{3.2}$$

The second terms on both sides of (3.2) vanish identically, since we can write these terms in a conservative form:

$$u\,f'(u)u_x = (G(u))_x$$

with $G' = u\,f'(u)$ and

$$u\,u_{xxx} = (u\,u_{xx} - \frac{1}{2}u_x^2)_x.$$

The estimate (3.1) follows therefore from (3.2). $\Box$

At this stage, in view of the assumption (A_1) and Lemma 3.1, we have a uniform control of u in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}))$ and $\varepsilon\,\beta(u_x)u_x$ in $L^1((0,T)\times\mathbb{R})$ for every $T > 0$. Our next aim is to derive a (non-uniform) estimate of u in L^∞ norm, which will be uniform in ε but not in δ. We first provide a second energy-type estimate.

Lemma 3.2. *Let F be defined by $F'(u) = f(u)$. For every $T > 0$, we have*

$$\begin{aligned}
&\frac{\delta}{2}\int_{\mathbb{R}} u_x^2(x,T)\,\mathrm{d}x - \int_{\mathbb{R}} F(u(x,T))\,\mathrm{d}x + \varepsilon\,\delta \int_0^T \int_{\mathbb{R}} u_{xx}^2\beta'(u_x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{\delta}{2}\int_{\mathbb{R}} u_{0,x}^2(x)\,\mathrm{d}x - \int_{\mathbb{R}} F(u_0(x))\,\mathrm{d}x + \varepsilon \int_0^T \int_{\mathbb{R}} f'(u)\beta(u_x)u_x\;\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.3}$$

Proof. Multiplying (1.1) by $f(u) + \delta u_{xx}$ we obtain the equality

$$\begin{aligned}
0 &= \int_{\mathbb{R}} \{F(u)_t + H(u)_x - \varepsilon f(u)\beta(u_x)_x + \delta f(u)u_{xxx}\}\,\mathrm{d}x \\
&+ \int_{\mathbb{R}} \{\delta\,u_{xx}u_t + \delta u_{xx}f(u)_x - \delta u_{xx}^2\,\beta'(u_x) + \delta u_{xxx}u_{xx}\}\,\mathrm{d}x
\end{aligned}$$

where we have set $H' = f f'$. After integration by parts in space, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int F(u)\,\mathrm{d}x - \frac{\delta}{2}\,u_x^2\ \mathrm{d}x + \int_{\mathbb{R}} \varepsilon f'(u)\beta(u_x)u_x\ \mathrm{d}x - \varepsilon\,\delta \int_{\mathbb{R}} u_{xx}^2\beta'(u_x)\,\mathrm{d}x = 0,$$

which gives the desired conclusion. □

Using the bound for $\sqrt{\delta}\ u_x$ in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}))$ that follows from Lemma 3.2, we are now able to estimate u in the L^∞ norm.

Lemma 3.3. *If $m < 5$ in the assumption* (A_1), *there exists a constant $C > 0$ such that*

$$|u(x,t)| \le C\,\delta^{-\frac{1}{5-m}} \tag{3.4}$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

Proof. In view of (A_2) and for all $u \in \mathbb{R}$, we have

$$|F(u)| \le C\,(1 + |u|^{m+1}).$$

The main idea is to use (3.3) to control $\delta\,u_x^2$ in terms of $F(u)$, the latter being estimated from the above growth condition. We deduce from (3.3) that

$$\begin{aligned}
&\frac{\delta}{2}\int_{\mathbb{R}} u_x^2(x,T)\,\mathrm{d}x + \varepsilon\,\delta \int_0^T \int_{\mathbb{R}} u_{xx}^2\,\beta'(u_x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \frac{\delta}{2}\int_{\mathbb{R}} u_{0,x}^2(x)\,\mathrm{d}x + \int_{\mathbb{R}} \big(F(u(x,T)) - F(u_0(x))\big)\,\mathrm{d}x + \varepsilon \int_0^T \int f'(u)\beta(u_x)u_x\ \mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\ +\ C\,\|u(\cdot,T)\|_{L^\infty(\mathbb{R})}^{m-1} \left(\|u(\cdot,T)\|_{L^2(\mathbb{R})}^2 + \varepsilon \int_0^T \int_{\mathbb{R}} \beta(u_x)u_x\ \mathrm{d}x\,\mathrm{d}t \right),
\end{aligned}$$

