Convergence to Equilibrium for the Relaxation Approximations of Conservation Laws

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Abstract

We study the Cauchy problem for 2×2 semilinear and quasilinear hyperbolic systems with a singular relaxation term. Special comparison and compactness properties are established by assuming the subcharacteristic condition. Therefore we can prove the convergence to equilibrium of the solutions of these problems as the singular perturbation parameter tends to zero. This research was strongly motivated by the recent numerical investigations of S. Jin and Z. Xin on the relaxation schemes for conservation laws.

1. Introduction

In this paper we are interested to the relaxation behaviour of the following system of hyperbolic conservation laws with a singular perturbation source

$$(1.1) \qquad \begin{cases} \partial_t u + \partial_x v = 0 \ , \\ \partial_t v + \partial_x \sigma(u) = -\frac{1}{\varepsilon}\left(v - f(u)\right) \qquad (\varepsilon > 0), \end{cases}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$. Here σ, f are some given smooth functions such that $\sigma'(u) \geq \nu$ $(\nu > 0)$, $f(0) = 0$.

The system (1.1) is equivalent to the one-dimensional perturbed wave equation

$$(1.2) \qquad \partial_{tt} w - \partial_x \sigma(\partial_x w) = -\frac{1}{\varepsilon}\left(\partial_t w + f(\partial_x w)\right)$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$, where w is given by

$$(1.3) \qquad \partial_t w = -v \quad , \quad \partial_x w = u \ .$$

The problem (1.1), or (1.2), can be considered as a singular perturbation problem and the solutions are expected to converge, as ε tends to zero, to some solutions of the equilibrium (or reduced) equation

$$(1.4) \qquad \partial_t u + \partial_x f(u) = 0 \ , \quad v = f(u)$$

or, respectively,

$$(1.5) \qquad \partial_t w + f(\partial_x w) = 0 \ .$$

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Relaxation systems are important in many physical situations. They arise, for example, in gases not in thermodynamic equilibrium [29] , kinetic theory [5], [6], [22], chromatography [24], river flows, traffic flows and more general waves [30].

The relaxation limit for 2×2 systems of conservation laws was first analyzed by Liu [19], who justified some nonlinear stability criteria for diffusion waves, expansion waves and travelling waves.

The main stability criterion can be (formally) derived by using the Chapman-Enskog expansion, which in the case of system (1.1) reads

$$v^\varepsilon = f(u^\varepsilon) + \varepsilon v_1^\varepsilon + O(\varepsilon^2) \tag{1.6}$$

Introducing (1.6) in the system (1.1), we obtain, up to $O(\varepsilon)$

$$v_1^\varepsilon = -\left(\sigma'(u^\varepsilon) - (f'(u^\varepsilon))^2\right)\partial_x u^2$$

which leads to the first–order correction to the relaxation limit (1.4) by dropping all the higher order terms

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x \left(\beta(u)\partial_x u\right)\ ,\ v = f(u) - \varepsilon\beta(u)\partial_x u\ , \tag{1.7}$$

for $\beta(u) = \sigma'(u) - (f'(u))^2$.

The first equation in (1.7) will be of parabolic type under the following stability condition

$$|f'(u)|^2 \leq \sigma'(u)\ . \tag{1.8}$$

It turns out that this condition preserves also the causality principle for the original system. In fact the characteristic speeds for (1.1) and for the equilibrium equation (1.4) are respectively $\pm\sqrt{\sigma'(u)}$ and $f'(u)$. So, we expect again to find condition (1.8), also called the subcharacteristic condition, verified in the relaxation limit.

As long as the solutions for the limit equation are smooth, the stability, in a suitable norm, and the convergence of the perturbed problems, as the relaxation parameter ε tends to zero, can be completely justified, see [28], [25], [32] and references therein.

In fact, in [28], the authors were concerned with the slight different problem

$$\partial_{tt} w - \alpha\partial_{xx} w = -\frac{1}{\varepsilon}\left(A\partial_t w + B\partial_x w + C\right) \tag{1.9}$$

where α, A, B, C may depend on t, x and u, $\alpha > 0$, and there holds the analogous condition

$$A > 0 \quad ,\ |B| < A\sqrt{\alpha} \tag{1.10}$$

In [25] the stability and convergence properties for the perturbed solutions, under a suitable version of condition (1.10), were extended to various multidimensional systems of wave equations by using energy methods. Other results for general first order hyperbolic systems were later obtained in [32].

Unfortunately, in the general case the solutions of the equilibrium equation (1.4) may become discontinuous in a finite time. The rigorous justification of the relaxation approximation to equilibrium solutions containing shock waves was started in [7] and [8], where the authors considered some special models, including the system (1.1), by using the methods of compensated compactness [27]. They proved that uniformly bounded sequences of perturbed solutions converge strongly to some weak equilibrium solutions as the relaxation parameter tends to zero, when the subcharacteristic condition (1.8) is verified, for initial data close to the equilibrium. Observe that, for system (1.1), the uniform boundedness of relaxing solutions was recovered just under the supplementary assumption that the function f is constant out of a fixed bounded set.

On the contrary one of the main results of this paper is to show that *uniform estimates are just a consequence of the stability condition (1.8)*, without any further requirement on f.

First, in Sections 3 – 6, we shall consider the semilinear case $\sigma(u) = \alpha u$ ($\alpha > 0$). This model was proposed in [15] for numerical purposes. It supplies a new and powerful approximation to equilibrium conservation laws, which possesses some important properties: simplicity and generality, finite speed of propagation, regularity of the approximating solutions. It was shown in [15] that, from a numerical point of view, general systems of higher dimensions can be treated in the same way as the scalar one–dimensional equation. The main advantage of the linear hyperbolic structure of the relaxing systems lies in fact in the possibility to avoid the resolution of the Riemann problems in the approximation of conservation laws.

Here we first give some rigorous results on this problem. In Section 3 we show that there exists a constant $\alpha_0 > 0$, only depending on the L^∞ norm of the initial data and on the flux function f, such that for $\alpha \geqq \alpha_0$ the solutions of the relaxing problems are uniformly bounded (independently on ε). This follows by using some comparison tools already developed for general weakly coupled systems of quasilinear hyperbolic equations in [14] (see Section 2 for the semilinear case). In fact it is possible to show that the stability condition (1.8) is actually equivalent to the quasimonotone condition (see Definition 2.2), which yields the monotone properties of the problem. Observe also that, for this problem, it is not possible to apply the standard theory of the invariant regions of [9], usually employed to establish L^∞ estimates.

Therefore, in Section 4, we show that the *same* stability condition implies also the compactness properties of the approximating sequence: uniform estimates on the L^1 modulus of continuity in space and equicontinuity in time with values in L^1. This yields the strong convergence to the equilibrium, shortly proved in Section 5 without using the compensated compactness methods.

The Section 6 is devoted to the study of the entropy conditions for the limit functions. As well–known [17], the unique "good" weak solution to problem

(1.4) has to verify some supplementary inequalities, the so–called *entropy conditions*. Checking of these conditions is found to be quite delicate for the singular perturbation term on the right-hand side. It turns out that in general some slight supplementary assumptions on the flux function f are needed to establish entropy inequalities for the limit function.

Finally, in Section 7, we establish similar uniform L^∞ bounds for the quasilinear problem (1.1), by assuming only the stability (or subcharacteristic) condition (1.8). The convergence to the equilibrium is made, for small initial data, just by applying the results of [8].

Observe that in the quasilinear case we need to use the compensated compactness framework, as in [8], which only gives the almost convergence everywhere of the relaxing sequence. Moreover in the present case the limit function verifies only a finite number of entropy conditions. It is an open problem to show that this limit function is actually an entropy solution (and in this case the unique one) in the sense of Kružkov [16].

Let us note that our approach shares some ideas with the kinetic approximations and the related fluid dynamical limits. In particular we found some analogies with the artificial kinetic approximations to conservation laws given in [13], [21] and [18]. On the other hand we expect that some of our ideas could be applied fruitfully to the study of the fluid dynamical limits for discrete Boltzmann models, as for example the Broadwell equations [4], [3], [31]. It could be equally interesting to apply these approximations to the investigation of the qualitative properties of the entropy solutions to the (equilibrium) conservation laws. Finally let us point out that related numerical schemes, which incorporate the monotonicity properties of the approximation (1.1), are studied in [1].

2. Comparison Results

Let us recall some general results concerning semilinear (diagonal) hyperbolic systems. The proofs are quite classical and will be omitted (see for example [14] for a complete theory of the quasilinear weakly coupled case).

Consider the Cauchy problem

$$\partial_t u_i + \lambda_i \partial_x u_i = g_i(U) \ , \quad i = 1, \dots, N \ , \tag{2.1}$$

$$U(x,0) = U^0(x) \ , \tag{2.2}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$. Here $U = (u_1, \dots, u_N)$, $\lambda_i \in \mathbb{R}$ $(i = 1, \dots, N)$ and $G(U) = (g_1(U), \dots, g_N(U))$ is a given locally Lipschitz continuous function; $U^0 = (u_1^0, \dots, u_N^0) \in L^\infty(\mathbb{R})^N$.

Definition 2.1. *A function* $U \in L^\infty(\mathbb{R} \times (0,T))^N$ $(T > 0)$ *is a (weak) solution of the Cauchy problem* (2.1)–(2.2) *if for all* $\varphi \in C_0^\infty(\mathbb{R} \times (0,T))$ *and every* $i = 1, \dots, N$*,*

$$\iint [u_i(\partial_t\varphi + \lambda_i\partial_x\varphi) + g_i(U)\varphi]\ \mathrm{d}x\,\mathrm{d}t = 0 \tag{2.3}$$

and for any interval $I \subseteq \mathbb{R}$

$$\lim_{T\to 0}\frac{1}{T}\int_0^T\int_I |u_i(x,t) - u_i^0(x)|\,\mathrm{d}x\,\mathrm{d}t = 0\ . \tag{2.4}$$

THEOREM 2.2. *For any* $U^0 \in L^\infty(\mathbb{R})^N$, *there is* $T > 0$ *(depending only on* $\|U^0\|_\infty$*) such that in* $\mathbb{R}\times(0,T)$ *there exists a unique (weak) solution* U *of* (2.1)–(2.2) *and* $U \in C\left([0,T];\, L^1_{\mathrm{loc}}(\mathbb{R})^N\right)$. *Moreover only two possibilities occur: either* $U \in L^\infty\left(\mathbb{R}\times(0,T)\right)^N$ *for any* $T > 0$ *or there exists* $T^* < +\infty$ *such that, for any* $T < T^*$, U *is defined on* $\mathbb{R}\times(0,T)$ *and*

$$\lim_{T\to T^*_-}\|U\|_{L^\infty(\mathbb{R}\times(0,T))^N} = +\infty\ .$$

Define

$$L = L(\lambda) = \max_{1\le i\le N}|\lambda_i|\ .$$

For any given vectors $A,\ B \in \mathbb{R}^N$, set $A \le B$ if $a_i \le b_i$ $(i = 1,\dots,N)$; if $A \le B$, denote the corresponding interval by $[A,B]$. For any fixed interval $[A,B]$ (with non empty interior) set

$$M = M(A,B,G) = \max_{1\le i,j\le N}\|\partial_{u_j}g_i\|_{L^\infty([A,B])}\ .$$

Also, for each interval $I = (\alpha,\beta) \subseteq \mathbb{R}$, we set $\tau = \frac{\beta-\alpha}{2L}$.

