Convergence of Relaxation Schemes for Conservation Laws

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Abstract. We study the stability and the convergence for a class of relaxing numerical schemes for conservation laws. Following the approach recently proposed by S. Jin and Z. Xin, we use a semilinear local relaxation approximation, with a stiff lower order term, and we construct some numerical first and second order accurate algorithms, which are uniformly bounded in the L^∞ and BV norms with respect to the relaxation parameter. The relaxation limit is also investigated.

Key words and phrases: Relaxation schemes, conservation laws, shock waves, entropy conditions, hyperbolic singular perturbations.

1. Introduction

In this paper we investigate a new class of numerical schemes, which are based on the local relaxation approximation of conservation laws. Consider the initial value problem

$$\partial_t u + \partial_x f(u) = 0 \ , \tag{1.1}$$

$$u(x,0) = \overline{u}_0(x) \tag{1.2}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$. Here f is a given (say C^1) smooth function such that $f(0) = f'(0) = 0$.

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We want to approximate this problem by a sequence of semilinear hyperbolic problems with a stiff relaxation term, when $\varepsilon \to 0$ and for some $a > 0$ to be fixed later:

$$\left\{ \begin{array}{l} \partial_t u^\varepsilon + \partial_x v^\varepsilon = 0 \ , \\ \partial_t v^\varepsilon + a\, \partial_x u^\varepsilon = -\frac{1}{\varepsilon}\,(v^\varepsilon - f(u^\varepsilon)) \qquad (\varepsilon > 0), \end{array} \right. \tag{1.3}$$

$$\left\{ \begin{array}{l} u^\varepsilon(x,0) = u_0^\varepsilon(x) \\ v^\varepsilon(x,0) = v_0^\varepsilon(x) \ . \end{array} \right. \tag{1.4}$$

This kind of approximation was first proposed in [JX] in the general setting of the quasilinear systems of, possibly multidimensional, hyperbolic conservation laws. It possedes some very interesting features: finite speed of propagation, great generality (in particular it is possible to avoid the use of Riemann problems), strong physical motivations.

In fact, it is possible to trace back the origin of this kind of approximation on one side from the investigations concerning hyperbolic conservation laws with relaxing source [Li], [CL], [CLL]. In these papers the authors consider some relaxing models arising in applications (see the references inside of these quoted papers) and derive some stability conditions for convergence.

On the other hand this approximation is strongly connected with the study of fluid-dynamical limits [Ce], [CIP], and in particular for the discrete Boltzmann equations [PI], as the Broadwell model [CP]. In fact the main idea here is to replace the physically underlying non–equilibrium Boltzmann kinetic, with an artificial "minimalist" model, which recover in the limit the same equilibrium hydrodynamic configuration. We recall, in this spirit, the Boltzmann schemes for linear systems [HLL] and a number of similar kinetic schemes developed for compressible Euler equations (see [Br], [Per] and references therein). Also, from a computational point of view, there are many relations with the lattice Boltzmann methods proposed by McNamara and Zanetti, see for instance the collection of papers in [Do].

The analytical study of the (equilibrium) scalar case was started in [GM] and [PT] for the *continuous Boltzmann approximation* (see also [LPT] for some further theoretical developments) and in [Na] for the *discrete case* (1.3) (see Section 2 for a summary of the main results in that paper).

Let us also observe that for system (1.3) we can eliminate the unknown v to obtain a scalar conservation law with a second–order perturbation term

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = -\varepsilon(\partial_{tt} u^\varepsilon - a\partial_{xx} u^\varepsilon) \ .$$

Therefore we can say that our schemes correspond to an "hyperbolic" regularization of the first–order equation.

From a numerical point of view, hyperbolic conservation laws with stiff (relaxing) source were extensively studied in [Ji], [JL], [Pe1,2], [CJR]. The main emphasis in these investigations was on numerical schemes which can work in the underresolved domain, i.e.: when the relaxation parameter ε is small with respect with the time and space mesh–length.

In the present paper we take advantage on some monotonicity properties of (1.3). Let us write (1.3)–(1.4) in the following diagonal form:

$$(1.5) \qquad \begin{cases} \partial_t w^\varepsilon + \sqrt{a}\, \partial_x w^\varepsilon = \frac{1}{\varepsilon}\, G_a(w^\varepsilon, z^\varepsilon) \\ \partial_t z^\varepsilon - \sqrt{a}\, \partial_x z^\varepsilon = -\frac{1}{\varepsilon}\, G_a(w^\varepsilon, z^\varepsilon) \end{cases}$$

$$(1.6) \qquad \begin{cases} w^\varepsilon(x,0) = w_0^\varepsilon(x) \\ z^\varepsilon(x,0) = z_0^\varepsilon(x) \ . \end{cases}$$

where

$$(1.7) \qquad G_a(w,z) = \frac{-w+z}{2} - f\left(-\frac{w+z}{2\sqrt{a}}\right) \ ,$$

and $(w^\varepsilon, z^\varepsilon)$ (respectively $(w_0^\varepsilon, z_0^\varepsilon)$) are given by

$$(1.8) \qquad w^\varepsilon = -v^\varepsilon - \sqrt{a}\, u^\varepsilon \quad , \quad z^\varepsilon = v^\varepsilon - \sqrt{a}\, u^\varepsilon$$

(respectively $w_0^\varepsilon = -v_0^\varepsilon - \sqrt{a}\, u_0^\varepsilon \quad , \quad z_0^\varepsilon = v_0^\varepsilon - \sqrt{a}\, u_0^\varepsilon$). The main argument used in this paper is the following: under the Whitham's subcharacteristic condition [Wh], [Li] (see also Theorem 2.4 in Section 2) the system (1.5) is a quasimonotone weakly coupled system in the sense of Definition 2.2 below (see also [HN] for the more general quasilinear case) then allowing comparison properties. Therefore we can construct first and second order numerical schemes preserving this property. This is done by considering a fractional–step scheme, where the homogeneous (linear) part is treated by some monotone or even higher order scheme (as Goodman–LeVeque) and then the source term is solved exactly thanks to its particular structure.

The paper is actually organized as follows. In Section 2 we recall some results about quasimonotone weakly coupled systems and the analytical results about problem (1.3)–(1.4).

In Section 3 we present some numerical schemes for the diagonal problem (1.5)–(1.6): a family of monotone schemes and a scheme based on the second order Goodman–LeVeque scheme. In both cases we are able to prove the uniform stability with respect to ε of the schemes in the L^∞, L^1, BV norms and the convergence for fixed ε.

In Section 4 we prove that for a fixed time step, the approximated solutions of (1.3)–(1.4) converge as $\varepsilon \to 0$, to the numerical approximations given by a TVD, L^∞–stable discretization of the limit conservation law (1.1)–(1.2). Convergence to solutions of (1.1)–(1.2) is proved. Numerical experiments are also presented. Comparison with schemes presented in [JX] is done. Although our relaxing schemes are different from those proposed in [JX], it appears that some relaxed schemes ($\varepsilon = 0$) coincide.

2. Theoretical Backgrounds

Let us recall some general results concerning semilinear (diagonal) hyperbolic systems. The proofs are quite classical and will be omitted (see for example [HN] for a complete theory in the quasilinear weakly coupled case).

Consider the Cauchy problem

$$\partial_t u_i + \lambda_i \partial_x u_i = g_i(U) \ , \quad i = 1, \ldots, N \ , \tag{2.1}$$

$$U(x,0) = U^0(x) \ , \tag{2.2}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$. Here $U = (u_1, \ldots, u_N)$, $\lambda_i \in \mathbb{R}$ $(i = 1, \ldots, N)$ and $G(U) = (g_1(U), \ldots, g_N(U))$ is a given locally Lipschitz continuous function; $U^0 = (u_1^0, \ldots, u_N^0) \in L^\infty(\mathbb{R})^N$.

Definition 2.1. *A function $U \in L^\infty\left(\mathbb{R} \times (0,T)\right)^N$ $(T > 0)$ is a (weak) solution of the Cauchy problem* (2.1)–(2.2) *if for all $\varphi \in C_0^\infty\left(\mathbb{R} \times (0,T)\right)$ and every $i = 1, \ldots, N$,*

$$\iint \left[u_i(\partial_t \varphi + \lambda_i \partial_x \varphi) + g_i(U)\varphi\right] \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{2.3}$$

and for any interval $I \subseteq \mathbb{R}$

$$\lim_{T\to 0} \frac{1}{T}\int_0^T \int_I |u_i(x,t) - u_i^0(x)|\,\mathrm{d}x\,\mathrm{d}t = 0\ . \tag{2.4}$$

Theorem 2.1. *For any* $U^0 \in L^\infty(\mathbb{R})^N$, *there is* $T > 0$ *(depending only on* $\|U^0\|_\infty$*) such that in* $\mathbb{R}\times(0,T)$ *there exists a unique (weak) solution* U *of* (2.1)–(2.2) *and* $U \in C\left([0,T];\, L^1_{\mathrm{loc}}(\mathbb{R})^N\right)$.