where we used (A_2) as well. (We do not explicitly write the terms involving the initial data as it is assumed to be a smooth function.) Therefore in view of (3.1), we arrive at

$$\frac{\delta}{2}\int_{\mathbb{R}} u_x^2(x,T)\,\mathrm{d}x + \varepsilon\,\delta \int_0^T \int_{\mathbb{R}} u_{xx}^2\,\beta'(u_x)\,\mathrm{d}x\,\mathrm{d}t \le C\ + C\,\|u(\cdot,T)\|_{L^\infty(\mathbb{R})}^{m-1}.$$

Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
|u(x,t)|^2 &\le 2\int_{-\infty}^x |u(y,t)u_x(y,t)|\,\mathrm{d}y \\
&\le \frac{2}{\sqrt{\delta}}\|u(\cdot,t)\|_{L^2(\mathbb{R})}\sqrt{\delta}\,\|u_x(\cdot,t)\|_{L^2(\mathbb{R})} \\
&\le \frac{C}{\sqrt{\delta}}\|u_0\|_{L^2(\mathbb{R})}\left(1 + \|u(\cdot,t)\|_{L^2(\mathbb{R})}^{m-1}\right)^{1/2}.
\end{aligned}$$

Hence, for all $t > 0$,

$$\|u(\cdot,t)\|^4_{L^\infty(\mathbb{R})} \le \frac{C}{\delta}\left(1 + \|u(\cdot,t)\|^{m-1}_{L^\infty(\mathbb{R})}\right). \tag{3.5}$$

Now, since $m < 5$, the growth of the left hand side of (3.5) exceeds the growth of the right hand side. So we can check that (3.5) implies

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} &\le \max\left(1, \left(\frac{2C}{\delta}\right)^{\frac{1}{5-m}}\right)\\ &\le C\,\delta^{-\frac{1}{5-m}}.\end{aligned}$$

Namely, setting $y = \|u(\cdot,t)\|_{L^\infty}$, we have for $y > 0$

$$y^4 \le \frac{C}{\delta}\left(1 + y^{m-1}\right). \tag{3.6}$$

If $y \le 1$ we have the conclusion. Otherwise, suppose we had $y > \left(\frac{2C}{\delta}\right)^{\frac{1}{5-m}}$, then we would deduce that

$$y^4 > \frac{2C}{\delta}\,y^{m-1} > \frac{C}{\delta}\left(1 + y^{m-1}\right),$$

which would contradict the inequality (3.6). □

In view of the proof of Lemma 3.3 we also state the following result:

Lemma 3.4. *For any $T > 0$ we have*

$$\frac{1}{2}\int_{\mathbb{R}} u_x^2(x,T)\,\mathrm{d}x + \varepsilon\int_0^T\int_{\mathbb{R}} u_{xx}^2\beta'(u_x)\,\mathrm{d}x\ \mathrm{d}t \le C\,\delta^{-\frac{4}{5-m}}. \tag{3.7}$$

Finally we derive uniform bounds in $L^\infty((0,T);\ L^q(\mathbb{R}))$ with $q \le 5$. The following result provides a uniform estimate in the Lebesgue space $L^5(\mathbb{R})$ (so improving upon the L^2 bound in Lemma 3.1) by taking advantage of the nonlinearity property of β (for large values of $\lambda = u_x$) as stated in (B_2).

Proposition 3.5. *Assume that the assumption (B_2) holds and $m < 3$ in the assumption (A_1). There exists a constant $C > 0$, which depends only on the initial data, such that, for all δ and ε small enough and for every $T > 0$, we have*

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|^5_{L^5(\mathbb{R})} \le C\left(T + \varepsilon^{-1}\delta^{\frac{3-m}{5-m}}\right). \tag{3.8}$$

Proof. Set $\eta(u) = |u|^5$. Multiply (1.1) by $\eta'(u)$ and integrate on $\mathbb{R} \times (0,T)$. It follows easily that

$$\begin{aligned}(3.9)\qquad &\int_{\mathbb{R}} \eta(u(x,T))\,\mathrm{d}x + \varepsilon \int_0^T \int_{\mathbb{R}} \eta''(u)\beta(u_x)u_x \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \eta(u_0(x))\,\mathrm{d}x - \frac{\delta}{2}\int_0^T \int_{\mathbb{R}} \eta'''(u)u_x^3 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