THEOREM 2.3. *Let* U *and* $\tilde U$ *be two weak solutions of the Cauchy problem* (2.1)–(2.2) *in* $\mathbb{R}\times(0,T)$ $(T>0)$, *for the right-hand side* G *and the initial data* U^0, $\tilde G$ *and* $\tilde U^0$ *respectively. Let* $[A,B] \subseteq \mathbb{R}^N$ *be an interval with non empty interior such that* $U,\ \tilde U \in [A,B]$. *Then for each interval* $(\alpha,\beta)\subseteq\mathbb{R}$ *and for all* $t \in (0,\min(\tau,T))$, *we have*

$$\sum_{i=1}^N\int_{\alpha+tL}^{\beta-tL}|u_i(x,t) - \tilde u_i(x,t)|\,\mathrm{d}x \le \sum_{i=1}^N\int_\alpha^\beta |u_i^0(x) - \tilde u_i^0(x)|\,\mathrm{d}x$$
$$+\sum_{i=1}^N\int_0^t \mathrm{d}s\int_{\alpha+sL}^{\beta-sL}\mathrm{sgn}\ (u_i(x,s) - \tilde u_i(x,s))\left(g_i(U(x,s)) - \tilde g_i(\tilde U(x,s))\right)\mathrm{d}x\ .$$

To obtain comparison properties we have to restrict our analysis to quasi-monotone interactions.

Definition 2.4. *Let Ω be a convex subset of $\mathbb{R}^N$. A function $G : \Omega \longrightarrow \mathbb{R}^N$ is quasimonotone (non decreasing) if each component g_i of G is non decreasing in u_j for $i \neq j$.*

The quasimonotonicity is a necessary and sufficient condition for a flow generated by the ordinary differential equation $\dot{U} = G(U)$, to preserve the partial ordering in $\mathbb{R}^n$ (see [26]) and it is also well–known for reaction–diffusion flows [23], [11]. For weakly coupled systems of quasilinear hyperbolic equations it was studied in [14]. In the present case we have the following useful comparison property.

Theorem 2.5. *Let U and $\tilde{U}$ be two weak solutions of problem* (2.1)–(2.2) *in $\mathbb{R} \times (0,T)$ for the initial data U^0 and $\tilde{U}^0$ respectively. Let $[A,B] \in \mathbb{R}^N$ be an interval (with non empty interior) such that*

$$G \text{ is quasimonotone in } [A,B] \ , \tag{2.5}$$

$$U,\, \tilde{U} \in [A,B] \text{ almost everywhere in } \mathbb{R} \times (0,T) \ . \tag{2.6}$$

If $U^0 \leq \tilde{U}^0$ for almost every $x \in \mathbb{R}$, then $U \leq \tilde{U}$ for almost every $(x,t) \in \mathbb{R} \times (0,T)$.

3. $\mathbf{L^\infty}$ Estimates in the Semilinear Case

In this section we shall establish a uniform supremum norm bound for the solutions $(u^\varepsilon, v^\varepsilon)$ of the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x v = 0 \ , \\ \partial_t v + \alpha \partial_x u = -\frac{1}{\varepsilon}(v - f(u)) \quad (\varepsilon > 0) \ , \end{cases} \tag{3.1}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$, $\alpha > 0$, for the initial data

$$u^\varepsilon(x,0) = u_0^\varepsilon(x) \quad , \quad v^\varepsilon(x,0) = v_0^\varepsilon(x), \tag{3.2}$$

for $x \in \mathbb{R}$. Let us consider the following assumptions:

(H_1) f is a locally Lipschitz continuous function with $f(0) = f'(0) = 0$;

(H_2) the functions $(u_0^\varepsilon, v_0^\varepsilon)$ are uniformly bounded in $L^\infty(\mathbb{R})^2$ by

$$N_0 = \max\left(\sup_{\varepsilon>0} ||u_0^\varepsilon||_\infty, \sup_{\varepsilon>0} ||v_0^\varepsilon||_\infty\right) \ .$$

Moreover the sequence $(u_0^\varepsilon, v_0^\varepsilon)$ converges in $L^1_{\rm loc}(\mathbb{R})^2$ to some $(\overline{u}_0, \overline{v}_0) \in L^\infty(\mathbb{R})^2$, as $\varepsilon \to 0^+$.

The following supplementary assumption on the initial data will be also of use in the sequel:

(H_3) for any bounded closed interval $K \subseteq \mathbb{R}$, it holds

$$\lim_{\varepsilon \to 0^+} \|v_0^\varepsilon - f(u_0^\varepsilon)\|_{L^1(K)} = 0 \ .$$

Finally let us denote, for any $N > 0$

$$F(N) := \sup_{|\xi| \leq N} |f(\xi)| \ , \tag{3.3}$$

$$B(N) := 2N + F(2N) \tag{3.4}$$

and

$$M(N) := \sup_{|\xi| \leq B(N)} |f'(\xi)| \ . \tag{3.5}$$

Next we give our global existence and boundedness result for problem (3.1)–(3.2).

Theorem 3.1. *Assume* (H_1) *and* (H_2). *For any* $N_0 > 0$ *and* $\varepsilon > 0$, *if*

$$\sqrt{\alpha} > M(N_0) \ , \tag{3.6}$$

then there exists a unique globally bounded solution $(u^\varepsilon, v^\varepsilon)$ *to* (3.1)–(3.2) *in* $C([0,\infty);\ L^1_{\rm loc}(\mathbb{R})^2)$ *and we have*

$$\|v^\varepsilon \pm \sqrt{\alpha}\, u^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,\infty))} \leq \sqrt{\alpha}\, B(N_0) \ . \tag{3.7}$$

Moreover there holds the subcharacteristic inequality

$$|f'(u^\varepsilon(x,t))| < \sqrt{\alpha} \tag{3.8}$$

for all $\varepsilon > 0$ *and for almost every* $(x,t) \in \mathbb{R} \times (0,\infty)$.

Proof: Assume first $(u_0^\varepsilon, v_0^\varepsilon) \in C_0^1(\mathbb{R})^2$. This implies, by the standard regularity theory for the semilinear hyperbolic problems, that $(u^\varepsilon, v^\varepsilon) \in C^1(\mathbb{R} \times (0,T))^2$ for some $T > 0$. For simplicity we shall omit the index ε in the proof.

To use the results of Section 2, we put our system in a diagonal form. Let

$$r_+ = v - \sqrt{\alpha}\, u \quad , \quad r_- = -v - \sqrt{\alpha}\, u \tag{3.9}$$

be the Riemann invariants with respect to the characteristic fields $\lambda_\pm = \pm\sqrt{\alpha}$.

They verify the system

$$
\left\{
\begin{array}{l}
\partial_t r_+ - \sqrt{\alpha}\,\partial_x r_+ = -\frac{1}{\varepsilon}\,G(r) \\
\partial_t r_- + \sqrt{\alpha}\,\partial_x r_- = \frac{1}{\varepsilon}\,G(r)
\end{array}
\right.
\tag{3.10}
$$

where

$$
G(r) = \frac{r_+ - r_-}{2} - f\left(-\frac{r_+ + r_-}{2\sqrt{\alpha}}\right) \ , \tag{3.11}
$$

with the initial conditions

$$
\begin{array}{c}
r_+(x,0) = v_0(x) - \sqrt{\alpha}\,u_0(x) = r_+^0(x) \ , \\
r_-(x,0) = -v_0(x) - \sqrt{\alpha}\,u_0(x) = r_-^0(x) \ .
\end{array}
\tag{3.12}
$$

From Theorem 2.2, the problem (3.10)–(3.12) has a (maximal) local solution on some interval $\mathbb{R} \times (0, \overline{T})$, $\overline{T} > 0$, and the same is true for the original problem (3.1)–(3.2) setting

$$
v = \frac{r_+ - r_-}{2} \quad , \quad u = -\frac{r_+ + r_-}{2\sqrt{\alpha}} \ . \tag{3.13}
$$

Therefore to conclude the proof, it is enough to establish the inequalities (3.7)–(3.8).

Let us observe now that the right–hand side in (3.10) is quasimonotone, according to Definition 2.4, in any domain $\Omega \subseteq \mathbb{R}^2$ such that

$$
\sup_{(r_+, r_-) \in \Omega} \left| f'\left(-\frac{r_+ + r_-}{2\sqrt{\alpha}}\right)\right| \leq \sqrt{\alpha} \ . \tag{3.14}
$$

In fact, from (3.11), the condition of quasimonotonicity holds if

$$
\frac{\partial G}{\partial r_-} \leq 0 \quad , \quad \frac{\partial G}{\partial r_+} \geq 0 \ , \tag{3.15}
$$

which actually yields (3.14).

Let $\delta > 0$ be such that

$$
\sqrt{\alpha} \geq M(N_0) + \delta \ .
$$

Let

$$
T_0^\delta = \sup \left\{ T \in (0, \overline{T}) \mid \sqrt{\alpha} \geq \|f'(u^\varepsilon)\|_{L^\infty(\mathbb{R} \times (0,T))} + \frac{\delta}{2} \right\} \ .
$$

Since $\sqrt{\alpha} \geq \sup_{|\xi| \leq N_0} |f'(\xi)| + \delta$, then $T_0^\delta > 0$. Moreover, from (3.14), system (3.10) is quasimonotone in the strip $\mathbb{R} \times (0, T_0^\delta)$.