Moreover only two possibilities occur: either $U \in L^\infty\left(\mathbb{R}\times(0,T)\right)^N$ *for any* $T > 0$ *or there exists* $T^* < +\infty$ *such that, for any* $T < T^*$, U *is defined on* $\mathbb{R}\times(0,T)$ *and*

$$\lim_{T\to T^*_-} \|U\|_{L^\infty(\mathbb{R}\times(0,T))^N} = +\infty\ .$$

Define

$$L = L(\lambda) = \max_{1\le i\le N} |\lambda_i|\ .$$

For any given vectors A, $B \in \mathbb{R}^N$, set $A \le B$ if $a_i \le b_i$ $(i = 1,\dots,N)$; if $A \le B$, denote the corresponding interval by $[A,B]$. For any fixed interval $[A,B]$ (with non empty interior) set

$$M = M(A,B,G) = \max_{1\le i,j\le N} \|\partial_{u_j} g_i\|_{L^\infty([A,B])}\ .$$

Also, for each interval $I = (a,\beta) \subseteq \mathbb{R}$, we set $\tau = \frac{\beta - a}{2L}$.

Theorem 2.2 (Stability)**.** *Let* U *and* $\tilde U$ *be two weak solutions of the Cauchy problem* (2.1)–(2.2) *in* $\mathbb{R}\times(0,T)$ $(T>0)$, *for the right-hand side* G *and the initial data* U^0, $\tilde G$ *and* $\tilde U^0$ *respectively. Let* $[A,B] \subseteq \mathbb{R}^N$ *be an interval with non empty interior such that* U, $\tilde U \in [A,B]$. *Then for each interval* $(a,\beta) \subseteq \mathbb{R}$ *and for all* $t \in (0, \min(\tau,T))$, *we have*

$$\sum_{i=1}^N \int_{a+tL}^{\beta - tL} |u_i(x,t) - \tilde u_i(x,t)|\ \mathrm{d}x \le \sum_{i=1}^N \int_a^\beta |u_i^0(x) - \tilde u_i^0(x)|\,\mathrm{d}x$$
$$+ \sum_{i=1}^N \int_0^t \mathrm{d}s \int_{a+sL}^{\beta - sL} \mathrm{sgn}\ (u_i(x,s) - \tilde u_i(x,s))\left(g_i(U(x,s)) - \tilde g_i(\tilde U(x,s))\right)\ \mathrm{d}x\ .$$

To obtain comparison properties we have to restrict our analysis to quasimonotone interactions.

Definition 2.2. *Let Ω be a convex subset of $\mathbb{R}^N$. A function $G : \Omega \to \mathbb{R}^N$ is quasimonotone (non decreasing) if each component g_i of G is non decreasing in u_j for $i \neq j$.*

The quasimonotonicity is a necessary and sufficient condition for a flow generated by the ordinary differential equation $\dot{U} = G(U)$, to preserve the partial ordering in $\mathbb{R}^n$ (see [Sm]) and it is also well–known for reaction–diffusion flows [PW], [Fi]. For weakly coupled systems of quasilinear hyperbolic equations it was studied in [HN]. In the present case we have the following useful property.

Theorem 2.3 (Comparison). *Let U and $\tilde{U}$ be two weak solutions of problem (2.1)–(2.2) in $\mathbb{R} \times (0,T)$ for the initial data U^0 and $\tilde{U}^0$ respectively. Let $[A,B] \in \mathbb{R}^N$ be an interval (with non empty interior) such that*

(2.5) $\qquad G$ is quasimonotone in $[A,B]$,

(2.6) $\qquad U,\, \tilde{U} \in [A,B]$ almost everywhere in $\mathbb{R} \times (0,T)$.

If $U^0 \leq \tilde{U}^0$ for almost every $x \in \mathbb{R}$, then $U \leq \tilde{U}$ for almost every $(x,t) \in \mathbb{R} \times (0,T)$.

In the remainder of this section we shall recall some results obtained in [Na] about the analytical properties of problem (1.3)–(1.4), which will be of use in the sequel.

Let us consider the following assumptions:

($\mathcal{H}_1$) f is a C^1 function with $f(0) = f'(0) = 0$;

($\mathcal{H}_2$) the functions $(u_0^\varepsilon, v_0^\varepsilon)$ verify $v_0^\varepsilon = f(u_0^\varepsilon)$, for all $\varepsilon > 0$, and are uniformly bounded in $L^\infty(\mathbb{R})^2$ by

$$N_0 = \max\left(\sup_{\varepsilon>0} \|u_0^\varepsilon\|_\infty,\ \sup_{\varepsilon>0} \|v_0^\varepsilon\|_\infty\right)$$

($\mathcal{H}_3$) The sequence $(u_0^\varepsilon, v_0^\varepsilon)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R})^2$ to some limit $(u_0, f(u_0)) \in L^\infty(\mathbb{R})^2$, as $\varepsilon \to 0^+$.

Finally let us denote, for any $N > 0$,

$$F(N) := \sup_{|\xi| \leq N} |f(\xi)| \ , \tag{2.7}$$

$$B(N) := 2N + F(2N) \tag{2.8}$$

and

$$M(N) := \sup_{|\xi| \le B(N)} |f'(\xi)| \ . \tag{2.9}$$

First we recall the main global existence and boundedness result for problem (1.3)–(1.4).

Theorem 2.4. *Assume* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$*. For any* $N_0 > 0$ *and* $\varepsilon > 0$*, if*

$$\sqrt{a} > M(N_0) \ , \tag{2.10}$$

then there exists a unique globally bounded solution $(u^\varepsilon, v^\varepsilon)$ *to* (1.3)–(1.4) *in* $C([0,\infty);\ L^1_{\mathrm{loc}}(\mathbb{R})^2)$ *and we have*

$$\|v^\varepsilon \pm \sqrt{a}\, u^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,\infty))} \le \sqrt{a}\, B(N_0) \ . \tag{2.11}$$

Moreover there holds the subcharacteristic inequality

$$|f'(u^\varepsilon(x,t))| < \sqrt{a} \tag{2.12}$$

for all $\varepsilon > 0$ *and for almost every* $(x,t) \in \mathbb{R}\times(0,\infty)$.

It turns out that, as $\varepsilon \to 0^+$, the sequence $\{(u^\varepsilon)\}$ converges towards the entropy solution $u = u(x,t)$, in the sense of Kružkov [Kr], of the scalar problem (1.1)–(1.2). More precisely we have the following statement.

Theorem 2.5. *Assume* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *and* (2.10) *and let* $(u^\varepsilon, v^\varepsilon)$ *be the global solution to problem* (1.3)–(1.4) *given by Theorem 2.4. Then there exists a weak solution* u *to* (1.1)–(1.2) *and a subsequence, still denoted* $(u^\varepsilon, v^\varepsilon)$*, such that*

$$u^\varepsilon \longrightarrow u \qquad \text{in } C([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R})) \ , \tag{2.13}$$

as $\varepsilon \to 0^+$,

$$v^\varepsilon \longrightarrow f(u) \qquad \text{in } C([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R})) \ , \tag{2.14}$$

as $\varepsilon \to 0^+$.

Assume moreover that one of the following hypothesis holds true:

(2.15) $\sup_{u\in\mathbb{R}} |f'(u)| < \infty$;

(2.16) $f \in C^2$ and there is $M > 0$ such that if $|u| \ge M$, then $|f''(u)| > 0$.

Then, for every $N_0 > 0$ there exists $a_0 > 0$, with $\sqrt{a_0} > M(N_0)$, such that, for any $a \geq a_0$ the weak solution u of (1.1)–(1.2), given by the first part of the statement, is actually the unique entropy solution in the sense of Kružkov.

3. Estimates on Relaxing Schemes

3.1. The numerical scheme.

In order to approximate the solutions of the initial value problem (1.3)–(1.4), we actually discretize (1.5)–(1.6). We denote $Z^\varepsilon = (w^\varepsilon, z^\varepsilon)$ and $Z_0^\varepsilon = (w_0^\varepsilon, z_0^\varepsilon)$ and we make the following assumptions:

(H_1) f is a C^1 function with $f(0) = f'(0) = 0$;

(H_2) $u_0^\varepsilon \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, $v_0^\varepsilon = f(u_0^\varepsilon)$ and there exist constants $\mu_\infty > 0$, $\mu_1 > 0$, $\nu > 0$ not depending on ε such that:

$$\| u_0^\varepsilon \|_\infty \leq \mu_\infty, \qquad \| u_0^\varepsilon \|_1 \leq \mu_1, \qquad TV(u_0^\varepsilon) \leq \nu$$

(H_3) $u_0^\varepsilon \to \overline{u}_0$ in $L^1(\mathbb{R})$.