On the other hand, according to (B_2), there exists a constant C such that for all $\lambda \in \mathbb{R}$

$$|\lambda|^3 \le C\,(1 + \beta(\lambda)\lambda).$$

Using the latter in the energy estimate (3.1), we are able to control the second term in the right hand side of (3.9):

$$\begin{aligned}\left|\int_0^T \int \eta'''(u)u_x^3 \,\mathrm{d}x\,\mathrm{d}t\right| &\le C_1 \int_0^T \int_{\mathbb{R}} |u|^2|u_x|^3 \,\mathrm{d}x\,\mathrm{d}t \\ &\le C_2 \int_0^T \int_{\mathbb{R}} |u|^2\big(1 + \beta(u_x)u_x\big) \,\mathrm{d}x\,\mathrm{d}t \\ &\le C_3\,T \sup_{t\in(0,T)} \|u(t)\|_{L^2(\mathbb{R})} + \frac{C}{\varepsilon}\|u\|^2_{L^\infty(\mathbb{R}\times(0,T))} \\ &\le C\,T + C\,\left(1 + \varepsilon^{-1}\delta^{\frac{3-m}{5-m}}\right).\end{aligned}$$

where the latter inequality follows from the L^∞ bound in Lemma 3.3. Returning to (3.9), we obtain (3.8). □

Under the stronger assumption (B_3), we have a sharper estimate.

Proposition 3.6. *Assume that* (B_3) *holds for a given* $r \ge 1$ *and that* $m < q \equiv 5 - 1/r$ *in* (A_1)*. There exists a constant* $C > 0$*, which depends only on the initial data, such that, for all* δ *and* ε *small enough and every* $T > 0$*,*

$$(3.10)\qquad \sup_{t\in(0,T)} \|u(\cdot,t)\|^q_{L^q(\mathbb{R})} \le C\,\left(T^{1-1/r} + \varepsilon^{-1/r}\delta^{-\frac{q-m}{5-m}}\right).$$

Proof. As in the proof of Proposition 3.5, we consider the formula (3.9) but now with $\eta(u) = |u|^q$ with $q = 5 - 1/r$. The assumption (B_3) yields, using (3.1) and (3.4),

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}} \eta'''(u) u_x^3 \, \mathrm{d}x \, \mathrm{d}t \right| &\leq C \int_0^T \int_{\mathbb{R}} |u|^{q-3} |u_x|^3 \, \mathrm{d}x \, \mathrm{d}t \\ &\leq C \left(\int_0^T \int_{\mathbb{R}} |u|^{r'(q-3)} \, \mathrm{d}x \, \mathrm{d}t \right)^{1/r'} \left(\int_0^T \int_{\mathbb{R}} \beta(u_x) u_x \, \mathrm{d}x \, \mathrm{d}t \right)^{1/r} \\ &\leq C \, \varepsilon^{-1/r} \|u\|_{L^\infty(\mathbb{R})}^{1/r} \, T^{1/r'} \sup_{t \in (0,T)} \|u(\cdot,t)\|_{L^2(\mathbb{R})}^{2/r} \\ &\leq C \, T^{1/r'} \, \varepsilon^{-1/r} \delta^{-\frac{1}{r(5-m)}}, \end{aligned}$$

where $1/r + 1/r' = 1$ and since $r'(q-3) = 2 + r'/r$. Now (3.10) follows from the formula (3.9). □

Observe that the estimates (3.8) and (3.10) hold only thanks to the nonlinear form of the viscosity term. Note that when $\delta = 0$, these estimates reduce to bound that are uniform in ε, which is expected for conservation laws with viscosity but no dispersion. On the other hand, most of our estimates blow-up when taking $\varepsilon = 0$ and provide no control of L^q norms. Observe however that the L^2 bound in Lemma 3.1 is uniform for δ arbitrary and $\varepsilon \to 0$. Loosely speaking, this is also the case of the estimate in Proposition 3.6 if r could be chosen to be $r = \infty$, in which case (3.10) becomes a uniform L^5 estimate. Such a choice of r is not allowed however, cf. (B_3).