Consider now the system of ordinary differential equations

$$\begin{cases} \dot{p}_+ = -\frac{1}{\varepsilon}\, G(p) \ , \\ \dot{p}_- = \frac{1}{\varepsilon}\, G(p) \ , \end{cases} \tag{3.16}$$

with $p_+(0) = p_-(0) = R_0$. It is easy to check that the *global* solution of this system is given by

$$p_\pm(t) = R_0 \pm (1 - e^{-\frac{t}{\varepsilon}}) f\left(-\frac{R_0}{\sqrt{\alpha}}\right) \ . \tag{3.17}$$

Let $(p_+^\pm, p_-^\pm)$ be the solution corresponding to the initial data

$$R_0^\pm := \pm(1+\sqrt{\alpha})N_0 \ . \tag{3.18}$$

From (3.17)-(3.18) we obtain that for all $t \geq 0$, the solutions verify (3.14), since

$$\left| f'\left(-\frac{p_+^\pm + p_-^\pm}{2\sqrt{\alpha}}\right)\right| = \left| f'\left(\mp\frac{1+\sqrt{\alpha}}{\sqrt{\alpha}} N_0\right)\right| \leq M(N_0) \ .$$

Hence, using Theorem 2.3, we have

$$p_\pm^-(t) \leq r_\pm(x,t) \leq p_\pm^+(t) \tag{3.19}$$

for every $(x,t) \in \mathbb{R} \times (0, T_0^\delta)$.

This yields

$$|r_\pm(x,t)| = |v^\varepsilon(x,t) \pm \sqrt{\alpha}\, u^\varepsilon(x,t)| \leq \sqrt{\alpha} B(N_0).$$

Therefore, from (3.13) we have

$$|u^\varepsilon(x,t)| \leq B(N_0) \ , \tag{3.20}$$

$$|v^\varepsilon(x,t)| \leq \sqrt{\alpha}\, B(N_0) \ , \tag{3.21}$$

for every $(x,t) \in \mathbb{R} \times (0, T_0^\delta)$.

Since $\sqrt{\alpha} \geq M(N_0) + \delta$, then for all $(x,t) \in \mathbb{R} \times (0, T_0^\delta)$ we have

$$\sqrt{\alpha} \geq |f'(u^\varepsilon(x,t))| + \delta$$

Therefore, by standard continuation arguments, $\overline{T} = T_0^\delta = +\infty$ and (3.7)–(3.8) hold in $\mathbb{R} \times (0, \infty)$.

Finally we extend our result to general initial data by approximating in $L^1_{\mathrm{loc}}(\mathbb{R})$ the functions u_0, v_0 by smooth functions u_0^ν, v_0^ν with compact support and applying Theorem 2.3.

4. Compactness Estimates

In this section we establish some uniform stability estimates in $L^1_{\rm loc}$ for the solutions of system (3.1). Our main goal is to establish compactness properties of the approximating sequences: uniform estimates on the L^1_{loc} modulus of continuity in space and equicontinuity in time with values in L^1_{loc}.

THEOREM 4.1. *Under the assumptions of Theorem 3.1, let $\sqrt{\alpha} > M(N_0)$ and $\varepsilon > 0$. Let $(u^\varepsilon, v^\varepsilon)$ and $(\tilde u^\varepsilon, \tilde v^\varepsilon)$ be the solutions of (3.1)–(3.2) for the initial data $(u_0^\varepsilon, v_0^\varepsilon)$ and $(\tilde u_0^\varepsilon, \tilde v_0^\varepsilon)$ respectively. Then, for any interval $(a,b) \subseteq \mathbb{R}$ and for every $t \geq 0$ we have*

$$\begin{aligned}
&\int_a^b |u^\varepsilon(x,t) - \tilde u^\varepsilon(x,t)| + |v^\varepsilon(x,t) - \tilde v^\varepsilon(x,t)| \,\mathrm{d}x \\
&\leq \frac{(1+\sqrt{\alpha})^2}{\sqrt{\alpha}} \int_{a-\sqrt{\alpha}\,t}^{b+\sqrt{\alpha}\,t} |u_0^\varepsilon(x) - \tilde u_0^\varepsilon(x)| + |v_0^\varepsilon(x) - \tilde v_0^\varepsilon(x)| \,\mathrm{d}x\ .
\end{aligned} \tag{4.1}$$

Proof: As previously we shall omit the index ε in this proof. Let r_+, r_- $(\tilde r_+, \tilde r_-)$ be the Riemann invariants for u, v $(\tilde u, \tilde v)$ defined by (3.9). Then, by Theorem 2.3, we have the following inequality for any interval $(a,b) \subseteq \mathbb{R}$ and almost every $t \geq 0$

$$\begin{aligned}
&\int_a^b |r_+(x,t) - \tilde r_+(x,t)| + |r_-(x,t) - \tilde r_-(x,t)| \,\mathrm{d}x \\
&\quad \leq \int_{a-\sqrt{\alpha}\,t}^{b+\sqrt{\alpha}\,t} |r_+^0(x) - \tilde r_+^0(x)| + |r_-^0(x) - \tilde r_-^0(x)| \,\mathrm{d}x \\
&\quad - \int_0^t \int_{a-\sqrt{\alpha}\,(t-s)}^{b+\sqrt{\alpha}\,(t-s)} \frac{1}{\varepsilon} \left(\mathrm{sgn}\,(r_+(x,s) - \tilde r_+(x,s)) - \mathrm{sgn}\,(r_-(x,s) - \tilde r_-(x,s)) \right) \cdot \\
&\quad \cdot \left(G(r_+(x,s), r_-(x,s)) - G(\tilde r_+(x,s), \tilde r_-(x,s)) \right) \mathrm{d}x\, \mathrm{d}s\ .
\end{aligned}$$

Observe now that the second term is nonpositive. In fact, set

$$\begin{aligned}
&I := \left(\mathrm{sgn}\,(r_+ - \tilde r_+) - \mathrm{sgn}\,(r_- - \tilde r_-) \right) \left(G(r_+, r_-) - G(\tilde r_+, \tilde r_-) \right) \\
&= \left(\mathrm{sgn}\,(r_+ - \tilde r_+) - \mathrm{sgn}\,(r_- - \tilde r_-) \right) \left[\left(\int_0^1 \frac{\partial G}{\partial r_+}(\theta r_+ + (1-\theta)\tilde r_+, r_-)\,\mathrm{d}\theta \right) (r_+ - \tilde r_+) \right. \\
&\quad \left. + \left(\int_0^1 \frac{\partial G}{\partial r_-}(\tilde r_+, \gamma r_- + (1-\gamma)\tilde r_-)\,\mathrm{d}\gamma \right) (r_- - \tilde r_-) \right] .
\end{aligned}$$

From (3.8), (3.15) holds true and just by a direct check one verifies that $I \geq 0$. The conclusion follows by (3.9) and (3.13).

Corollary 4.2. *Under the assumptions of Theorem 3.1, let $\sqrt{\alpha} > M(N_0)$ and $\varepsilon > 0$. Let $(u^\varepsilon, v^\varepsilon)$ be the solution of the Cauchy problem (3.1)–(3.2). Then for any interval $(a,b) \subseteq \mathbb{R}$ and for any $T > 0$ there exists a continuous nondecreasing function $\omega \in C([0,T))$, not depending on ε and with $\omega(0) = 0$, such that, for every $t \in (0,T)$*

$$\int_a^b |u^\varepsilon(x+h,t) - u^\varepsilon(x,t)| + |v^\varepsilon(x+h,t) - v^\varepsilon(x,t)| \, \mathrm{d}x \le \omega(|h|) \ , \tag{4.2}$$

for any $|h| \le h_0$ ($h_0 > 0$).

Corollary 4.3. *Under the assumptions of Corollary 4.2, suppose that the initial data $(u_0^\varepsilon, v_0^\varepsilon)$ are locally of bounded variation. Then there exists a constant $C = C(\alpha)$, not depending on ε, such that for any interval $(a,b) \subseteq \mathbb{R}$ and for every $t \ge 0$*

$$\|(u^\varepsilon(\cdot,t), v^\varepsilon(\cdot,t)\|_{BV((a,b))} \le C \|(u_0^\varepsilon, v_0^\varepsilon)\|_{BV\left((a-\sqrt{\alpha}\,t, b+\sqrt{\alpha}\,t)\right)} \ . \tag{4.3}$$

As a consequence, thanks to the Fréchet-Kolmogorov compactness theorem (see for instance [2]), we have that the sequence $(u^\varepsilon, v^\varepsilon)$ lies in a compact set of L^1_{loc} for all $t \ge 0$. To establish the equicontinuity in time of the sequence u^ε we need the following important lemma, originally due to Kružkov ([16]).

Lemma 4.4. *Let u be a measurable bounded function in $[0,T] \times (-R-h_0, R+h_0)$ $(T, R, h_0 > 0)$ and $\omega_R \in C([0,h_0))$ a nondecreasing function, with $\omega_R(0) = 0$, such that, for every $t \in (0,T)$, $|h| < h_0$*

$$\int_{-R-h_0}^{R+h_0} |u(x+h,t) - u(x,t)| \, \mathrm{d}x \le \omega_R(|h|) \ . \tag{4.4}$$

Moreover, let the following condition be satisfied

$$\left| \int_{-R}^{R} (u(x,t+\tau) - u(x,t)) \varphi(x) \, \mathrm{d}x \right| \le C_R \tau \|\varphi\|_{C^2} \tag{4.5}$$

for any t, $t+\tau \in (0,T)$ $(\tau > 0)$, for any $\varphi \in C_0^2([-R,R])$ and some constant $C_R > 0$. Then for any $0 \le t \le t+\tau \le T$ there holds

$$\int_{-R}^{R} |u(x,t+\tau) - u(x,t)| \, \mathrm{d}x \le \tilde{\omega}_R(\tau) \ , \tag{4.6}$$

where

$$\tilde{\omega}_R(\tau) = C_R \min_{|h| \le h_0} \left[|h| + \omega_R(|h|) + \frac{\tau}{h^2} \right] \ .$$

We can state now.