Then v_0^ε is bounded in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and $v_0^\varepsilon \to f(u_0)$ in $L^1(\mathbb{R})$.

For any given Δt, Δx set $\lambda = \frac{\sqrt{a}\Delta t}{\Delta x}$ and construct the numerical solution Z_Δ^ε of (1.5)–(1.6) on a strip $\mathbb{R} \times [0, T]$. Z_Δ^ε is constant on each interval $]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[\, \times [t_n, t_{n+1}[$, where

$$x_{j-\frac{1}{2}} = x_j - \frac{\Delta x}{2} \quad , \quad x_j = j\,\Delta x \quad , \quad j \in \mathbb{Z}\,,$$

$$t_n = n\,\Delta t \quad , \quad n \in \{0, \ldots, N\} \quad \text{with} \quad [N\Delta t] = T\,.$$

At this point we drop superscript ε as far as there is no ambiguity. We set

$$Z_\Delta(x,t) = \sum_{n=0}^{N} \sum_{j\in} Z_j^n \; \mathbf{1}_{]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[}(x) \; \mathbf{1}_{[t_n, t_{n+1}[}(t)$$

and

$$Z_\Delta^n(x) = \sum_{j\in} Z_j^n \; \mathbf{1}_{]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[}(x) \quad , \quad Z^n = (Z_j^n)_{j\in}$$

and similar notations are used for variables (u, v).

Z^0 is obtained in following way: u_j^0 is the average of u_0 on the cell $I_j =]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[$:

$$u_j^0 = \frac{1}{\Delta x} \int_{I_j} u_0(x)dx$$

It is well known that the discretization error is of order Δx. More precisely :

$$\|u_\Delta^0 - u_0\|_1 \leq \Delta x \, TV(u_0) \tag{3.1}$$

We take then $v_j^0 = f(u_j^0)$. By (H$_1$)–(H$_2$) we obtain:

$$\| v_\Delta^0 - v_0 \|_1 \leq \sup_{\xi \leq \mu_\infty} | f'(\xi) | \| u_\Delta^0 - u_0 \|_1 \leq M(\mu_\infty)\nu\Delta x$$

and

$$\begin{aligned} \| Z_\Delta^0 - Z_0 \|_1 &:= \| w_\Delta^0 - w_0 \|_1 + \| z_\Delta^0 - z_0 \|_1 \\ &\leq 2(M(\mu_\infty) + \sqrt{a})\nu\Delta x \end{aligned}$$

System (1.5) is splitted into a linear hyperbolic part and an ordinary differential system.

For given Z_Δ^n, $Z_\Delta^{n+\frac{1}{2}}$ is an approximate solution at time t_{n+1} of the system

$$\begin{cases} \partial_t w + \sqrt{a}\ \partial_x w = 0 \\ \partial_t z - \sqrt{a}\ \partial_x z = 0 \\ z(t_n, x) = Z_\Delta^n(x) \end{cases}$$

$Z_\Delta^{n+\frac{1}{2}}$ is obtained by discretizing each equation by two different ways. The first is a consistent monotone scheme written in the conservative form:

$$Z_j^{n+\frac{1}{2}} = Z_j^n - \lambda(\phi_{j+\frac{1}{2}}^n - \phi_{j-\frac{1}{2}}^n) \tag{3.2}$$

where $\phi_{j+\frac{1}{2}}^n = \Big(\varphi(w_{j-k+1}^n, \ldots, w_{j+k}^n), -\psi(z_{j-k+1}^n, \ldots, z_{j+k}^n)\Big)$, φ, ψ are Lipschitz continuous fluxes and $\lambda \leq 1$. Generally we take the same scheme for each equation so that $\varphi = \psi$. Of course such a scheme is only first order accurate.

We now study the case of a second order Goodman-LeVeque scheme [GL]. Using standard notation we denote:

$$\begin{cases} \sigma_j^n = \frac{1}{\Delta x} \quad \min\bmod(w_{j+1}^n - w_j^n, w_j^n - w_{j-1}^n) \\ \tau_j^n = \frac{1}{\Delta x} \quad \min\bmod(z_{j+1}^n - z_j^n, z_j^n - z_{j-1}^n) \end{cases}$$

where the minmod function is defined by:

$$\min\text{mod}\,(a,b) = \frac{1}{2}[\,\text{sgn}\,(a) + \text{ sgn}\,(b)]\min(|a|,|b|)\ .$$

Then the scheme may be written, for $\lambda \leq 1$, as:

$$\left\{\begin{array}{l} w_j^{n+\frac{1}{2}} = w_j^n - \lambda(w_j^n - w_{j-1}^n) - \frac{1}{2}\lambda(1-\lambda)\Delta x(\sigma_j^n - \sigma_{j-1}^n) \\ z_j^{n+\frac{1}{2}} = z_j^n + \lambda(z_{j+1}^n - z_j^n) + \frac{1}{2}\lambda(-1+\lambda)\Delta x(\tau_{j+1}^n - \tau_j^n) \end{array}\right. \tag{3.3}$$

Note that the Riemann problem for a piecewise linear function can be solved exactly, providing formula(3.3).

We recall that (3.3) is a generalization of Godunov scheme obtained by application of the following three steps:

1) given data $\{w_j^n, z_j^n\}$ construct the piecewise linear function $\{\tilde{w}^n(x), \tilde{z}^n(x)\}$:

$$\left\{\begin{array}{ll} \tilde{w}^n(x) = w_j^n + (x - x_j)\sigma_j^n & \text{for } x \in]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \\ \tilde{z}^n(x) = z_j^n + (x - x_j)\tau_j^n & \end{array}\right.$$

2) solve the linear system exactly with this data to obtain $\{\tilde{w}(t_{n+1}, x), \tilde{z}(t_{n+1}, x)\}$
3) compute cell average of the resulting solution to obtain $\{w_j^{n+1/2}, z_j^{n+1/2}\}$.

This scheme may also be written in the same conservative form as (3.2) with numerical flux ϕ given by:

$$\left\{\begin{array}{l} \varphi_{j+\frac{1}{2}}^n = w_j^n + \frac{(1-\lambda)}{2}\Delta x\ \sigma_j^n \\ \psi_{j+\frac{1}{2}}^n = z_{j+1}^n + \frac{\lambda-1}{2}\Delta x\ \tau_{j+1}^n\ . \end{array}\right. \tag{3.4}$$

The nonlinear part is treated in both cases by solving exactly on $[t_n, t_{n+1}]$ the problem:

$$\left\{\begin{array}{l} \omega' = \frac{1}{\varepsilon}G_a(\omega,\zeta) \\ \zeta' = -\frac{1}{\varepsilon}G_a(\omega,\zeta) \\ \omega(t_n) = w_j^{n+1/2},\ \zeta(t_n) = z_j^{n+1/2} \qquad j \in \mathbb{Z}\ . \end{array}\right. \tag{3.5}$$

Denote by $S_t^a(\theta_0, (\omega_0, \zeta_0))$ the solution at time t of (3.5) with $(\omega(\theta_0), \zeta(\theta_0)) = (\omega_0, \zeta_0)$ and Z_j^{n+1} given by:

$$Z_j^{n+1} := S_{t_{n+1}}^a(t_n, Z_j^{n+1/2})\ .$$

Observe that the solution of (3.5) is explicitly given by

$$\begin{cases} \omega(t) = \omega(t_n) + (1 - e^{-\frac{t-t_n}{\varepsilon}}) G_a(\omega(t_n), \zeta(t_n)) \\ \zeta(t) = \zeta(t_n) - (1 - e^{-\frac{t-t_n}{\varepsilon}}) G_a(\omega(t_n), \zeta(t_n)) \ . \end{cases}$$

Now we can write our scheme in its explicit form:

$$\begin{cases} w_j^{n+1} = w_j^{n+\frac{1}{2}} + (1 - e^{\frac{-\Delta t}{\varepsilon}}) G_a(w_j^{n+\frac{1}{2}}, z_j^{n+\frac{1}{2}}) \\ z_j^{n+1} = z_j^{n+\frac{1}{2}} - (1 - e^{\frac{-\Delta t}{\varepsilon}}) G_a(w_j^{n+\frac{1}{2}}, z_j^{n+\frac{1}{2}}) \ . \end{cases} \tag{3.6}$$

It is easy to obtain the following useful relation

$$G_a(\omega(t), \zeta(t)) = e^{-\frac{t-t_n}{\varepsilon}} \quad G_a(\omega(t_n), \zeta(t_n)) \ . \tag{3.7}$$

In Sections 3.2 to 3.5 we shall give various estimates for the scheme (3.2)–(3.6) and prove convergence for fixed $\varepsilon > 0$. The Section 3.6 is devoted to the uniform estimates for the second order scheme (3.3)–(3.6).

3.2. The supremum norm bound.

For a large enough we find here a uniform bound for the numerical approximations given by the scheme (3.2)–(3.6).