For the sake of completeness, we finally state an analogous estimate for linear diffusions which was proved by Schonbek [27].

Proposition 3.7. *Let $\beta(\lambda) = \lambda$ and take $m = 2$ in (A_1). For any $T > 0$, there exists a constant $C_T > 0$, which depends only on the initial data, such that for $\delta \leq \nu \, \varepsilon^3$*

$$\sup_{t \in (0,T)} \|u(\cdot,t)\|_{L^4(\mathbb{R})} \leq C_T. \tag{3.11}$$

4. Convergence Results

In Section 3, we have established several uniform bounds for the sequence $\{u^{\varepsilon,\delta}\}$ of solutions to the Cauchy problem (1.1)-(1.2) under certain assumptions on the functions

f, β, and the parameters ε and δ. Assume again that the initial data $u_0^{\varepsilon,\delta}$ are smooth with compact support and that there exists a limiting function $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$ and a suitable $q > 1$ (specified below) such that, if $\delta = O(\varepsilon)$,

$$\lim_{\varepsilon \to 0} u_0^{\varepsilon,\delta} = u_0 \quad \text{in } L^1(\mathbb{R}) \cap L^q(\mathbb{R}). \tag{4.1}$$

Returning to the proofs of Section 3, it is not hard to see that the following conditions on the initial data are sufficient for the estimates therein to hold uniformly with respect to a class of initial data:

$$\|u_0^{\varepsilon,\delta}\|_{L^2(\mathbb{R})} + \|u_0^{\varepsilon,\delta}\|_{L^q(\mathbb{R})} \delta^{1/2} \, \|u_{0,x}^{\varepsilon,\delta}\|_{L^2(\mathbb{R})} \leq C.$$

In this section we prove the strong convergence of the sequence $u^{\varepsilon,\delta}$.

Theorem 4.1. *Assume that* (B_1) *holds and* $m < 3$ *in* (A_1)*. Let* $u^{\varepsilon,\delta}$ *be a sequence of smooth solutions to* (1.1)-(1.2) *on* $\mathbb{R} \times (0,T)$ *(for a given* $T > 0$*), which vanish at infinity and are associated with initial data satisfying* (4.1) *with* $q = 5$*. If there is a constant* $C > 0$ *such that* $\delta \leq C\,\varepsilon^{\frac{5-m}{3-m}}$*, then the (whole) sequence* $u^{\varepsilon,\delta}$ *converges to a function* $u \in L^\infty((0,T); L^5(\mathbb{R}))$*, which is the unique entropy solution to* (2.3)-(2.4).

Theorem 4.2. *Assume that* (B_1) *and* (B_3) *hold and* $m < 5 - 1/r$ *in* (A_1)*. Let* $u^{\varepsilon,\delta}$ *be a sequence of smooth solutions to* (1.1)-(1.2) *on* $\mathbb{R} \times (0,T)$ *(for a given* $T > 0$*), which vanish at infinity and are associated with initial data satisfying* (4.1) *with* $q = 5 - 1/r$*. If there is a constant* $C > 0$ *such that* $\delta \leq C\,\varepsilon^{\frac{5-m}{r(5-m)-1}}$*, then the (whole) sequence converges to a function* $u \in L^\infty((0,T); L^q(\mathbb{R}))$*,* $q = 5 - 1/r$*, which is the unique entropy solution to* (2.3)-(2.4).

Let us give an analogous statement in the case $\beta(\lambda) = \lambda$, which improve upon [27] (Cf. Theorem 5.1 therein).

Theorem 4.3. *Assume* $m = 2$ *in* (A_1) *and let* $\beta(\lambda) = \lambda$*. If* $\delta \leq C\,\varepsilon^3$*, the whole sequence* $u^{\varepsilon,\delta}$ *of solutions of* (1.1)-(1.2) *converges to a function* $u \in L^\infty((0,T); L^4(\mathbb{R}))$*, which is the unique entropy solution to* (2.3)-(2.4).

Similar convergence results can be proven for the case $m = 3$ or for some special flux-functions along the lines of what was done in [27] (Cf. Sections 4 and 5 therein). Theorems 4.1-4.2 follow easily by simply using the following general result and the L^q bounds derived in Section 3, Proposition 3.5 and Proposition 3.6 respectively.