THEOREM 4.5. *Under the assumptions of Theorem 3.1, let $\sqrt{\alpha} > M(N_0)$ and $\varepsilon > 0$. Let $(u^\varepsilon, v^\varepsilon)$ be the solution of the Cauchy problem (3.1)–(3.2). Then for any interval $(a,b) \subseteq \mathbb{R}$ and for any $T > 0$ there exists a continuous nondecreasing function $\tilde{\omega} \in C([0,T])$, not depending on ε and with $\tilde{\omega}(0) = 0$, such that for every $0 \le t \le t + \tau \le T$ $(\tau > 0)$ it holds*

$$\int_a^b |u^\varepsilon(x,t+\tau) - u^\varepsilon(x,t)|\,\mathrm{d}x \le \tilde{\omega}(\tau)\ . \tag{4.7}$$

Proof: First observe that inequality (4.5) is verified by u^ε. In fact, for any $\varphi \in C_0^2([a,b])$

$$\begin{aligned}
&\left| \int_a^b (u^\varepsilon(x,t+\tau) - u^\varepsilon(x,t))\varphi(x)\,\mathrm{d}x \right| \\
&\quad = \left| \int_a^b \left(\int_t^{t+\tau} \partial_t u^\varepsilon(x,s)\,\mathrm{d}s \right) \varphi(x)\,\mathrm{d}x \right| \\
&\quad = \left| \int_a^b \left(-\int_t^{t+\tau} \partial_x v^\varepsilon(x,s)\,\mathrm{d}s \right) \varphi(x)\,\mathrm{d}x \right| \\
&\quad = \left| \int_a^b \left(\int_t^{t+\tau} v^\varepsilon(x,s)\,\mathrm{d}s \right) \partial_x\varphi(x)\,\mathrm{d}x \right| \\
&\quad \le N\sqrt{\alpha}(b-a)\tau ||\varphi||_{C^1([a,b])}\ .
\end{aligned}$$

Therefore the conclusion follows directly by Lemma 4.4 and Corollary 4.2.

The equicontinuity in time of the sequence of v^ε is more delicate and it follows from the next result.

THEOREM 4.6. *Under the assumptions of Theorem 4.5, for any $0 < \nu < T$ there exists a continuous nondecreasing function $\tilde{\omega}^\nu \in C([0, T-\nu))$, not depending on ε and with $\tilde{\omega}^\nu(0) = 0$, such that for every $\nu \le t \le t+\tau \le T$ $(\tau > 0)$ there holds*

$$\int_a^b |v^\varepsilon(x,t+\tau) - v^\varepsilon(x,t)|\,\mathrm{d}x \le \tilde{\omega}^\nu(\tau)\ . \tag{4.8}$$

Proof: As in the proof of Theorem 4.5 we have just to establish inequality (4.5) uniformly for the sequence $\{v^\varepsilon\}$. We can assume, without loss of generality, the sequence is smooth. Then, from the second equation in (3.1) we can write

$$v(x,t) = e^{-\frac{t}{\varepsilon}} v_0(x) + \int_0^t \left[\frac{1}{\varepsilon} f(u(x,s)) - \alpha(\partial_x u)(x,s)\right] e^{-\frac{t-s}{\varepsilon}}\, \mathrm{d}s \ ,$$

where for simplicity the index ε is omitted for u and v.

Then

$$\begin{aligned} v(x,t+\tau) - v(x,t) &= \left(e^{-\frac{t+\tau}{\varepsilon}} - e^{-\frac{t}{\varepsilon}}\right) v_0(x) \\ &+ \frac{1}{\varepsilon}\left(\int_0^{t+\tau} f(u(x,s)) e^{-\frac{t+\tau-s}{\varepsilon}}\, \mathrm{d}s - \int_0^t f(u) e^{-\frac{t-s}{\varepsilon}}\, \mathrm{d}s\right) \\ &- \alpha\left(\int_0^{t+\tau} (\partial_x u)(x,s) e^{-\frac{t+\tau-s}{\varepsilon}}\, \mathrm{d}s - \int_0^t (\partial_x u) e^{-\frac{t-s}{\varepsilon}}\, \mathrm{d}s\right) \ . \end{aligned}$$

Let $\varphi \in C_0^2([a,b])$. We obtain

$$\begin{aligned} I(t) &= \left|\int_a^b (v(x,t+\tau) - v(x,t))\varphi(x)\, \mathrm{d}x\right| \\ &\le \left|\int_a^b (e^{-\frac{t+\tau}{\varepsilon}} - e^{-\frac{t}{\varepsilon}}) v_0(x)\varphi(x)\, \mathrm{d}x\right| \\ &+ \frac{1}{\varepsilon}\left|\int_a^b \left(\int_0^{t+\tau} f(u(x,s)) e^{-\frac{t+\tau-s}{\varepsilon}}\, \mathrm{d}s - \int_0^t f(u) e^{-\frac{t-s}{\varepsilon}}\, \mathrm{d}s\right)\varphi(x)\, \mathrm{d}x\right| \\ &+ \alpha\left|\int_a^b \left(\int_0^{t+\tau} u(x,s) e^{-\frac{t+\tau-s}{\varepsilon}}\, \mathrm{d}s - \int_0^t u(x,s) e^{-\frac{t-s}{\varepsilon}}\, \mathrm{d}s\right)\partial_x\varphi(x)\, \mathrm{d}x\right| \\ &= I_1 + \frac{1}{\varepsilon} I_2 + I_3 \ . \end{aligned}$$

First we have, for $\nu \le t \le T$

$$\begin{aligned} I_1 &\le \|v_0^\varepsilon\|_{L^\infty} \|\varphi\|_{C^0} (b-a) \left|e^{-\frac{t+\tau}{\varepsilon}} - e^{-\frac{t}{\varepsilon}}\right| \\ &\le \frac{M\tau}{\varepsilon}\, e^{-\frac{\nu}{\varepsilon}} (b-a)\|\varphi\|_{C^0} \ . \end{aligned}$$

Since f is locally Lipschitz continuous and thanks to Theorem 4.5, the terms I_2 and I_3 are of the following type:

$$\tilde{I}(t) = \left|\int_a^b \left(\int_0^{t+\tau} w(x,s) e^{-\frac{t+\tau-s}{\varepsilon}}\, \mathrm{d}s - \int_0^t w(x,s) e^{-\frac{t-s}{\varepsilon}}\, \mathrm{d}s\right)\zeta(x)\, \mathrm{d}x\right|$$

for $\zeta \in C_0^2(a,b)$ and for some bounded function w with $|w| \le \tilde{M}$ ($\tilde{M} > 0$) and such that for some constant $\tilde{C} > 0$ it holds

$$\left| \int_a^b (w(x,t+\tau) - w(x,t))\zeta(x)\,\mathrm{d}x \right| \le \tilde{C}\tau \|\zeta\|_{C^1} \ .$$

We have

$$\begin{aligned}
\tilde{I}(t) &\le \left| \int_a^b \int_0^t (w(x,s+\tau) - w(x,s))\, e^{-\frac{t-s}{\varepsilon}} \zeta(x)\,\mathrm{d}s\,\mathrm{d}x \right| \\
&+ \left| \int_a^b \int_0^\tau w(x,s) e^{-\frac{t+\tau-s}{\varepsilon}} \zeta(x)\,\mathrm{d}s\,\mathrm{d}x \right| \\
&\le \tilde{C}\tau\|\zeta\|_{C^1} \int_0^t e^{-\frac{t-s}{\varepsilon}}\,\mathrm{d}s + (b-a)\tilde{M}\|\zeta\|_{C^0} \int_0^\tau e^{-\frac{t+\tau-s}{\varepsilon}}\,\mathrm{d}s \\
&\le \varepsilon\,\tilde{C}\tau\|\zeta\|_{C^1} + (b-a)\tilde{M}\tau\|\zeta\|_{C^0} \frac{e^{-t/\varepsilon}}{\varepsilon} \\
&\le C\tau\|\zeta\|_{C^1}\left(\varepsilon + \frac{e^{-\frac{t}{\varepsilon}}}{\varepsilon}\right) \ .
\end{aligned}$$

Therefore, summing up the different terms we have that there exists a constant $C^\nu > 0$ such that, for any $t \ge \nu$

$$I(t) \le C^\nu \tau \|\varphi\|_{C^2} e^{-\frac{\nu}{\varepsilon}} \ .$$

This completes the proof.

It is also useful to estimate the deviation from the equilibrium in the L^1 norm.

PROPOSITION 4.7. *Under the assumptions of Theorem 4.5, suppose that the initial data $(u_0^\varepsilon, v_0^\varepsilon)$ are locally of bounded variation. Then, for any interval $(a,b) \subseteq \mathbb{R}$ and for every $t > 0$, it holds*

$$\begin{aligned}
\int_a^b |v^\varepsilon(x,t) - f(u^\varepsilon(x,t))|\,\mathrm{d}x &\le e^{-\frac{t}{\varepsilon}} \int_a^b |v_0^\varepsilon(x) - f(u_0^\varepsilon(x))|\,\mathrm{d}x \\
&+ \varepsilon\, C(\alpha)\|(u_0^\varepsilon, v_0^\varepsilon)\|_{BV(a-\sqrt{\alpha}t,\, b+\sqrt{\alpha}t)} \ .
\end{aligned} \tag{4.9}$$

Proof: Omitting the index ε for simplicity, take smooth regular initial data. Then the function $\xi = v - f(u)$ verifies

$$\xi_t + \frac{1}{\varepsilon}\xi = -(\alpha\,\partial_x u + f'(u)\partial_x v) \ ,$$

for $(x,t) \in \mathbb{R} \times (0,\infty))$ and the initial condition

$$\xi(x,0) = \xi_0(x) := v_0(x) - f(u_0(x)) ,$$

for $x \in \mathbb{R}$.

Then, by multiplying for a smooth approximation of the function $\operatorname{sgn}(\xi)$ and integrating on the interval (a,b), we have, by standard arguments and Corollary 4.3,

$$\int_a^b |\xi(x,t)| \,\mathrm{d}x \le e^{-\frac{t}{\varepsilon}} \int_a^b |\xi_0(x)| \,\mathrm{d}x \\ + \varepsilon\, C(\alpha) \int_{a-\sqrt{\alpha}\,t}^{b+\sqrt{\alpha}\,t} |\partial_x u_0(x)| + |\partial_x v_0(x)| \,\mathrm{d}x \ .$$

This implies (4.9) for general data of bounded variation.