Denote

$$N_\infty = \max(\sup_\varepsilon \parallel u_0^\varepsilon \parallel_\infty, \sup_\varepsilon \parallel v_0^\varepsilon \parallel_\infty)$$

so that

$$\max(\sup_j \mid w_j^0 \mid, \sup_j \mid z_j^0 \mid) \leq N_\infty(1 + \sqrt{a})$$

Define

$$\underline{Z}_0 = -N_\infty(1+\sqrt{a})\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\overline{Z}_0 = N_\infty(1+\sqrt{a})\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$a_0 = \sup_{|\xi| \leq N_\infty} |f'(\xi)|^2$$

and consider $a \geq a_0$. $\underline{Z}(t) = S_t^a(0, \underline{Z}_0)$, $\overline{Z}(t) = S_t^a(0, \overline{Z}_0)$ are given by the formulas:

$$\begin{cases} \underline{w}(t) = -N_\infty(1+\sqrt{a}) - (1 - e^{-t/\varepsilon}) f(N_\infty \frac{1+\sqrt{a}}{\sqrt{a}}) \ , \\ \underline{z}(t) = -N_\infty(1+\sqrt{a}) + (1 - e^{-t/\varepsilon}) f(N_\infty \frac{1+\sqrt{a}}{\sqrt{a}}) \ , \end{cases}$$

and respectively

$$\begin{cases} \overline{w}(t) = N_\infty(1+\sqrt{a}) - (1-e^{-t/\varepsilon}) f(-N_\infty \frac{1+\sqrt{a}}{\sqrt{a}}) \ , \\ \overline{z}(t) = N_\infty(1+\sqrt{a}) + (1-e^{-t/\varepsilon}) f(-N_\infty \frac{1+\sqrt{a}}{\sqrt{a}}) \ . \end{cases}$$

Therefore, using notation (2.7), we have

$$\begin{aligned} \max(|\overline{w}(t)|, |\overline{z}(t)|, |\underline{w}(t)|, |\underline{z}(t)|) &\leq 2N_\infty(1+\sqrt{a}) + F(N_\infty(1+\frac{1}{\sqrt{a_0}})) \\ &\leq M_\infty(1+\sqrt{a}) \end{aligned}$$

Denoting $A_\infty = 2M_\infty(1+\sqrt{a}) + F(M_\infty(1+\frac{1}{\sqrt{a_0}}))$ and choosing A such that $A_\infty \leq A(1+\sqrt{a})$ we define

$$a_1 = \max(a_0, \sup_{|\xi| \leq A(\frac{1}{\sqrt{a_0}}+1)} |f'(\xi)|^2)$$

Proposition 3.1. *Suppose* (H$_1$)–(H$_2$) *and let* $a \geq a_1$.
(i) *for any* $(w,z) \in [-A(1+\sqrt{a}), A(1+\sqrt{a})]^2$ *the subcharacteristic condition is satisfied:*

$$\left| f'\left(\frac{-w-z}{2\sqrt{a}}\right)\right| \leq \sqrt{a} \tag{3.8}$$

(ii) *the interaction term* G_a *is quasi monotone on* $[-A(1+\sqrt{a}), A(1+\sqrt{a})]^2$.

Proof. For any $(w,z) \in [-A(1+\sqrt{a}), A(1+\sqrt{a})]^2$, we have

$$\left| f'\left(\frac{-w-z}{2\sqrt{a}}\right)\right| \leq \sup_{|\xi| \leq \frac{A(1+\sqrt{a})}{\sqrt{a}}} |f'(\xi)| \leq \sup_{|\xi| \leq A(\frac{1}{\sqrt{a_0}}+1)} |f'(\xi)| \leq \sqrt{a_1} \leq \sqrt{a}\,.$$

(ii) is an immediate consequence of (i).

□

Theorem 3.2 (L^∞ Bound). *Suppose* (H$_1$)–(H$_2$) *are satisfied and* $a \geq a_1$. *For all* $n \in \{0, \ldots, N\}$, *it holds*

$$\|Z_\Delta^n\|_{L^\infty} \leq 2A(1+\sqrt{a}). \tag{3.9}$$

where A is a constant depending only on μ_∞.

Remark. Observe that this bound does not depend on T.

Proof. We shall prove that for $n \geq 0$ and $j \in \mathbb{Z}$:

$$Z_j^n \in [\underline{Z}(n\,\Delta t), \overline{Z}(n\,\Delta t)] \ .$$

We have $Z_j^0 \in [\underline{Z}_0, \overline{Z}_0]$ and extremas are not modified by the scheme (3.2) so that $Z_j^{1/2} \in [\underline{Z}_0, \overline{Z}_0]$ and, for any $t > 0$,

$$\begin{aligned} S_t^a(0, Z_j^{1/2}) &\in [-M_\infty(1+\sqrt{a}), M_\infty(1+\sqrt{a})]^2 \\ &\subseteq [-A(1+\sqrt{a}), A(1+\sqrt{a})]^2 \ . \end{aligned}$$

The interaction term G_a is quasi monotone on this set and then Theorem 2.3 yields

$$S_t^a(0, Z_j^{1/2}) \in [S_t^a(0, \underline{Z}_0), S_t^a(0, \overline{Z}_0)] \quad , \quad t > 0 \ .$$

In particular for $t = \Delta T$ one finds

$$Z_j^1 \in [\underline{Z}(\Delta t), \overline{Z}(\Delta t)] \ .$$

Suppose now $Z_j^k \in [\underline{Z}(k\,\Delta t), \overline{Z}(k\,\Delta t)]$, $0 \leq k \leq n$, $j \in \mathbb{Z}$. Then $Z_j^{n+\frac{1}{2}} \in [\underline{Z}(n\,\Delta t), \overline{Z}(n\,\Delta t)] \subset [-M_\infty(1+\sqrt{a}), M_\infty(1+\sqrt{a})]^2$ as extremas are not modified by the scheme (3.2). Hence for any $t > t_n$:

$$S_t^a(t_n, Z_j^{n+\frac{1}{2}}) \in [-A(1+\sqrt{a}), A(1+\sqrt{a})]^2 \ .$$

Finally by Theorem 2.3, we conclude that $Z_j^{n+1} \in [\underline{Z}(t_{n+1}), \overline{Z}(t_{n+1})]$, which ends the proof.

□

Corollary 3.3. *Suppose $\underline{u}_0$, $\overline{u}_0$ satisfy* (H$_1$)–(H$_2$) *and take $a \geq a_1$. The scheme* (3.2)–(3.6) *is monotone i.e.: if $\underline{Z}_j^0 \leq \overline{Z}_j^0$ for any $j \in \mathbb{Z}$, then for any $j \in \mathbb{Z}$ and any $n \in \{0, \ldots, N\}$ we have*

$$\underline{Z}_j^n \leq \overline{Z}_j^n \ .$$

As a particular case, if the initial conditions (w_0, z_0) are nonnegative, also the solution Z_Δ is nonnegative.

In what follows we shall suppose $a \geq a_1$.

3.3. BV estimate.

Theorem 3.4. *Suppose* (H_1)–(H_2) *are satisfied and* $a \geq a_1$. *The numerical scheme* (3.2)–(3.6) *is TVD, i.e.:*

$$TV(Z_\Delta^{n+1}) \leq TV(Z_\Delta^n) \tag{3.10}$$

The proof of this theorem is based on the following lemma.

Lemma 3.5. *Let* $(\omega_{0,1}, \zeta_{0,1})$ *and* $(\omega_{0,2}, \zeta_{0,2})$ *be two initial conditions for system* (3.5) *with corresponding solutions* (ω_1, ζ_1) *and* (ω_1, ζ_2). *Then it holds*

$$|\omega_1(t) - \omega_2(t)| + |\zeta_1(t) - \zeta_2(t)| \leq |\omega_{0,1} - \omega_{0,2}| + |\zeta_{0,1} - \zeta_{0,2}| \ .$$

The proof follows easily by using the quasimonotone properties of G and so it is omitted.