Theorem 4.4. *Assume that* (B_1) *holds. Let* $u^{\varepsilon,\delta}$ *be a sequence of smooth solutions to* (1.1)-(1.2) *on* $\mathbb{R}\times(0,T)$ *(for a given* $T>0$*) associated with initial data satisfying* (4.1)*. If the sequence is uniformly bounded in* $L^\infty((0,T);L^q(\mathbb{R}))$ *for* $q>m$ *and* $\delta=o(\varepsilon^{1/r})$*, then the (whole) sequence converges to a function* $u\in L^\infty((0,T);L^q(\mathbb{R}))$*, which is the unique entropy solution to* (2.3)-(2.4).

Proof of Theorem 4.4. First of all let us establish that, for any convex function $\eta=\eta(u)$ such that η', η'', η''' are uniformly bounded on $\mathbb{R}$, we have

$$\Lambda^{\varepsilon,\delta}=\partial_t\eta(u^{\varepsilon,\delta})+\partial_x Q(u^{\varepsilon,\delta})\rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}\times\mathbb{R}_+), \tag{4.2}$$

where $Q'=f'\eta'$. To begin with, observe that

$$\begin{aligned}\Lambda^{\varepsilon,\delta}&=\varepsilon(\eta'(u)\beta(u_x))_x-\varepsilon\eta''(u)\beta(u_x)u_x-\delta(\eta'(u)u_{xx})_x+\delta\eta''(u)u_xu_{xx}\\&=T_1+T_2+T_3+T_4.\end{aligned}$$

To estimate T_1, we use the assumption (B_1), which implies $|\beta(\lambda)|\le C\left(1+|\lambda|^{3r-1}\right)$ for all λ. For any given $\theta\in C_0^\infty(\mathbb{R}\times(0,T))$, $\theta\ge0$, and for $p=\frac{3r}{3r-1}>1$ (p' being the conjugate exponent of p), we get

$$\begin{aligned}\langle T_1,\theta\rangle&=\left|\iint\varepsilon\,\theta_x\eta'(u)\beta(u_x)\,\mathrm{d}x\ \mathrm{d}t\right|\\&\le C\,\varepsilon\|\theta_x\|_{L^1(\mathbb{R}\times(0,T))}+C\,\varepsilon\|\theta_x\|_{L^{p'}(\mathbb{R}\times(0,T))}\left(\iint|u_x|^{p(3r-1)}\right)^{1/p},\end{aligned}$$

so using the derivative estimate in (3.1) and (B_1) again:

$$\langle T_1,\theta\rangle\le C\left(\varepsilon+\varepsilon^{1/3r}\right)\le C\,\varepsilon^{1/3r}. \tag{4.3}$$

The second term T_2 is nonpositive, namely

$$\langle T_2,\theta\rangle=-\iint\varepsilon\eta''(u)\beta(u_x)u_x\theta\ \mathrm{d}x\,\mathrm{d}t\le0. \tag{4.4}$$

To estimate T_3, we use the energy estimate (3.1) and the fact that β is at least quadratic.

We write

$$
\begin{aligned}
\langle T_3, \theta\rangle &= \delta \iint_{\mathbb{R}\times(0,T)} \theta_x \eta'(u) u_{xx}\, \mathrm{d}x\, \mathrm{d}t \\
&= \delta \iint_{\mathbb{R}\times(0,T)} \theta_x \left((\eta'(u)u_x)_x - \eta''(u) u_x^2 \right) \mathrm{d}x\, \mathrm{d}t \\
&\le -\delta \iint_{\mathbb{R}\times(0,T)} \theta_{xx} \eta'(u) u_x + \delta \iint_{\mathbb{R}\times(0,T)} |\theta_x|\, u_x^2\, \mathrm{d}x\, \mathrm{d}t \\
&\le C\,\delta + C\,\delta \left(\iint_{\operatorname{supp}\theta} |u_x|^{3r}\, \mathrm{d}x\, \mathrm{d}t \right)^{1/3r} + C\,\varepsilon^{-2/3r},
\end{aligned}
$$

where $\operatorname{supp}\theta$ denotes the support of the function in $\mathbb{R}\times(0,T)$, thus

$$
\langle T_3, \theta\rangle \le C\,\delta \left(1 + C\,\varepsilon^{-2/3r} + \varepsilon^{-1/3r}\right) \le C \left(1 + C\,\varepsilon^{-2/3r}\right). \tag{4.5}
$$