5. The Strong Convergence

We say that $w = w(x,t)$ is a weak solution to the scalar problem

$$\partial_t w + \partial_x f(w) = 0 \ , \tag{5.1}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$, with the initial condition

$$w(x,0) = w_0(x) \ , \tag{5.2}$$

for $x \in \mathbb{R}$, if $w \in L^\infty_{\mathrm{loc}}(\mathbb{R} \times (0,\infty))$ and for any smooth function $\varphi \in C_0^\infty(\mathbb{R} \times (0,\infty))$, it holds

$$\iint \{w\partial_t\varphi + f(w)\partial_x\varphi\} \,\mathrm{d}x \,\mathrm{d}t = 0;$$

for any interval $I \subseteq \mathbb{R}$

$$\lim_{T\to 0^+} \frac{1}{T} \int_0^T \int_I |w(x,t) - w_0(x)| \,\mathrm{d}x \,\mathrm{d}t = 0 \ .$$

Theorem 5.1. *Assume* (H_1), (H_2) *and* (3.6) *and let* $(u^\varepsilon, v^\varepsilon)$ *be the global solution to problem* (3.1)–(3.2) *given by Theorem* 3.1. *Then there exists a weak solution* w *to* (5.1)–(5.2) *and a subsequence, still denoted* $(u^\varepsilon, v^\varepsilon)$*, such that*

$$u^\varepsilon \longrightarrow w \qquad \text{in } C([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R})) \ , \tag{5.3}$$

as $\varepsilon \to 0^+$,

$$v^\varepsilon \longrightarrow f(w) \qquad \text{in } C([\nu,\infty); L^1_{\mathrm{loc}}(\mathbb{R})) \ , \tag{5.4}$$

as $\varepsilon \longrightarrow 0^+$ and for any $\nu > 0$. Moreover, if (H_3) *is also verified, then in* (5.4) *we can take $\nu = 0$.*

Proof: First let us take the initial data $(u_0^\varepsilon, v_0^\varepsilon)$ in a bounded subset of $BV_{\mathrm{loc}}(\mathrm{I\!R})^2$. Then, thanks to Theorems 3.1, 4.5, 4.6, Corollaries 4.2, 4.3 and Proposition 4.7 and by standard compactness arguments (see for instance [12]), there exists $w \in L^\infty(\mathrm{I\!R} \times (0,\infty))$ such that (5.3)–(5.4) hold true and w is a weak solution to (5.1)–(5.2).

Moreover, from Proposition 4.7, we have that for any interval $(a,b) \subseteq \mathrm{I\!R}$

$$\int_a^b |v^\varepsilon(x,t) - f(u^\varepsilon(x,t))|\,\mathrm{d}x \le e^{-\frac{t}{\varepsilon}} \int_a^b |v_0^\varepsilon(x) - f(u_0^\varepsilon(x))|\,\mathrm{d}x$$
$$+ \varepsilon\, C(\alpha) \|(u_0^\varepsilon, v_0^\varepsilon)\|_{BV((a-\sqrt{\alpha}\,t, b+\sqrt{\alpha}\,t))} \,.$$

then, if (H_3) is also verified, we can choose $\nu = 0$ in (5.4).

Let us consider now a sequence of initial data $(u_0^\varepsilon, v_0^\varepsilon)$ which are not of bounded variation. By the same arguments there exist $w, \tilde{v} \in L^\infty(\mathrm{I\!R} \times (0,\infty)))$ and a subsequence still denoted $(u^\varepsilon, v^\varepsilon)$, such that

$$u^\varepsilon \longrightarrow w \qquad \text{in } C([0,\infty); L^1_{\mathrm{loc}}(\mathrm{I\!R}))$$

and, for any $\nu > 0$,

$$v^\varepsilon \longrightarrow \tilde{v} \qquad \text{in } C([\nu,\infty); L^1_{\mathrm{loc}}(\mathrm{I\!R}))$$

as $\varepsilon \longrightarrow 0^+$.

Furthermore, by standard regularization, there exists a sequence $(u_0^{\varepsilon,\delta}, v_0^{\varepsilon,\delta})$ $(\delta > 0)$ of smooth functions with compact support such that

i)
$$(u_0^{\varepsilon,\delta}, v_0^{\varepsilon,\delta}) \longrightarrow (u_0^\varepsilon, v_0^\varepsilon)$$
in $L^1_{\mathrm{loc}}(\mathrm{I\!R})$ as $\delta \longrightarrow 0^+$;

ii)
$$\|u_0^{\varepsilon,\delta}\|_\infty \le \|u_0^\varepsilon\|_\infty \ ,$$
$$\|v_0^{\varepsilon,\delta}\|_\infty \le \|v_0^\varepsilon\|_\infty \ ;$$

iii) for any compact subset $K \subseteq \mathrm{I\!R}$
$$\|(u_0^{\varepsilon,\delta}, v_0^{\varepsilon,\delta})\|_{BV(K)} \le \frac{1}{\delta}\|(u_0^\varepsilon, v_0^\varepsilon)\|_{L^1(K+B^\delta(0))} \ ,$$
where $B^\delta(0) = \{x \in \mathrm{I\!R}\, | \,|x| \leqq \delta\}$.

Let $(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})$ be the sequence of global solutions to (3.1)-(3.2) corresponding to the initial data $(u_0^{\varepsilon,\delta}, v_0^{\varepsilon,\delta})$.

Then, from Theorem 4.1, inequality (3.8) and Proposition 4.7 we have, for any fixed $\varepsilon > 0$ and any $\delta > 0$

$$
\begin{aligned}
&\int_a^b |v^\varepsilon(x,t) - f(u^\varepsilon(x,t))| \,\mathrm{d}x \le \int_a^b |v^\varepsilon(x,t) - v^{\varepsilon,\delta}(x,t)| \,\mathrm{d}x \\
&\quad + \int_a^b |f(u^\varepsilon(x,t)) - f(u^{\varepsilon,\delta}(x,t))| \,\mathrm{d}x \\
&\quad + \int_a^b |v^{\varepsilon,\delta}(x,t) - f(u^{\varepsilon,\delta}(x,t))| \,\mathrm{d}x \\
&\quad \le (1+\sqrt{\alpha})^2 \int_{a-\sqrt{\alpha}\,t}^{b+\sqrt{\alpha}\,t} \{|v_0^\varepsilon(x) - v_0^{\varepsilon,\delta}(x)| + \sqrt{\alpha}\,|u_0^\varepsilon(x) - u_0^{\varepsilon,\delta}(x)|\} \,\mathrm{d}x \\
&\quad + e^{-\frac{t}{\varepsilon}} \int_a^b |v_0^{\varepsilon,\delta}(x) - f(u_0^{\varepsilon,\delta}(x))| \,\mathrm{d}x \\
&\quad + \frac{\varepsilon\, C(\alpha)}{\delta} \int_{a-\sqrt{\alpha}\,t-\delta}^{b+\sqrt{\alpha}\,t+\delta} (|u_0^\varepsilon(x)| + |v_0^\varepsilon(x)|) \,\mathrm{d}x \ .
\end{aligned}
$$

Now (5.4) follows easily and the proof is complete.

6. Entropy Solutions

As well known (see for instance [17]) weak solutions to the Cauchy problem (5.1)–(5.2) are in general not unique and, to select a reasonable solution, we have to restrict the class of the admissible solutions giving the so–called entropy conditions of Kružkov [16].

Definition 6.1. *A function $w \in L^\infty(\mathbb{R} \times (0,T))$ $(T > 0)$ is an entropy solution of problem* (5.1)–(5.2) *if w is a weak solution and for any $k \in \mathbb{R}$ and $\varphi \in C_0^\infty(\mathbb{R} \times (0,T))$, $\varphi \ge 0$, we have*

$$\iint \{|w-k|\partial_t\varphi + \mathrm{sgn}\,(w-k)(f(w)-f(k))\partial_x\varphi\} \,\mathrm{d}x\,\mathrm{d}t \ge 0 \ .$$

We recall that existence and uniqueness results for entropy solutions were first proved in [16] (see also the reference text [12]). Let us note that w verifies the entropy conditions if and only for any smooth pair (η, q), η strictly convex and $q = \int^u \eta'(s) f'(s) \,\mathrm{d}s$, we have

$$\partial_t \eta(w) + \partial_x q(w) \le 0 \tag{6.1}$$

in the sense of distributions. The pair (η, q) is called an entropy pair for the equation (5.1).

In this section we shall prove that the limit function, obtained in Section 5 when $\varepsilon \to 0^+$, is actually an entropy solution under two different sets of assumptions:

a) for a fixed $\alpha > 0$ and small initial data in the L^∞ norm (Theorem 6.2);

b) for arbitrarily large initial data and large enough $\alpha > 0$, under some technical conditions on f at infinity (Theorem 6.3); namely, $f = f(u)$ is required to be genuinely nonlinear out of a bounded interval $\mathcal{M} \subseteq \mathbb{R}$.

Note that the smallness condition a) is in the spirit of [8] (see also Section 7). However, concerning condition b), the argument is more subtle and involves a quite delicate analysis of the behaviour of the entropy pairs for system (3.1).

Let us state now our results.

THEOREM 6.2. *Assume* (H_1), (H_2) *and let* $\alpha > 0$ *be fixed. Then there exists* $N_0 > 0$ *such that the stability condition* (3.6) *holds and the weak solution* w *of* (5.1)–(5.2)*, given by Theorem* 5.1*, is an entropy solution of the same problem.*

THEOREM 6.3. *Assume* (H_1), (H_2) *and one of the following hypothesis:*

(6.2) $\sup_{u\in\mathbb{R}} |f'(u)| < \infty$;

(6.3) $f \in C^2$ and there is $M > 0$ such that if $|u| \geq M$, then $|f''(u)| > 0$.

Then, for every $N_0 > 0$ *there exists* $\alpha_0 > 0$ *such that, for any* $\alpha \geq \alpha_0$ *the stability condition* (3.6) *is satisfied and the weak solution* w *of* (5.1)–(5.2)*, given by Theorem* 5.1*, is an entropy solution of the same problem.*

Observe that under the assumptions of Theorem 6.2 or 6.3, and thanks to the uniqueness result for the entropy solutions, the *whole* sequence $\{(u^\varepsilon, v^\varepsilon)\}$ in Theorem 5.1 is actually converging.