□

Proof of Theorem 3.4. The scheme (3.2) (on the linear part) being TVD, we have

$$TV(Z^{n+\frac{1}{2}}) \leq TV(Z^n) \ . \tag{3.11}$$

Then we apply Lemma 3.5, with

$$(\omega_{0,1}, \zeta_{0,1}) = (w_j^{n+\frac{1}{2}}, z_j^{n+\frac{1}{2}}) \quad \text{and} \quad (\omega_{0,2}, \zeta_{0,2}) = (w_{j-1}^{n+\frac{1}{2}}, z_{j-1}^{n+\frac{1}{2}}) \ ,$$

and we obtain, for all $j \in \mathbb{Z}$;

$$|w_j^{n+1} - w_{j-1}^{n+1}| + |z_j^{n+1} - z_{j-1}^{n+1}| \leq |w_j^{n+\frac{1}{2}} - w_{j-1}^{n+\frac{1}{2}}| + |z_j^{n+\frac{1}{2}} - z_{j-1}^{n+\frac{1}{2}}| \ . \tag{3.12}$$

□

As a corollary, using (H_1)–(H_2) and the fact that $TV(u_\Delta^0) \leq TV(u_0)$, there exists $V > 0$ not depending on ε and a such that

$$TV(Z_\Delta^n) \leq V(1 + \sqrt{a}) := V_a \tag{3.13}$$

3.4. L^1 stability.

Lemma 3.6. *Suppose* (H_1)–(H_2) *satisfied and* $a \geq a_1$. *There exists a positive constant* C, *not depending on* ε *and* a, *such that*

$$\sum_j |G_a(Z_j^{n+\frac{1}{2}})| \leq \sum_j |G_a(Z_j^n)| + C\lambda\, V_a \ . \tag{3.14}$$

Proof. We have

$$\begin{aligned}
G_a(Z_j^{n+\frac{1}{2}}) &= \frac{-w_j^{n+\frac{1}{2}} + z_j^{n+\frac{1}{2}}}{2} - f\left(\frac{-w_j^{n+\frac{1}{2}} - z_j^{n+\frac{1}{2}}}{2\sqrt{a}}\right) \\
&= \frac{-w_j^n + \lambda(\varphi_{j+\frac{1}{2}}^n - \varphi_{j-\frac{1}{2}}^n) + z_j^n + \lambda(\psi_{j+\frac{1}{2}}^n - \psi_{j-\frac{1}{2}}^n)}{2} \\
&\quad - f\left(\frac{-w_j^n + \lambda(\varphi_{j+\frac{1}{2}}^n - \varphi_{j-\frac{1}{2}}^n) - z_j^n - \lambda(\psi_{j+\frac{1}{2}}^n - \psi_{j-\frac{1}{2}}^n)}{2\sqrt{a}}\right) \\
&= G_a(Z_j^n) + \frac{\lambda}{2}\left[\left(\varphi_{j+\frac{1}{2}}^n - \varphi_{j-\frac{1}{2}}^n\right) + \left(\psi_{j+\frac{1}{2}}^n - \psi_{j-\frac{1}{2}}^n\right)\right] \\
&\quad - f'(\xi_j^n)\frac{\lambda}{2\sqrt{a}}\left[\left(\varphi_{j+\frac{1}{2}}^n - \varphi_{j-\frac{1}{2}}^n\right) + \left(\psi_{j+\frac{1}{2}}^n - \psi_{j-\frac{1}{2}}^n\right)\right] \ .
\end{aligned}$$

Since the subcharacteristic condition (3.8) is satisfied, it follows

$$|G_a(Z_j^{n+\frac{1}{2}})| \leq |G_a(Z_j^n)| + \lambda|\varphi_{j+\frac{1}{2}}^n - \varphi_{j-\frac{1}{2}}^n| + \lambda|\psi_{j+\frac{1}{2}}^n - \psi_{j-\frac{1}{2}}^n| \ .$$

The scheme (3.2) is TVD, and φ,ψ are Lipschitz continuous, which yields inequality (3.14).

□

Let us present our main stability result in L^1.

Theorem 3.7. *Suppose* $(u_0^\varepsilon, v_0^\varepsilon)$, $(\tilde{u}_0^\varepsilon, \tilde{v}_0^\varepsilon)$ *satisfy* (H_1)–(H_2) *and let* $a \geq a_1$. *Let* Z_Δ^ε, $\tilde{Z}_\Delta^\varepsilon$ *be two numerical approximations given by the scheme* (3.2)–(3.6) *with respect to the initial data* u_0 *and* $\tilde{u}_0$ *respectively. Then there exists a constant* C_a *not depending on* ε, $C_a \leq C(1+\sqrt{a})(1+V_a)$, *such that*

$$\|Z_\Delta^\varepsilon(t) - \tilde{Z}_\Delta^\varepsilon(t)\|_1 \leq C_a \|u_0^\varepsilon - \tilde{u}_0^\varepsilon\|_1 \ ; \tag{3.15}$$

$$\|v_\Delta^\varepsilon(t) - f(u_\Delta^\varepsilon(t))\|_1 \leq \varepsilon\, C_a \ , \tag{3.16}$$

for all $t \in [0,T]$*;*

$$\|Z^\varepsilon_\Delta(t) - Z^\varepsilon_\Delta(t')\|_1 \le C_a(\Delta t + |t-t'|) \tag{3.17}$$

for all $t, t' \in [0,T]$.

Proof. By assumptions (H_1)–(H_2) and the fact that $\|u^{\varepsilon,0}_\Delta - \tilde u^{\varepsilon,0}_\Delta\|_1 \le \|u^\varepsilon_0 - \tilde u^\varepsilon_0\|_1$ we have:

$$\|Z^{\varepsilon,0}_\Delta - \tilde Z^{\varepsilon,0}_\Delta\|_1 \le C(1+\sqrt{a})\|u^\varepsilon_0 - \tilde u^\varepsilon_0\|_1$$

By Lemma 3.5:

$$|w^{n+1}_j - \tilde w^{n+1}_j| + |z^{n+1}_j - \tilde z^{n+1}_j| \le |w^{n+\frac12}_j - \tilde w^{n+\frac12}_j| + |z^{n+\frac12}_j - \tilde z^{n+\frac12}_j| \; .$$

The scheme (3.2) on the linear part being L^1 contracting [CM] (3.15) follows.

In particular, choosing $\tilde u^\varepsilon_0 = 0$, we have

$$\|Z^\varepsilon_\Delta(t)\|_1 \le C_a\|u^\varepsilon_0\|_1 \le C_a\mu_1 \; . \tag{3.15$'$}$$

To prove (3.16) we use Lemma 3.6; denoting $\|v^n_\Delta - f(u^n_\Delta)\|_1 = \|G_a(Z^n_\Delta)\|_1 := A_n$ and $\beta = C\sqrt{a}V_a$ and using also (3.7):

$$A_{n+1}e^{\frac{\Delta t}{\varepsilon}} \le A_n + \beta\Delta t \tag{3.18}$$

from which we deduce, by a recursive argument:

$$A_{n+1} \le e^{-\frac{(n+1)\Delta t}{\varepsilon}}A_0 + \beta\Delta t\,\frac{e^{-\Delta t/\varepsilon}}{1-e^{-\Delta t/\varepsilon}}\left(1 - e^{-(n+1)\frac{\Delta t}{\varepsilon}}\right) \; .$$

We conclude noting that for $s>0 : \frac{se^{-s}}{1-e^{-s}} \le 1$.

To show the equicontinuity property (3.17) we need to estimate $\|Z^{n+1}_\Delta - Z^n_\Delta\|_1$. Using Lemma 3.5 and the property of L^1 contraction of (3.2) we obtain:

$$\|Z^{n+1}_\Delta - Z^n_\Delta\|_1 \le \|Z^1_\Delta - Z^0_\Delta\| \tag{3.19}$$

As in Lemma 3.6 we have:

$$\Delta x \sum_j |Z^{\frac12}_j - Z^0_j| \le C\sqrt{a}\Delta t V_a$$

and, by (3.14),

$$\Delta x \sum_j |Z_j^1 - Z_j^{\frac{1}{2}}| \leq 2\Delta x(1 - e^{-\frac{\Delta t}{\varepsilon}}) \sum_j |G_a(Z_j^{\frac{1}{2}})|$$
$$\leq 2\Delta x \left(\sum_j |G_a(Z_j^0)| + C\lambda\, V_a \right) .$$

Hence we obtain

$$\|Z_\Delta^{n+1} - Z_\Delta^n\|_1 \leq C_a \Delta t$$

If $t \in [t_n, t_{n+1}[$ and $t' \in [t_m, t_{m+1}[$, we have

$$\|Z_\Delta(t) - Z_\Delta(t')\|_{L^1} = \|Z_\Delta^n - Z_\Delta^m\|_{L^1} \leq |m-n| \cdot \|Z_\Delta^1 - Z_\Delta^0\|_{L^1} .$$

Using the fact that $|m-n|\Delta t \leq \Delta t + |t - t'|$, we conclude the proof. $\square$

Remark. The inequality (3.17) yields

$$\lim_{\varepsilon \to 0} v_\Delta^\varepsilon - f(u_\Delta^\varepsilon) = 0 \quad \text{in} \quad L^\infty([0,T]; L^1(\mathbb{R}))$$

The above estimates allow us to prove the convergence of our numerical scheme. For this we reintroduce superscript ε in the notations as in paragraphs 1, 2.