Finally we deal with T_4 as follows:

$$
\begin{aligned}
|\langle T_4, \theta\rangle| &= \left| \delta \iint \eta''(u) u_x\, u_{xx} \theta\, \mathrm{d}x\, \mathrm{d}t \right| \\
&= \left| \delta \iint \theta_x \eta''(u) u_x^2 + \delta \iint \theta \eta'''(u) \frac{1}{2} u_x^3 \right| \\
&\le C\,\delta \left(\iint_{\operatorname{supp}\theta} |u_x|^2 + |u_x|^3\, \mathrm{d}x\, \mathrm{d}t \right),
\end{aligned}
$$

so

$$
|\langle T_4, \theta\rangle| \le C\,\delta \left(\varepsilon^{-2/3r} + \varepsilon^{-1/r}\right) \le C\,\varepsilon^{-1/r}. \tag{4.6}
$$

Therefore, if $\delta = o(\varepsilon^{1/r})$, (4.2) follows immediately from the estimate (4.3)-(4.6). To apply Corollary 2.5 we have to show that (2.5) and (2.6) are satisfied for a Young measure ν associated with the sequence $u^{\varepsilon,\delta}$. It is a standard matter to deduce, for all convex entropy pairs,

$$
\partial_t \langle \nu_{(\cdot)}, \eta(\lambda)\rangle + \partial_x \langle \nu_{(\cdot)}, Q(\lambda)\rangle \le 0
$$

from the convergence property (4.2). The inequality (2.5) for all $k \in \mathbb{R}$ then follows by using a standard regularization of the function $|u-k|$. Concerning the initial data and in order to establish (2.6), we now combine the entropy inequalities and the weak consistency property as was suggested by DiPerna [7]. We follow the detailled arguments given in [28].

Consider the function $g(\lambda) = |\lambda|^r$ for $1 < r < \min(2,q)$, and set

$$
\begin{aligned}
G(\lambda, \lambda_0) &\equiv g(\lambda) - g(\lambda_0) - g'(\lambda_0)(\lambda - \lambda_0) \\
&\geq \frac{r(r-1)}{2} \, \frac{(\lambda - \lambda_0)^2}{(1 + |\lambda| + |\lambda_0|)^{2-r}}.
\end{aligned} \tag{4.7}
$$

Let $I \subseteq \mathbb{R}$ be be a closed and bounded interval. Using the Jensen inequality, the Cauchy-Schwartz inequality, and the above convexity inequality (4.7), it is easily checked that

$$
\begin{aligned}
&\frac{1}{T} \int_0^T \int_I \langle \nu_{(x,t)}, |\lambda - u_0(x)| \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\leq C_I \left(\frac{1}{T} \int_0^T \int_I \langle \nu_{(x,t)}, G(\lambda, u_0(x)) \rangle \, \mathrm{d}x \; \mathrm{d}t \right)^{1/2}.
\end{aligned} \tag{4.8}
$$

Let $\{\psi_n\} \in C_0^\infty(\mathbb{R})$ be a sequence of test-functions such that

$$
\lim_{u \to \infty} \psi_n = g'(u_0) = \quad \text{in } L^{r'}(\mathbb{R}),
$$

where $1 = 1/r + 1/r'$. Using the uniform bound in L^q available for the sequence $\{u_0^{\varepsilon,\delta}\}$, we get

$$
\begin{aligned}
&\int_0^T \int_I \langle \nu_{(x,t)}, G(\lambda, u_0(x)) \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_0^T \int_{\mathbb{R}} \langle \nu_{(x,t)}, u_0 - \lambda \rangle \psi_n \; \mathrm{d}x \; \mathrm{d}t + T \int_{\mathbb{R} \quad I} |u_0|^2 \, \mathrm{d}x \\
&+ 2\,T \, \|u_0\|_{L^r(\mathbb{R})} \|g'(u_0) - \psi_n\|_{L^{r'}(\mathbb{R})}.
\end{aligned} \tag{4.9}
$$