To prove that these limit solutions are entropy solutions to (5.1)–(5.2), we shall follow the strategy first proposed in [8]. Recall that an entropy pair for the system (3.1), on the set $\Omega \subseteq \mathbb{R}^2$, is given by two smooth functions $\Phi, \Psi : \Omega \to \mathbb{R}$ such that Φ is convex and

$$\partial_u \Phi = \partial_v \Psi \quad , \quad \alpha \partial_v \Phi = \partial_u \Psi \,. \tag{6.4}$$

It is easy to check that smooth solutions $(u^\varepsilon, v^\varepsilon)$ of (3.1) satisfy

$$\partial_t \Phi(u^\varepsilon, v^\varepsilon) + \partial_x \Psi(u^\varepsilon, v^\varepsilon) = -\frac{1}{\varepsilon} \partial_v \Phi(u^\varepsilon, v^\varepsilon) \cdot (v^\varepsilon - f(u^\varepsilon)) \,. \tag{6.5}$$

For any given (scalar) entropy pair (η, q) for (5.1), we are going to construct an entropy pair (Φ, Ψ) for the system (3.1) on some open set Ω such that:

$$\Phi(u, f(u)) = \eta(u) \quad , \quad \Psi(u, f(u)) = q(u) \tag{6.6}$$

for any u such that $(u, f(u)) \in \Omega$.

First we consider the case of small initial data for a fixed $\alpha > 0$, namely the assumptions of Theorem 6.2.

Proposition 6.4. *Let (η, q) be a strictly convex entropy pair for* (5.1). *For any fixed $\alpha > 0$ there exists $N_0 > 0$ and an entropy pair (Φ, Ψ) for the system* (3.1) *on the set*

$$\Omega^{\alpha}_{N_0} := \{(u,v) \in \mathbb{R}^2 \,\big|\, \quad |v \pm \sqrt{\alpha}\, u| \le \sqrt{\alpha}\; B(N_0)\}, \tag{6.7}$$

with B given by (3.4)*, which verifies* (6.6) *and such that:*

$$\partial_v \Phi(u,v) \cdot (v - f(u)) \ge 0 \tag{6.8}$$

for all $(u,v) \in \Omega^{\alpha}_{N_0}$ and the equality holds if and only if $v = f(u)$;

$$\partial_{vv}\Phi > 0 \quad , \quad \partial_{uu}\Phi \partial_{vv}\Phi - (\partial_{uv}\Phi)^2 > 0 \tag{6.9}$$

for all $(u,v) \in \Omega^{\alpha}_{N_0}$.

Proof: Observe first that (6.8) follows immediately from the first part of (6.9) and the requirement

$$\partial_v \Phi(u, f(u)) = 0 \ .$$

Then, from (6.4), (6.6), it suffices to find our (convex) function Φ as the solution of the following linear Cauchy problem, for some fixed α, N_0,

$$\begin{cases} \partial_{uu}\Phi - \alpha \partial_{vv}\Phi = 0 \ , \\ \Phi(u, f(u)) = \eta(u) \ , \\ \partial_v \Phi(u, f(u)) = 0 \ , \end{cases} \tag{6.10}$$

in the set $\Sigma^{\alpha}_{N_0} = \{(u,v) \in \Omega^{\alpha}_{N_0};\ |f'(u)| < \sqrt{\alpha}\}$.

First we observe that the curve $v = f(u)$ is space–like thanks to the stability condition (3.6). It is also immediate to get an explicit solution to (6.10) in $\Sigma^{\alpha}_{N_0}$, taking

$$\Phi(u,v) = a_+(v + \sqrt{\alpha}\, u) + a_-(v - \sqrt{\alpha}\, u),$$

where a_+ and a_- verify

$$\begin{aligned} a'_+(f(u) + \sqrt{\alpha}\, u) - a'_-(f(u) - \sqrt{\alpha}\, u) &= \frac{\eta'(u)}{\sqrt{\alpha}} \ , \\ a'_+(f(u) + \sqrt{\alpha}\, u) + a'_-(f(u) - \sqrt{\alpha}\, u) &= 0 \ . \end{aligned}$$

This yields

$$a'_{\pm}(f(u) \pm \sqrt{\alpha}\, u) = \pm \frac{\eta'(u)}{2\sqrt{\alpha}} \ .$$

Set

$$h_{\pm}(u) = f(u) \pm \sqrt{\alpha}\, u \ .$$

In view of the subcharacteristic condition we can invert the function $h_{\pm}(u)$ as long as $|f'(u)| < \sqrt{\alpha}$. Let $h_{\pm}^{-1}$ be its inverse function. This yields

$$a'_{\pm}(r) = \pm \frac{\eta'(h_{\pm}^{-1}(r))}{2\sqrt{\alpha}}$$

for all $r \in \mathbb{R}$ such that the stability condition

$$|f'(h_{\pm}^{-1}(r))| < \sqrt{\alpha} \tag{6.11}$$

is verified.

Therefore we have

$$\begin{aligned} \partial_v \Phi(u,v) &= a'_{+}(v + \sqrt{\alpha}\, u) + a'_{-}(v - \sqrt{\alpha}\, u) \\ &= \frac{1}{2\sqrt{\alpha}} \left[\eta'(h_{+}^{-1}(v + \sqrt{\alpha}\, u)) - \eta'(h_{-}^{-1}(v - \sqrt{\alpha}\, u)) \right] \end{aligned} \tag{6.12}$$

and

$$\begin{aligned} \partial_u \Phi(u,v) &= \sqrt{\alpha} \left(a'_{+}(v + \sqrt{\alpha}\, u) - a'_{-}(v - \sqrt{\alpha}\, u) \right) \\ &= \frac{1}{2} \left[\eta'(h_{+}^{-1}(v + \sqrt{\alpha}\, u)) + \eta'(h_{-}^{-1}(v - \sqrt{\alpha}\, u)) \right] \ . \end{aligned} \tag{6.13}$$

Since $|f'(u)| < \sqrt{\alpha}$, from the Implicit Function Theorem there exist two functions $\theta_{\pm}(u,v) \in C^1$ and a small neighbourhood $U^{\alpha} \subseteq \mathbb{R}^2$ of the curve $v = f(u)$ such that:

$$\theta_{\pm}(u, f(u)) = u \ ; \tag{6.14a}$$

$$f(\theta_{\pm}) \pm \sqrt{\alpha}\, \theta_{\pm} = v \pm \sqrt{\alpha}\, u \quad , \quad (u,v) \in U^{\alpha} \ ; \tag{6.14b}$$

$$|f'(\theta_{\pm})| < \sqrt{\alpha} \quad , \quad (u,v) \in U^{\alpha} \ ; \tag{6.14c}$$

$$\partial_v \theta_{\pm} = (f'(r_{\pm}) \pm \sqrt{\alpha}\,)^{-1} \quad , \quad \partial_u \theta_{\pm} = \pm\sqrt{\alpha}\,(f'(\theta_{\pm}) \pm \sqrt{\alpha}\,)^{-1} \ . \tag{6.14d}$$

Moreover, taking N_0 small enough we can include $\Sigma_{N_0}^{\alpha}$ in U^{α}. Then for such N_0 we obtain, for any $(u,v) \in \Sigma_{N_0}^{\alpha}$,

$$h_{\pm}^{-1}(v \pm \sqrt{\alpha}\, u) = \theta_{\pm}(u,v) \ ,$$

which yields

$$\partial_v \Phi(u,v) = \frac{1}{2\sqrt{\alpha}} \left[\eta'(\theta_+(u,v)) - \eta'(\theta_-(u,v))\right] \; .$$

Then, using (6.14c), (6.14d) and the convexity of η, we obtain

$$\begin{aligned}\partial_{vv}\Phi &= \frac{1}{2\sqrt{\alpha}} \left[\eta''(\theta_+)\partial_v\theta_+ - \eta''(\theta_-)\partial_v\theta_-\right] \\ &= \frac{1}{2\sqrt{\alpha}} \left[\frac{\eta''(\theta_+)}{f'(\theta_+)+\sqrt{\alpha}} - \frac{\eta''(\theta_-)}{f'(\theta_-)-\sqrt{\alpha}}\right] > 0\end{aligned}$$

for $(u,v) \in \Sigma^\alpha_{N_0}$.

The second inequality in (6.9) follows by arguing in the same way.

Proof of Theorem 6.2: Assume smooth initial data. Let $\alpha > 0$ be fixed and $N_0 > 0$, as given by Proposition 6.4, small enough to satisfy (3.6). Let $\{(u^\varepsilon, v^\varepsilon)\}$ be the converging subsequence of global solutions of problem (3.1)–(3.2) given by Theorem 5.1. Then for any entropy pair (η, q), η strictly convex, Proposition 6.4 yields an entropy pair (Φ, Ψ) for the system (3.1) which verifies (6.6) and (6.8). Since $(u^\varepsilon, v^\varepsilon) \in \Sigma^\alpha_{N_0} = \{(u,v) \in \mathbb{R}^2 \,|\, |f'(u)| \le \sqrt{\alpha},\; |v \pm \sqrt{\alpha}\,u| \le \sqrt{\alpha}\,B(N_0)\}$, we obtain,

$$\begin{aligned}\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) = {}& \partial_t \left(\Phi(u^\varepsilon, f(u^\varepsilon)) - \Phi(u^\varepsilon, v^\varepsilon)\right) \\ &+ \partial_x \left(\Psi(u^\varepsilon, f(u^\varepsilon)) - \Psi(u^\varepsilon, v^\varepsilon)\right) \\ &- \frac{1}{\varepsilon}\partial_v \Phi(u^\varepsilon, v^\varepsilon)\cdot(v^\varepsilon - f(u^\varepsilon)) \; .\end{aligned}$$

For any $\varphi \in C_0^\infty(\mathbb{R} \times (0,T))$, $\varphi \ge 0$, the estimate (4.9) yields

$$\begin{aligned}&\left| \iint \partial_t \left(\Phi(u^\varepsilon, f(u^\varepsilon)) - \Phi(u^\varepsilon, v^\varepsilon)\right)\varphi \;\mathrm{d}x\;\mathrm{d}t \right| \\ &\qquad \le C\|\partial_t\varphi\|_{L^\infty} \iint_{\mathrm{supp}\;\varphi} |v^\varepsilon - f(u^\varepsilon)|\,\mathrm{d}x\;\mathrm{d}t = O(\varepsilon)\end{aligned}$$

In the same way

$$\left| \iint \partial_x \left(\Psi(u^\varepsilon, f(u^\varepsilon)) - \Psi(u^\varepsilon, v^\varepsilon)\right)\varphi \;\mathrm{d}x\;\mathrm{d}t \right| = O(\varepsilon) \; .$$

Since the term $\partial_v\Phi(u^\varepsilon, v^\varepsilon)\cdot(v^\varepsilon - f(u^\varepsilon))$ is nonnegative for all $\varepsilon > 0$, this concludes the proof.