Theorem 3.8. *Let $\varepsilon > 0$, suppose (H_1)–(H_2) satisfied and $a \geq a_1$. For any $T > 0$, let $\lambda = \frac{\sqrt{a}\Delta t}{\Delta x}$ be constant. As $\Delta t \to 0$, the sequence $Z_\Delta^{(\varepsilon)}$ converges in $L^\infty(0,T; L^1(\mathbb{R})^2)$ to the unique solution $Z^{(\varepsilon)}$ of* (1.5)–(1.6)*, $Z^{(\varepsilon)} \in C^0([0,T], L^1(\mathbb{R})^2) \cap L^\infty(0,T; L^\infty(\mathbb{R})^2)$ and:*

$$\sup_{t \in (0,T)} \|Z^{(\varepsilon)}(t)\|_1 \leq C_a \|u_0^\varepsilon\|_1 \tag{3.21}$$

$$\|Z^{(\varepsilon)}(t) - Z^{(\varepsilon)}(t')\|_1 \leq C_a |t - t'| \tag{3.22}$$

$$\sup_{t \in [0,T]} \|v^\varepsilon(t) - f(u^\varepsilon(t))\|_1 \leq \varepsilon C_a . \tag{3.23}$$

As a corollary, we obtain a proof of the global existence for the solutions of (1.5)–(1.6) (see [Na] for a direct proof).

Proof of Theorem 3.8. The proof of convergence is classical using the above estimates (see for example [CM]). We just give here the sketch. Without loss of generality we can consider compactly supported initial data.

1) $\forall t \in [0,T]$ $H^\varepsilon_\Delta(t) = \{Z^\varepsilon_\Delta(t); \Delta t > 0\}$ is bounded in $L^1(\mathbb{R}) \cap BV(\mathbb{R})^2$ and then relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R})^2$ by Frechet-Kolmogorov theorem.
   Also, by the finite speed of propagation, there exists a compact set $K \subset \mathbb{R}$ containing $\operatorname{supp} Z^\varepsilon_\Delta(t)$ for $t \in [0,T]$, so that $H^\varepsilon_\Delta(t)$ is a relatively compact set in $L^1(\mathbb{R})^2$.
2) By a diagonal process, for all dense countable set $\{t_k\} \subset [0,T]$, one can find a subsequence of Z^ε_Δ converging for all t_k.
3) We use the equicontinuity property (3.17) as in the proof of Ascoli-Arzela's theorem and find a subsequence of Z^ε_Δ which converges in $L^\infty(0,T;L^1(\mathbb{R})^2)$ and a.e. This subsequence is uniformly bounded from Theorem 3.2.
4) We then obtain (3.21),(3.22),(3.23) as a consequence of Theorem 3.7. Z^ε is a weak solution of (1.5) by Lax-Wendroff theorem and using (3.22) we conclude that $Z^\varepsilon \in C^0([0,T],L^1(\mathbb{R}))^2$ and Z is the unique solution of (1.5)–(1.6) in the sense of definition 2.1.

□

3.6. Estimates for Goodman-LeVeque based scheme (3.3)–(3.6).

The estimates for scheme (3.3)–(3.6) rely on the following properties:

Proposition 3.9 [GL]. *The scheme* (3.3) *is second order accurate in smooth regions, at least away from extreme points. It is TVD, L^∞-stable and extreme values are not modified.*

Since constant functions are invariant by scheme (3.3) and extreme values are not modified, for scheme (3.3)–(3.6) we can follow the proof of Theorem 3.2 without changes. The homogeneous part being TVD, (3.3)–(3.6) is also TVD as in Theorem 3.4.

Proposition 3.10. *Suppose* (H_1)–(H_2) *are satisfied and $a \geq a_1$. The estimates (3.9), (3.10), (3.13), (3.14), (3.16), (3.17) hold for the numerical solution given by scheme* (3.3)–(3.6). *Moreover there exists a constant C_a,*

$C_a \leq C(1+\sqrt{a})(1+V_a)$, *such that:*

$$\|Z^{\varepsilon}_{\Delta}(t)\|_1 \leq C_a \ (T + \|u_0^{\varepsilon}\|_1) \tag{3.24}$$

Proof of Proposition 3.10. According to the above remarks, (3.8), (3.9), (3.10),(3.13) hold. It remains to estimate the L^1 norm.

We denote:

$$\begin{cases} \alpha_j^n = w_j^n - w_{j-1}^n + \frac{1}{2}(1-\lambda)\Delta x(\sigma_j^n - \sigma_{j-1}^n) \\ \beta_j^n = z_{j+1}^n - z_j^n + \frac{1}{2}(\lambda-1)\Delta x(\tau_{j+1}^n - \tau_j^n) \ . \end{cases} \tag{3.25}$$

so that

$$\begin{cases} w_j^{n+\frac{1}{2}} = w_j^n - \lambda\alpha_j^n \\ z_j^{n+\frac{1}{2}} = z_j^n + \lambda\beta_j^n \ . \end{cases}$$

We obtain as in Lemma 3.6:

$$\left|G_a(Z_j^{n+\frac{1}{2}})\right| \leq G_a(Z_j^n) + \lambda(|\alpha_j^n| + |\beta_j^n|) \ . \tag{3.26}$$

Noting that

$$\sum_j (|\alpha_j^n| + |\beta_j^n|) \leq C \ TV(Z_{\Delta}^n) \tag{3.27}$$

and using (3.13), we obtain (3.14). Consequently, as in the proof of Theorem 3.7, one obtains (3.16).

We have no L^1 contraction property for scheme (3.3)–(3.6) so that (3.15) is no longer true. Nevertheless (3.24) holds. By lemma 3.5:

$$\begin{aligned} \|Z_{\Delta}^{n+1}\|_1 &\leq \|Z_{\Delta}^{n+\frac{1}{2}}\|_1 \\ &\leq \|Z_{\Delta}^n\|_1 + \sqrt{a}\Delta t \sum_j (|\alpha_j^n| + |\beta_j^n|) \end{aligned}$$

(3.27) and (3.13) allow us to conclude.

It remains to prove (3.17). We have, using again (3.27) and (3.13), and using also (3.14) and (3.16):

$$\begin{aligned} \|Z_{\Delta}^{n+1} - Z_{\Delta}^n\|_1 &\leq \|Z_{\Delta}^{n+1} - Z_{\Delta}^{n+\frac{1}{2}}\|_1 + \|Z_{\Delta}^{n+\frac{1}{2}} - Z_{\Delta}^n\|_1 \\ &\leq 2(1 - e^{\frac{-\Delta t}{\varepsilon}})\left(\|G_a(Z_{\Delta}^n)\|_1 + C\sqrt{a}\Delta t V_a\right) + V_a C \Delta t \sqrt{a} \\ &\leq C'_a \left(\Delta t + (1 - e^{\frac{-\Delta t}{\varepsilon}})\varepsilon\right) \\ &\leq C_a \Delta t \ . \end{aligned}$$

which concludes the proof.

□

Remark. The proof of (3.17) for the scheme (3.3)–(3.6) furnishes an alternative proof for this inequality in the case of the monotone scheme (3.2)–(3.6).

We are now able to give a convergence result for scheme (3.3)–(3.6).

Theorem 3.11. *Let $\varepsilon > 0$, suppose* (H_1)–(H_2) *are satisfied and* $a \geq a_1$. *For any* $T > 0$, $\lambda = \frac{\sqrt{a}\Delta t}{\Delta x}$ *constant and* $\Delta t \to 0$, $Z_\Delta^{(\varepsilon)}$ *converges to the unique solution* $Z^{(\varepsilon)}$ *of* (1.5)–(1.6) *on* $L^\infty(0,T;L^1(\mathrm{I\!R})^2)$, $Z^{(\varepsilon)} \in C^0([0,T],L^1(\mathrm{I\!R})^2) \cap L^\infty(0,T;L^\infty(\mathrm{I\!R})^2)$. *There hold (3.22), (3.23) and*

$$\|Z^\varepsilon(t)\|_1 \leq C_a \ (T + \|u_0^\varepsilon\|_1) \tag{3.28}$$

4. Relaxed Schemes

In this section we are interested in the behaviour of the above numerical schemes as the relaxation parameter ε tends to zero. A good criteria to appreciate the quality of the numerical solution in the underresolved domain $\varepsilon \ll \Delta t$ is that the relaxed schemes obtained by letting $\varepsilon \to 0$ should be a stable and consistent discretization of the limit conservation law (1.1) ([JX]).

Here the results of section 3 allow us to prove such a property for both families of schemes (3.2)–(3.6) and (3.3)–(3.6). Thanks to our estimates, which are uniform in ε , we need no additional calculations, on the contrary of [JX], to obtain L^∞ stability and TVD properties of the relaxed schemes. In fact the relaxed schemes associated with the monotone scheme (3.2)–(3.6) are monotone. We assume here that u_0^ε tends in L^1 norm to a function $\overline{u}_0$ (assumption (H_3)) and we prove convergence of the relaxed approximations to solutions of (1.1)–(1.2) with initial data $\overline{u}_0$.

4.1. Convergence to the relaxed numerical solution.

Theorem 4.1. *Suppose* (H_1)–(H_3) *and let* $a \geq a_1$.