Taking an increasing sequence of compact sets K_i covering $\mathbb{R}$, i.e. such that $I \subset K_1 \subset K_2 \subset \dots$ and $\bigcup_{i=1}^\infty K_i = \mathbb{R}$, we have

$$
\int_0^T \int_I \langle \nu_{(x,t)}, G(\lambda, u_0(x)) \rangle \, \mathrm{d}x \, \mathrm{d}t \leq \int_0^T \int_{K_i} \langle \nu_{(x,t)}, G(\lambda, u_0(x)) \rangle \, \mathrm{d}x \, \mathrm{d}t,
$$

which, together with (4.9) where I is replaced by K_i, yields

$$
\begin{aligned}
&\frac{1}{T} \int_0^T \int_I \langle \nu_{(x,t)}, Q(\lambda, u_0(x)) \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \frac{1}{T} \int_0^T \int_{\mathbb{R}} \langle \nu_{(x,t)}, u_0(x) - \lambda \rangle \psi_n \; \mathrm{d}x \, \mathrm{d}t + 2 \, \|u_0\|_{L^r(\mathbb{R})} \|g'(u_0) - \psi_n\|_{L^{r'}(\mathbb{R})},
\end{aligned} \tag{4.10}
$$

since

$$\lim_{i\to\infty}\int_{\mathbb{R}\quad K_i} |u_0|^2\,\mathrm{d}x \;=\; 0.$$

Therefore, in view of (4.8) and (4.10), the strong consistency property (2.6) will be established if we show that

$$\lim_{T\to 0+}\frac{1}{T}\int_0^T\int_{\mathbb{R}}\langle \nu_{(x,t)}, u_0(x)-\lambda\rangle\psi_n\;\mathrm{d}x\;\mathrm{d}t \le 0 \tag{4.11}$$

for all $n\in\mathbb{N}$. By definition of the Young measure (Cf. (2.2)), we have

$$\frac{1}{T}\int_0^T\int_{\mathbb{R}}\langle \nu_{(x,t)}, u_0(x)-\lambda\rangle\,\psi_n\;\;\mathrm{d}x\,\mathrm{d}t = \lim_{\varepsilon,\delta\to 0}\frac{1}{T}\int_0^T\int\big(u_0(x)-u^{\varepsilon,\delta}(x,t)\big)\,\psi_n\;\mathrm{d}x\,\mathrm{d}t.$$

On the other hand, we can write

$$\begin{aligned}
&= \int_{\mathbb{R}}\big(u_0(x)-u_0^{\varepsilon,\delta}(x,t)\big)\,\psi_n(x)\,\mathrm{d}x - \frac{1}{T}\int_0^T\int_{\mathbb{R}}\left(\int_0^t \partial_s\, u^{\varepsilon,\delta}(x,s)\,\mathrm{d}s\right)\psi_n(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\equiv A+B.
\end{aligned}$$

The term A tends to zero as $\varepsilon\to 0$ in view of the weak consistency property (4.1). Furthermore, by arguing as in the derivation of (4.3), we have

$$\begin{aligned}
B &= -\frac{1}{T}\int_0^T\int_{\mathbb{R}}\left(\int_0^t(-\partial_x f(u^{\varepsilon,\delta})+\varepsilon\partial_x(\beta(u_x^{\varepsilon,\delta}))-\delta\partial_x^3 u^{\varepsilon,\delta}\right)\psi_n(x)\,\mathrm{d}x\,\mathrm{d}t\\
&= -\frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t\big(f(u^{\varepsilon,\delta})\partial_x\psi_n - \varepsilon\beta(u_x^{\varepsilon,\delta})\partial_x\psi_n + \delta u^{\varepsilon,\delta}\partial_x^3\psi_n\big)\;\mathrm{d}s\;\mathrm{d}x\,\mathrm{d}t\\
&\le C_n\,T.
\end{aligned}$$

This leads to the inequality (4.11). The proof of Theorem 4.4 is completed. □

The proof of Theorem 4.3 is based on slightly modified estimates in the inequalities (4.5) and (4.6), which can be derived by arguing as in [27]. The details of the proof are omitted.

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