We note that the key points in the above proof of Theorem 6.2 are the uniform L^∞ estimates of $(u^\varepsilon, v^\varepsilon)$ and the construction of suitable entropy pairs for the system (3.1). The smallness assumption on the initial data allows this construction without any further requirement on the flux function f. Now, to deal with arbitrarily large initial data, we need to study more carefully the functions θ_+ and θ_- given by (6.14).

Definition 6.5. *For any fixed $N > 0$ and $\alpha > 0$ set*

$$U^+(\pm\sqrt{\alpha}, N) = \{\theta \geq 0 \mid \forall s \in [0, \theta] \ |f(s) \pm \sqrt{\alpha}\, s| \leq \sqrt{\alpha}\, B(N)\}$$

and

$$U^-(\pm\sqrt{\alpha}, N) = \{\theta \leq 0 \mid \forall s \in [\theta, 0] \ |f(s) \pm \sqrt{\alpha}\, s| \leq \sqrt{\alpha}\, B(N)\} \ ,$$

Proposition 6.6. *Let* (H_1) *and* (6.2) *or* (6.3) *be verified. Then for every* $N_0 > 0$ *there exists* $\alpha_0 > 0$ *such that, for all* $\alpha \geq \alpha_0$ *and* $\theta \in U^+(\sqrt{\alpha}, N_0) \cup U^-(\sqrt{\alpha}, N_0)$ *(resp.:* $U^+(-\sqrt{\alpha}, N_0) \cup U^-(-\sqrt{\alpha}, N_0)$*) it holds*

$$|f'(\theta)| \leq \sqrt{\alpha} \ . \tag{6.15}$$

To prove Proposition 6.6 we need the results given by the next two lemmas.

Lemma 6.7. *Assume* (H_1) *and let* $M > 0$ *be such that* $f''(u) > 0$ *for any* $u < -M$*. Then for every* $N_0 > 0$ *there exists* $\alpha_0 > 0$ *such that, for all* $\alpha \geq \alpha_0$*, if* $\theta \in U^-(\sqrt{\alpha}, N_0)$ *then* (6.15) *holds.*

Proof: If $\sup_{\theta \leq 0} |f'(\theta)| < +\infty$ the conclusion is immediate. Otherwise we have

$$\lim_{u \to -\infty} f'(u) = -\infty \ . \tag{6.16}$$

Set

$$\theta_\alpha := \inf\{\theta \leq 0; \ f'(s) \geq -\sqrt{\alpha} \quad \forall s \in (\theta, 0]\} \ .$$

Clearly

$$f'(\theta_\alpha) = -\sqrt{\alpha} \ .$$

Moreover, for $s \in [\theta_\alpha, 0]$, the function $f(s) + \sqrt{\alpha}\, s$ is monotone increasing. Then

$$\max_{[\theta_\alpha, 0]} (f(s) + \sqrt{\alpha}\, s) = 0 \ ,$$
$$\min_{[\theta_\alpha, 0]} (f(s) + \sqrt{\alpha}\, s) = f(\theta_\alpha) - f'(\theta_\alpha)\theta_\alpha \ .$$

Let us show that

$$\lim_{\alpha\to\infty} \frac{f(\theta_\alpha)+\sqrt{\alpha}\,\theta_\alpha}{\sqrt{\alpha}} = -\infty \ . \tag{6.17}$$

Observe first that the sequence $\{\theta_\alpha\}$ is monotone decreasing with respect to α. In fact, if $s\in(\theta_\alpha,0)$, it follows that

$$f'(s) \geq -\sqrt{\alpha} > -\sqrt{\beta}$$

and then $s>\theta_\beta$.

Let

$$-\infty \leq L := \lim_{\alpha\to\infty} \theta_\alpha \ .$$

We have

$$L = -\infty \tag{6.18}$$

In fact

$$-\infty = \lim_{\alpha\to+\infty} -\sqrt{\alpha} = \lim_{\alpha\to+\infty} f'(\theta_\alpha) = \inf_{s\in(L,0)} f'(s) \ .$$

If $L>-\infty$ this last term is finite and this yields a contradiction.

Now

$$\theta_\alpha = (f')^{-1}(-\sqrt{\alpha}).$$

Then, setting $y=\theta_\alpha$ and

$$\psi(y) = \frac{f(y)-f'(y)y}{f'(y)} \ .$$

To prove (6.17) it suffices to show that

$$\lim_{y\to-\infty} \psi(y) = +\infty \ . \tag{6.19}$$

First observe that, from (6.16) we have

$$\lim_{u\to-\infty} f(u) = +\infty \ .$$

Then

$$\psi' = -\frac{f\,f''}{(f')^2} < 0$$

for $y<-\overline{M}$, $\overline{M}>0$ large enough and there exists $\lim_{y\to-\infty}\psi(y)>0$.

If the limit in (6.19) is finite then

$$\lim_{y\to-\infty}(f(y)-f'(y)y) = +\infty$$

and, from (6.16), we have at once that

$$\lim_{y\to-\infty}\psi(y)=\lim_{y\to-\infty}\frac{(f(y)-f'(y)y)'}{f''(y)}=\lim_{y\to-\infty}(-y)=+\infty,$$

which leads at a contradiction. Now, in view of (6.17), we have that for any $N_0>0$ there exists $\tilde\alpha>0$ such that, for any $\alpha\geq\tilde\alpha$

$$f(\theta_\alpha)+\sqrt{\alpha}\,\theta_\alpha\leq-\sqrt{\alpha}\,B(N_0)\ .$$

This yields

$$f'(s)\geq-\sqrt{\alpha}$$

for any $s\geq\theta_\alpha$. On the other hand, from (6.16) and (H_1) we have that

$$\overline{s}:=\sup_{\theta\leq 0}f'(\theta)<+\infty\ .$$

Therefore (6.15) follows choosing $\alpha_0=\max(\tilde\alpha,\overline{s}^2)$.

Lemma 6.8. *Assume* (H_1) *and let* $M>0$ *be such that* $f''(u)<0$ *for any* $u<-M$. *Hence for every* $N_0>0$ *there exists* $\alpha_0>0$ *such that, for all* $\alpha\geq\alpha_0$, *if* $\gamma\in U^-(\sqrt{\alpha},N_0)$, *then (6.15) holds.*

Proof: Let again $\sup_{\gamma\leq 0}|f'(\gamma)|=+\infty$, the opposite case being obvious. Then

$$\lim_{u\to-\infty}f'(u)=+\infty\ . \tag{6.20}$$

Let

$$\gamma_\alpha=\inf\{\gamma\leq 0;\ f'(s)\leq\sqrt{\alpha}\quad\forall s\in(\gamma,0]\}\ ,$$

which gives

$$f'(\gamma_\alpha)=+\sqrt{\alpha}\ .$$

It is easy to check, as in the previous lemma, that

$$\lim_{\alpha\to\infty}\gamma_\alpha=-\infty$$

and

$$\begin{aligned}\lim_{\alpha\to\infty}\frac{f(\gamma_\alpha)+\sqrt{\alpha}\,\gamma_\alpha}{\sqrt{\alpha}}&=\lim_{y\to-\infty}\frac{f(y)+f'(y)y}{f'(y)}\\&\leqq\lim_{y\to-\infty}y=-\infty\ .\end{aligned}$$

The conclusion follows as in Lemma 6.7.

Proof of Proposition 6.6: We can deal with the set $U^+(\sqrt{\alpha}\,, N_0)$ just replacing, in Lemmas 6.7, 6.8, θ by $-\theta$. In the same way the case of $U^{\pm}(-\sqrt{\alpha}\,, N_0)$ follows immediately just by replacing f by $-f$.

PROPOSITION 6.9. *Assume* (H_1) *and* (6.2) *or* (6.3). *Let* (η, q) *be a strictly convex entropy pair for* (5.1). *For any fixed* $N_0 > 0$, *there is* $\alpha_0 > 0$ *such that for any* $\alpha \geqq \alpha_0$ *there exists an entropy pair* (Φ, Ψ) *for the systems* (3.1), *on the set* $\Omega^\alpha_{N_0}$ *given by* (6.7), *which verifies* (6.6), (6.8) *and* (6.9).

Proof: We proceed as in Proposition 6.4. Let Φ be the solution on $\Sigma^\alpha_{N_0}$ of the Cauchy problem (6.10). Proposition 6.6 yields easily that, for any fixed $N_0 > 0$, there exists $\alpha_0 > 0$ such that, for all $\alpha \geq \alpha_0$

$$\Sigma^\alpha_{N_0} \subseteq \{(u, v) \mid \quad |f'(\theta_\pm(u, v))| < \sqrt{\alpha}\} \ ,$$

where $\theta_\pm$ are given by (6.14) and $\theta_\pm \in U^+(\pm\sqrt{\alpha}\,, N_0) \cup U^-(\pm\sqrt{\alpha}\,, N_0)$. Then the conclusion follows as in Proposition 6.4.

Proof of Theorem 6.3: The proof follows by arguing as in Theorem 6.2, just replacing Proposition 6.4 by Proposition 6.9.

7. The $\mathbf{L}^\infty$ Estimates in the Quasilinear Case

In this last section we shall be concerned with the general quasilinear system (1.1). This model has probably not a direct physical interpretation, but is a good mathematical model to understand relaxation and in this spirit it was already studied in [8]. Let us write again the system

$$\left\{ \begin{array}{l} \partial_t u + \partial_x v = 0 \ , \\ \partial_t v + \partial_x \sigma(u) = -\frac{1}{\varepsilon}(v - f(u)) \ , \end{array} \right. \tag{7.1}$$

for $(x, t) \in \mathbb{R} \times (0, \infty)$, with initial conditions

$$u(x, 0) = u_0^\varepsilon(x) \quad , \quad v(x, 0) = v_0^\varepsilon(x) \tag{7.2}$$

for $x \in \mathbb{R}$. Here σ, f are some given smooth functions such that

$$\begin{array}{l} \sigma \in C^2;\ \sigma'(u) \geq \nu > 0, \text{ for all } u \in \mathbb{R} \ ; \\ u\sigma''(u) > 0, \text{ for all } u \in \mathbb{R} \setminus \{0\} \ . \end{array} \tag{7.3}$$

$$f \in C^2 \quad , \quad f(0) = 0 \text{ and } \mathrm{meas}\{u | f''(u) = 0\} = 0 \ . \tag{7.4}$$

Roughly speaking, we are considering the simplest nonlinear model for one-dimensional elasticity, with a stiff relaxation term on the right–hand side. In this section we shall establish global existence of uniformly bounded entropy solution to the Cauchy problem (7.1)–(7.2). The relaxation limit will follows immediately by Theorem 4.1 in [8].