(i) *As $\varepsilon \to 0$ the solution of* (3.2)–(3.6) *converges in* $L^\infty(0,T;L^1_{\mathrm{loc}}(\mathrm{I\!R})^2)$ *to a limit* $\overline{Z}_\Delta$ *which satisfies:*

$$\|\overline{Z}_\Delta(t)\|_\infty \leq 2A(1+\sqrt{a}) \qquad \forall\, t \in [0,T] \tag{4.1}$$

(4.2) $$TV(\overline{Z}_\Delta^{n+1}) \leq TV(\overline{Z}_\Delta^n) \leq V_a$$

(4.3) $$\|\overline{Z}_\Delta(t) - \tilde{\overline{Z}}_\Delta(t)\|_1 \leq C_a \|\overline{u}_0 - \tilde{\overline{u}}_0\|_1$$

(4.4) $$\overline{v}_\Delta^n = f(\overline{u}_\Delta^n)$$

(4.5) $$\|\overline{Z}_\Delta(t) - \overline{Z}_\Delta(t')\|_1 \leq C_a(\Delta t + |t - t'|)$$

for all $t, t' \in [0, T]$.

(ii) *As* $\varepsilon \to 0$ *the solution of* (3.3)–(3.6) *converges in* $L^\infty(0, T; L^1_{\text{loc}}(\mathbb{R})^2)$ *to a limit* $\overline{Z}_\Delta$ *which satisfies* (4.1), (4.2), (4.4), (4.5) *and:*

(4.6) $$\|\overline{Z}_\Delta^n\|_1 \leq C_a \ (T + \|\overline{u}_0\|_1)$$

As a consequence of (4.1) the subcharacteristic condition holds:

(4.7) $$|f'(\overline{u}_\Delta(t))| \leq \sqrt{a}$$

Proof of Theorem 4.1.

(i) By estimates (3.10) and (3.15)', for $n \in \{0, \ldots, N\}$, $\{Z_\Delta^{n,\varepsilon}\}$ is bounded in $L^1(\mathbb{R})^2 \cap BV(\mathbb{R})^2$ and then there exists a sequence $\varepsilon_k \to 0$ such that $Z_\Delta^{n,\varepsilon_k}$ converges in $L^1_{\text{loc}}(\mathbb{R})^2$ to $\overline{Z}_\Delta^n = \sum\limits_j \overline{Z}_j^n \mathbf{1}_{]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[}$ and $Z_j^{n,\varepsilon} \to \overline{Z}_j^n$.

Define $\overline{Z}_\Delta(t, x) = \sum\limits_{n=0}^{N} \sum\limits_j \overline{Z}_j^n \ \mathbf{1}_{[t_n, t_{n+1}[}(t) \ \mathbf{1}_{]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[}(x)$. We have

$$Z_\Delta^{n,\varepsilon_k} \to \overline{Z}_\Delta \quad \text{in} \quad L^\infty(0, T; L^1_{\text{loc}}(\mathbb{R})^2) \ .$$

Estimates (4.1) to (4.5) are immediate consequences of (3.9), (3.10), (3.13), (3.15) to (3.17).

But $\overline{Z}_\Delta$ verifies

(4.8) $$\overline{Z}_j^0 = \left(-\overline{u}_j^0 \sqrt{a} - f(\overline{u}_j^0), \ -\overline{u}_j^0 \sqrt{a} + f(\overline{u}_j^0)\right)$$

and the limit scheme:

(4.9) $$\begin{cases} \overline{Z}_j^{n+\frac{1}{2}} = \overline{Z}_j^n - \lambda \left(\phi_{j+\frac{1}{2}}^n - \phi_{j-\frac{1}{2}}^n\right) \\ \overline{Z}_j^{n+1} = \overline{Z}_j^{n+\frac{1}{2}} + \binom{1}{-1} \ G(\overline{Z}_j^{n+\frac{1}{2}}) \end{cases}$$

Thus $\overline{Z}_\Delta$ is unique and the whole sequence converges.

(ii) For scheme (3.3)–(3.6) the same arguments hold true by using (3.24) instead of (3.15)' and the relaxed numerical solution satisfies also (4.8)–(4.9) with numerical flux given by formula (3.4).

4.2. Consistency.

We have found relaxed schemes which have conserved all the stability properties of (3.2)–(3.6) and (3.3)–(3.6). We are now going to prove their consistency with the limit conservation law (1.1).

We rewrite (4.9) in terms of (u, v).

$$\begin{cases} \overline{u}_j^{n+1} = \overline{u}_j^{n+\frac{1}{2}} \\ \overline{v}_j^{n+1} = \overline{v}_j^{n+\frac{1}{2}} - \left(\overline{v}_j^{n+\frac{1}{2}} - f\left(\overline{u}_j^{n+\frac{1}{2}}\right)\right) \end{cases}$$

which gives

$$\begin{cases} \overline{v}_j^{n+1} = f(\overline{u}_j^{n+1}) \\ \overline{u}_j^{n+1} = \overline{u}_j^n - \frac{\lambda}{2\sqrt{a}}\left(-\varphi_{j+\frac{1}{2}}^n + \varphi_{j-\frac{1}{2}}^n + \psi_{j+\frac{1}{2}}^n - \psi_{j-\frac{1}{2}}^n\right) \end{cases} \tag{4.10}$$

Note that we already had $\overline{v}_\Delta(t) = f(\overline{u}_\Delta(t))$ for $t \geq 0$ by (4.4).

The relaxed scheme may be written as:

$$\overline{u}_j^{n+1} = \overline{u}_j^n - \frac{\Delta t}{\Delta x}(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n) \tag{4.11}$$

with:

$$F_{j+\frac{1}{2}}^n = F(\overline{u}_{j-k+1}^n, \ldots, \overline{u}_{j+k}^n) \tag{4.12}$$

and:

$$\begin{aligned} F(U_1, \ldots, U_{2k}) = \frac{1}{2}\Big[&-\varphi\left(-U_1\sqrt{a} - f(U_1), \ldots, -U_{2k}\sqrt{a} - f(U_{2k})\right) \\ &+ \psi\left(-U_1\sqrt{a} + f(U_1), \ldots, -U_{2k}\sqrt{a} + f(U_{2k})\right)\Big] \end{aligned}$$

$$\begin{aligned} F(U, \ldots, U) &= \frac{1}{2}\left[-\left(-U\sqrt{a} - f(U)\right) + \left(-U\sqrt{a} + f(U)\right)\right] \\ &= f(U) \end{aligned}$$

by the consistency of schemes (3.2) and (3.3). Moreover it is easy to see that

$$\overline{u}_j^0 = \frac{1}{\Delta x}\int_{I_j} \overline{u}_0(x)dx\ , \quad j \in \mathbb{Z} \tag{4.13}$$

Therefore we can use the same arguments as in paragraph 3 to prove the convergence of the relaxed scheme towards a weak solution of (1.1)–(1.2):

Theorem 4.2. *Suppose* (H_1)–(H_3) *satisfied and* $a \geq a_1$. *For any* $T > 0$, $\lambda = \frac{\sqrt{a}\Delta t}{\Delta x}$ *constant and* $\Delta t \to 0$, *the numerical relaxed solution* $\overline{u}_\Delta$ *associated to scheme* (3.2)–(3.6) *or* (3.3)–(3.6) *converges to a weak solution* $\overline{u}$ *of* (1.1)–(1.2) *on* $L^\infty(0, T; L^1(\mathbb{R})^2)$.

4.3. Entropy conditions.

Proposition 4.3. *Suppose* (H_1)–(H_3) *satisfied and* $a \geq a_1$. *The relaxed scheme associated to* (3.2)–(3.6) *is monotone.*

Proof of Proposition 4.3. Rewrite scheme (3.2) in the form:

$$\begin{cases} w_j^{n+1} = H_1(w_{j-k}^n, \ldots, w_{j+k}^n) \\ z_j^{n+1} = H_2(z_{j-k}^n, \ldots, z_{j+k}^n)\ . \end{cases}$$

Then (4.9) becomes:

$$\begin{aligned} \overline{u}_j^{n+1} &= -\frac{1}{2\sqrt{a}}\Big[H_1(-\overline{u}_{j-k}^n\sqrt{a} - f(\overline{u}_{j-k}^n), \ldots, -\overline{u}_{j+k}^n\sqrt{a} - f(\overline{u}_{j+k}^n)) \\ &\quad + H_2(-\overline{u}_{j-k}^n\sqrt{a} + f(\overline{u}_{j-k}^n), \ldots, -\overline{u}_{j+k}^n\sqrt{a} + f(\overline{u}_{j+k}^n))\Big] \\ &= H(\overline{u}_{j-k}^n, \ldots, \overline{u}_{j+k}^n) \end{aligned}$$

We know that H_1 and H_2 are nondecreasing functions of each of their arguments. Then it suffices to show that for $U_1 \leq U_2$:

$$\begin{cases} -U_1\sqrt{a} - f(U_1) \geq -U_2\sqrt{a} - f(U_2) \\ -U_1\sqrt{a} + f(U_1) \geq -U_2\sqrt{a} + f(U_2) \end{cases}$$

which is ensured by the subcharacteristic condition (4.7).