The main tool will be again the quasimonotone structure of the relaxation term (see Sections 2 and 3) which yields a useful comparison principle even in the quasilinear case. In fact, as in Section 3, we are unable to use the theory of the invariant regions by Chueh, Conley and Smoller [9]. Nevertheless we have the following result

THEOREM 7.1. *Assume the functions $\sigma(u)$ and $f(u)$ satisfy* (7.3), (7.4) *and the following stability condition holds*

$$|f'(u)| \leq \sqrt{\sigma'(u)}\ , \tag{7.5}$$

for all $u \in \mathbb{R}$. Assume the initial data $(u_0^\varepsilon, v_0^\varepsilon)$ are uniformly bounded in L^∞.

Then there exists a global entropy solution $(u^\varepsilon, v^\varepsilon)$ for the Cauchy problem (7.1)-(7.2) for any fixed $\varepsilon > 0$. Moreover

$$\|v^\varepsilon \pm \psi(u^\varepsilon)\|_\infty \leq \sup_{|\lambda|\leq M_0} \left(|\lambda| + |f(\psi^{-1}(\lambda))|\right) =: M \tag{7.6}$$

where $\psi(u) := \int_0^u \sqrt{\sigma'(\xi)}\,\mathrm{d}\xi$ and

$$M_0 = \sup_{\varepsilon>0} \max\left(\|v_0^\varepsilon + \psi(u_0^\varepsilon)\|_\infty, \|v_0^\varepsilon - \psi(u_0^\varepsilon)\|_\infty\right)\ .$$

Remark. Observe that no growth assumptions are here needed on the flux function f, unlike condition (4.12) in [8]. Other uniform bounds were also found by Marcati [20] by using a quite different approach. Observe also that for fixed initial data it is enough to assume the stability condition (7.5) only for $|u| \leqq \psi^{-1}(2M)$.

Proof: Consider the following viscous approximation of system (1.1), for $\delta > 0$

$$\begin{cases} \partial_t u + \partial_x v = \delta \partial_{xx} u\ , \\ \partial_t v + \partial_x \sigma(u) = -\frac{1}{\varepsilon}(v - f(u)) + \delta \partial_{xx} v \end{cases} \tag{7.7}$$

and for some regular initial data

$$(u(x,0), v(x,0)) = (u_0^{\varepsilon\delta}(x), v_0^{\varepsilon\delta}(x)) \tag{7.8}$$

for all $x \in \mathbb{R}$, where $(u_0^\varepsilon, v_0^\varepsilon) = \lim_{\delta\to 0}(u_0^{\varepsilon\delta}, v_0^{\varepsilon\delta})$ in $L^1_{\mathrm{loc}}(\mathbb{R})$, and $\|(u_0^{\varepsilon\delta}, v_0^{\varepsilon\delta})\|_\infty \leq \|(u_0^\varepsilon . v_0^\varepsilon)\|_\infty$.

For any fixed $\delta > 0$, let $(u^{\varepsilon\delta}, v^{\varepsilon\delta})$ be the local smooth solution for (7.7)–(7.8) on the strip $\mathbb{R} \times (0, T^\delta)$.

Then the eigenvalues for the system (7.7) are

$$\lambda_\pm = \pm\sqrt{\sigma'(u^{\varepsilon\delta})}$$

and the associated Riemann invariants are

$$\begin{aligned} w = w(u^{\varepsilon\delta}, v^{\varepsilon\delta}) &= v^{\varepsilon\delta} - \psi(u^{\varepsilon\delta}) \ , \\ z = z(u^{\varepsilon\delta}, v^{\varepsilon\delta}) &= -v^{\varepsilon\delta} - \psi(u^{\varepsilon\delta}) \ . \end{aligned}$$

Rewriting the system in the Riemann coordinates we obtain

$$\left\{ \begin{array}{l} \partial_t w + \lambda_- \partial_x w = -\frac{1}{\varepsilon}\, G(w,z) + \delta \partial_{xx} w + \delta \frac{\sigma''(u^{\varepsilon\delta})}{2\sqrt{\sigma'(u^{\varepsilon\delta})}} (\partial_x u^{\varepsilon\delta})^2 \\ pa_t z + \lambda_+ \partial_x z = \frac{1}{\varepsilon}\, G(w,z) + \delta \partial_{xx} z + \delta \frac{\sigma''(u^{\varepsilon\delta})}{2\sqrt{\sigma'(u^{\varepsilon\delta})}} (\partial_x u^{\varepsilon\delta})^2 \end{array} \right. \tag{7.9}$$

where

$$G(w,z) = \frac{w-z}{2} - f\left(\psi^{-1}\left(-\frac{w+z}{2}\right)\right) \ . \tag{7.10}$$

Observe now that the system (7.9) can be compared with the following parabolic 2×2 system

$$\partial_t U + \Lambda \partial_x U - \delta \partial_{xx} U = \frac{1}{\varepsilon}\Gamma(U) \tag{7.11}$$

where $U = (u_1, u_2) \in \mathbb{R}$, $\Lambda = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}$ and $\Gamma(U) = \big(-G(u_1,u_2),\, G(u_1,u_2)\big)$. This system is quasimonotone if the right–hand side verifies the conditions given by Definition 2.4, which read now

$$\frac{\partial G}{\partial u_1} \geq 0 \quad , \quad \frac{\partial G}{\partial u_2} \leq 0 \ . \tag{7.12}$$

It is easy to check that (7.12) is equivalent to the stability condition (7.5).

Now, as in Section 3, we consider the system of ordinary differential equations

$$\left\{ \begin{array}{l} \dot p_1 = -\frac{1}{\varepsilon}\, G(p_1, p_2) \\ \dot p_2 = \frac{1}{\varepsilon}\, G(p_1, p_2) \ , \end{array} \right. \tag{7.13}$$

with $p_1(0) = p_2(0) = p_0$.

The solutions of (7.13) are spatially homogeneous solutions of the system (7.11) and are explicitly given by

$$p_1(t) = p_0 + f(\psi^{-1}(-p_0))(1 - e^{-\frac{t}{\varepsilon}}) ,$$
$$p_2(t) = p_0 - f(\psi^{-1}(-p_0))(1 - e^{-\frac{t}{\varepsilon}}) .$$

Therefore we can apply the standard quasimonotone comparison principle for parabolic systems (see [23], [11]) to the system (7.11) by observing that (w, z) are a subsolution of (7.11) for $w + z \geq 0$ and a supersolution for $w + z \leq 0$. So, if $(p_1^{\pm}, p_2^{\pm})$ are the solutions of (7.13) for the initial data

$$p_0^{\pm} = \pm \sup_{\varepsilon,\delta>0} \max \left(\|v^{\varepsilon\delta} + \psi(u_0^{\varepsilon\delta})\|_\infty, \|v_0^{\varepsilon\delta} - \psi(u_0^{\varepsilon\delta})\|_\infty \right) ,$$

we obtain

$$p_1^-(t) \leq w(x,t) \leq p_1^+(t), \quad p_2^-(t) \leq z(x,t) \leq p_2^+(t) ,$$

for all $(x,t) \in \mathbb{R} \times (0, T^\delta)$.

Then

$$|v^{\varepsilon\delta} \pm \psi(u^{\varepsilon\delta})| \leq M$$

where

$$M = M(\|u_0^\varepsilon\|_\infty, \|v_0^\varepsilon\|_\infty) = \sup_{|\lambda|\leq p_0^+} \left(|\lambda| + |f(\psi^{-1}(\lambda))| \right) .$$

Since

$$v^{\varepsilon\delta} = \frac{w - z}{2} \quad , \quad u^{\varepsilon\delta} = \psi^{-1}\left(\frac{w+z}{2}\right) ,$$

there exist two constants M_1, $M_2 > 0$, not depending on ε, δ, such that

$$|u^{\varepsilon\delta}(x,t)| \leq M_1 \quad , \quad |v^{\varepsilon\delta}(x,t)| \leq M_2 \tag{7.15}$$

for all $(x,t) \in \mathbb{R} \times (0, T^\delta)$. Hence standard continuation arguments yield the global existence of uniformly bounded solutions.

Finally, we can use the compactness framework given by DiPerna ([10]) for the system of elasticity to show that the smooth sequence $(u^{\varepsilon\delta}, v^{\varepsilon\delta})$ of the solutions of (7.7)–(7.8) converges strongly and almost everywhere to an entropy solution $(u^\varepsilon, v^\varepsilon)$ of (7.1)–(7.2), as $\delta \to 0^+$ and for any fixed $\varepsilon > 0$.

These limit solutions are obviously still uniformly bounded in ε by the same constants M_1, M_2. The proof is complete.

As a consequence of the uniform estimate (7.6) and thanks to Theorems 3.2 and 4.1 in [8], we obtain the following improved result of relaxation, without any restriction on the growth of the function $f(u)$.

THEOREM 7.2. *Assume the functions $\sigma(u)$ and $f(u)$ satisfy (7.3), (7.4) and the stability condition (7.5). Assume the initial data $(u_0^\varepsilon, v_0^\varepsilon) \in L^\infty \cap L^2$ and verify (H_2). Then there exists a constant $N_0 > 0$ such that, if*

$$||(u_0^\varepsilon, v_0^\varepsilon)||_{L^\infty} \leq N_0 \ ,$$

there exists a subsequence, still denoted $(u^\varepsilon, v^\varepsilon)$, of global weak solutions to the Cauchy problem (7.1)–(7.2) which converge pointwise almost everywhere:

$$(u^\varepsilon(x,t), v^\varepsilon(x,t)) \longrightarrow (u(x,t), v(x,t)) \ ,$$

as $\varepsilon \to 0^+$. The limit functions (u, v) satisfy
(i) *$v(x,t) = f(u(x,t))$ for almost all $(x,t) \in \mathbb{R} \times (0,\infty)$;*
(ii) *u is a weak solution to the Cauchy problem (5.1)–(5.2).*

Observe that, as in [8], it is actually possible to prove that the limit function u verifies a finite number of entropy inequalities, but we are not able to show that u is actually the unique entropy solution (then verifying an infinite number of inequalities) in the sense of Kružkov.

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