□

This proposition allows us to prove the convergence of relaxed (3.2)–(3.6) scheme towards the entropy solution of (1.1)–(1.2): actually a monotone consistent scheme is consistent with any entropy condition [CM].

Theorem 4.4. *Suppose* (H_1)–(H_3) *satisfied and* $a \geq a_1$. *The numerical relaxed solution* $\overline{u}_\Delta$ *related to* (3.2)–(3.6) *converges in* $L^\infty(0, T; L^1(\mathbb{R}))$ *to the unique entropy solution of* (1.1)–(1.2) *with* $u(0) = \overline{u}_0$, *as* Δt *tends to zero and* $\frac{\Delta t}{\Delta x}$ *kept constant.*

5. Numerical Experiments

We have tested here the method with upwind scheme for (3.2), and then with second order Goodman–LeVeque scheme. We also compare with schemes proposed in [JX].

By a straighforward calculation we obtain:

Proposition 5.1. *Suppose* (H_1)–(H_3) *satisfied and* $a \geq a_1$. *The relaxed scheme associated to* (3.2)–(3.6) *where* (3.2) *is the upwind scheme is given by* (4.13) *and:*

$$\left\{\begin{array}{l} \frac{\overline{u}_j^{n+1}-\overline{u}_j^n}{\Delta t} + \frac{f(\overline{u}_{j+1}^n)-f(\overline{u}_{j-1}^n)}{2\Delta x} = \frac{-2\overline{u}_j^n+\overline{u}_{j-1}^n+\overline{u}_{j+1}^n}{2\Delta x}\sqrt{a} \\ \overline{v}_j^{n+1} = f(\overline{u}_j^{n+1}) \ . \end{array}\right. \tag{5.1}$$

Proposition 5.2. *Suppose* (H_1)–(H_3) *satisfied and* $a \geq a_1$. *The relaxed scheme associated to* (3.3)–(3.6) *is given by* (4.13) *and:*

$$\begin{aligned} \overline{u}_j^{n+1} =& \overline{u}_j^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(f\left(\overline{u}_{j+1}^n\right) - f\left(\overline{u}_{j-1}^n\right)\right) + \frac{\sqrt{a}}{2}\frac{\Delta t}{\Delta x}\left(-2\overline{u}_j^n + \overline{u}_{j-1}^n + \overline{u}_{j+1}^n\right) \\ &+ \frac{1-\lambda}{4}\frac{\Delta t}{\Delta x}[\min\bmod(D_{j+1}^n, D_j^n) - \min\bmod(D_j^n, D_{j-1}^n) \\ &- \min\bmod(E_{j+2}^n, D_{j+1}^n) + \min\bmod(E_{j+1}^n, E_j^n)] \end{aligned} \tag{5.2}$$

where D_j^n *and* E_j^n *are defined by:*

$$D_j^n = -\sqrt{a}(\overline{u}_j^n - \overline{u}_{j-1}^n) - \left(f(\overline{u}_j^n) - f(\overline{u}_{j-1}^n)\right) \tag{5.3}$$

$$E_j^n = -\sqrt{a}(\overline{u}_j^n - \overline{u}_{j-1}^n) + \left(f(\overline{u}_j^n) - f(\overline{u}_{j-1}^n)\right) \tag{5.4}$$

Remark. (5.2) is the sum of a monotone part given by
(4.17)

$$H_j^M = \overline{u}_j^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(f\left(\overline{u}_{j+1}^n\right) - f\left(\overline{u}_{j-1}^n\right)\right) + \frac{\sqrt{a}}{2}\frac{\Delta t}{\Delta x}\left(-2\overline{u}_j^n + \overline{u}_{j-1}^n + \overline{u}_{j+1}^n\right)$$

which is the relaxed scheme of proposition 5.1, and of a "second order" part given by

$$\begin{aligned} H_j^G =& \frac{1-\lambda}{4}\frac{\Delta t}{\Delta x}[\min\bmod(D_{j+1}^n, D_j^n) - \min\bmod(D_j^n, D_{j-1}^n) \\ &- \min\bmod(E_{j+2}^n, D_{j+1}^n) + \min\bmod(E_{j+1}^n, E_j^n)] \end{aligned} \tag{5.5}$$

We present here some numerical tests for the (non convex) flux $f(u) = u^3$. The initial data is:

$$\overline{u}_0 = \left\{\begin{array}{lll} -1 & for & x < 0 \\ +1 & for & x > 0 \end{array}\right. \tag{5.6}$$

The exact solution of (1.1)–(1.2) is then given by:

$$\overline{u}(x,t) = \begin{cases} -1 & for \quad x/t < 3/4 \\ \sqrt{\frac{x}{3t}} & for \quad 3/4 < x/t < 3 \\ +1 & for \quad x/t > 3 \end{cases} \tag{5.7}$$

Note that $\overline{u}_0$ is not in $L^1(\mathbb{R})$ but it makes sense to consider the L^1 norm of the error on a bounded interval because on one hand we have finite propagation speed and on the other hand our schemes are exact for constant initial data.

The solution is calculated on the space-time domain $[-3.25, 3.25] \times [0,1]$ and we take $a = 9.9$.

5.1. Convergence to the relaxed numerical solution.

We first study the convergence of the relaxing numerical solution to the relaxed one (Table 1). For given Δt we denote

$$e_k = \sup_n \| u_\Delta^{\varepsilon_k, n} - u_\Delta^n \|_1, \quad \varepsilon_k = 2^{-k}$$

and γ_k is the convergence rate:

$$\frac{e_k}{e_{k+1}} = \left(\frac{\varepsilon_k}{\varepsilon_{k+1}}\right)^{\gamma_k}, \quad k \geq 1$$

Both schemes (3.2)–(3.6) and (3.3)–(3.6) are under consideration with $\Delta x = 0.01$ and $\lambda = 0.9$ ($\lambda = \sqrt{a}\frac{\Delta t}{\Delta x}$). The quantity $\exp(-\Delta t/\varepsilon)$ is involved. We remark that for $\Delta t/\varepsilon < 36.8$ we have $10^{-16} < \exp -(\Delta t/\varepsilon)$ and for $\Delta t/\varepsilon > 36.9$ we have $10^{-16} > \exp -(\Delta t/\varepsilon)$. We observe that this is the critical value for the relaxed solution to be obtained. The convergence rate increases as $\varepsilon \to 0$.

5.2. Convergence to the exact solution.

Let us consider the error between the exact solution and the numerical relaxed solution :

$$err = \sup_n \| u_\Delta^n - u(t_n) \|_1$$

Two points of view are under consideration.

First, for fixed Δx, we observe the variation of the error when λ decreases (Table 2). Here γ_k is defined by:

$$\frac{e_k}{e_{k+1}} = \left(\frac{\lambda_k}{\lambda_{k+1}}\right)^{\gamma_k}, \quad k \geq 1$$

It appears that for the first order scheme the error does not decay with λ. The best choice seems to be $\lambda = 1$ which corresponds to an exact resolution of the linear system

$$\begin{cases} \partial_t w + \sqrt{a}\ \partial_x w = 0 \\ \partial_t z - \sqrt{a}\ \partial_x z = 0 \\ z(t_n, x) = Z_\Delta^n(x) \end{cases}$$

although this value provides the maximal diffusion of (5.1) as already noted in [JX].

On the contrary the error decays with λ for the second order scheme with a maximal decay rate around $\lambda = 0.5$. Others experiments, not presented here, prove that this rate does not depend on Δx.

Secondly we study for fixed λ the variation of the error when Δt decreases (Table 3). Here γ_k is defined by:

$$\frac{e_k}{e_{k+1}} = \left(\frac{\Delta t_k}{\Delta t_{k+1}}\right)^{\gamma_k}, \quad k \geq 1$$

It is well known that for a monotone scheme as (5.1) the error bound is at least as $(\Delta t)^{0.5}$ (see [Ku]). Actually we find that it is around $(\Delta t)^{0.7}$.

5.3. Comparison with schemes proposed in [JX].

The relaxing schemes presented in [JX] are based on the method of lines. An upwind or a second order MUSCL type space discretization is associated to a second order Runge-Kutta splitting scheme in time. Relaxed schemes for the same space discretization and a forward Euler time discretization are also studied. Actually although our relaxing schemes are different from those of [JX], our relaxed schemes are identical to those obtained with this Euler method.

We compare here our schemes with those obtained by Runge-Kutta splitting scheme in time for the same space discretizations.

We give for $\Delta x = 0.005$ the solutions at $T = 1$ (see the figures in the Appendix).

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