A Discrete Approximation for Hyperbolic Systems with Quadratic Interaction Term

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Abstract

We investigate the stability and the convergence of a discrete implicit approximation to a class of hyperbolic semilinear kinetic systems in the one-dimensional case. In particular the uniform boundedness and the convergence of this approximation is proved under some suitable assumptions, which include Broadwell model for arbitrary large data.

Key words and phrases: discrete kinetic equations, finite differences approximation, hyperbolic semilinear systems.

1991 Mathematics Subject Classification. Primary 35L45; Secondary 35A35, 35A40, 76P05.

1 Introduction

In this paper we are concerned with the uniform convergence of a discrete numerical approximation for semilinear hyperbolic systems with quadratic interaction of the type

$$\begin{cases} \partial_t u_i + c_i \partial_x u_i = q_i\left(u\right), \qquad i = 1, \ldots, N, \\ q_i\left(u\right) = -\sum\limits_{j,k=1}^{N} A^i_{j,k} u_j u_k, \quad A^i_{j,k} \in \mathbb{R}, \end{cases} \tag{1.1}$$

with the initial conditions

$$u_i(0,x) = \varphi_i(x), \qquad\qquad i = 1, \ldots, N, \tag{1.2}$$

where $u = (u_1, \ldots, u_N)$ is a real function and $c_i \in \mathbb{R}$, $c_i \leq c_{i+1}$.

Systems of this type often originate from physics and represent useful models for wave interaction phenomena. Let us remember the kinetic theory of gas where a model such as (1.1) is obtained by allowing only a discrete set of velocity in the Boltzmann equation. In this case the quadratic nonlinearity corresponds to the interactions between particles through collisions.

Let us set $\Gamma = \{1, \ldots, N\}$. We suppose that the coefficients $A^i_{j,k}$ satisfy the following condition which is physical meaningful:

$$A^i_{j,k} = A^i_{k,j} \qquad\qquad \text{for all } i, j, k \in \Gamma, \tag{1.3}$$

that is particles during the collisions introduce a symmetric interaction. Let us consider the Cauchy problem (1.1)-(1.2). Here we prove, under suitable conditions which we shall state later, convergence of the numerical approximation, stability and L^∞-boundedness for small L^1-data. In addition, when the *H-Theorem* is valid, we obtain convergence of the numerical approximation and stability in L^∞.

Our proofs need an additional restriction on $A^i_{j,k}$:

$$A^i_{j,k} = 0 \qquad\qquad \text{if } c_j = c_k, \tag{1.4}$$

or alternatively

$$A^i_{j,k} \leq 0 \qquad\qquad \text{if } i \neq j \text{ and } i \neq k \tag{1.5}$$

and one of the following conditions:

- $\exists \lambda_i \geq 0$, $i \in \Gamma$ such that $c_j = c_k$ implies

$$\sum_{i\in\Gamma} \lambda_i A^i_{j,k} \geq \sum_{i\in\Gamma} \left|A^i_{j,k}\right|; \tag{1.6}$$

- $$A^i_{j,k} \geq 0 \qquad\qquad \text{if } c_i = c_j = c_k. \tag{1.7}$$

Note that (1.4) and (1.6)-(1.7) are not satisfied for the Carleman model

$$\begin{cases} (\partial_t + c\partial_x)\, u_1 = u_2^2 - u_1^2, \\ (\partial_t - c\partial_x)\, u_2 = u_1^2 - u_2^2, \end{cases} \tag{1.8}$$

while the Broadwell model

$$\begin{cases} (\partial_t + c\partial_x)\, u_1 = \frac{1}{\varepsilon}\left(u_3^2 - u_1 u_2\right), \\ (\partial_t - c\partial_x)\, u_2 = \frac{1}{\varepsilon}\left(u_3^2 - u_1 u_2\right), \\ \partial_t u_3 = \frac{2}{\varepsilon}\left(u_1 u_2 - u_3^2\right), \end{cases} \tag{1.9}$$

where $\varepsilon = \frac{1}{2\sigma}$ corresponds to the Knudsen number, satisfies (1.6)-(1.7) and not (1.4).

Moreover, when condition (1.4) is not satisfied, under conditions (1.6)-(1.7), if the initial data are nonnegative, we obtain again convergence of the numerical approximation, stability for small L^1-data and stability in L^∞ for arbitrary data under the hypothesis of *H-Theorem*.

An elegant way to prove a global existence result for the Cauchy problem (1.1)-(1.2) under the hypotheses (1.4) or (1.6)-(1.7) was given by Tartar in [22, 23]. Related to the results proved by Tartar are also the papers of Beale [3, 4]. A proof for a different class of discrete velocity models in the one-dimensional case was successively given by Bony [7]. Let us notice that our construction provides an alternative existence proof under the same assumptions. Existence results for the Carleman and related models can be found in Kolodner [18], Temam [24], J.L. Lions [20] and Tartar [23]. On the other hand, when the above assumptions are not verified, the solutions not need to exist for all positive times, see [2], [11] and the references therein.

There exists a remarkable quantity of review articles and books on the subject of discrete velocity models of the Boltzmann equation. Among these, we draw attention to the books of Gatignol [14], Monaco and Preziosi [21] and the papers of Platkowsky and Illner [17], Bellomo and Gustafsson [5].

The initial value problem for the discrete Boltzmann equation in the whole space $\mathbb{R}^3$ was studied first by Illner [16], Hamdache [15] and finally by Bony [8]. Unfortunately these results are far to be completely satisfactory. Bony proved in [8] global existence and uniqueness just for *small data*, but, to our knowledge, a global existence result for large initial data is not yet available in the literature.

Numerical approximations for the discrete kinetic equations have been previously considered by several authors. Among these we wish to mention Toscani [25], Emamirad [12], Aregba-Driollet [1], Gabetta and Pareschi [13], Caflisch and Russo [10], Caflisch, Jin and Russo [9], Bobylev, Gabetta and Pareschi [6], where the reader can also refer for a large amount of numerical tests.

Nevertheless, we do not know any complete theory for the rigorous convergence of numerical approximations to systems of the type (1.1). Here we present some results in this direction. We hope that our analysis could be useful also for a better understanding of the qualitative properties of the solutions of these systems.

The outline of the paper is the following. In the next section we present an approximating scheme and introduce functional setting for which the interaction products are stable. In Section 3 we discuss and prove the convergence of our discretization for small data, while in Section 4 we are concerned with arbitrarily large nonnegative initial data.

2 Preliminary results and the scheme

Here we present a discrete approximation for the first order semilinear hyperbolic system (1.1). Let us point out that many ideas in our construction proofs are largely inspired by the fundamental paper by Tartar [23], where existence results for the Cauchy problem (1.1)-(1.2) are given under the same assumptions of the present paper, by a direct method.

First we introduce the spatial grid points x_l, $l \in \mathbb{Z}$, with uniform mesh-length $\Delta x = h = x_{l+1} - x_l$ for all l. The time level t_n, with $t_0 = 0$, are also spaced uniformly with mesh-length $\Delta t = k = t_{n+1} - t_n$ for $n \in \mathbb{N}$. We denote by λ the *CFL* ratio $\lambda = k/h$, which is taken constant throughout the paper. The uniform grid is obviously assumed only for simplicity. For a good definition of the discretization we will assume, for any $i \in \Gamma$, $c_i\lambda \in \mathbb{Z}$.

For any given function $f : \mathbb{R}_x \times \mathbb{R}_t^+ \longrightarrow \mathbb{R}^N$, $f(\cdot, t_n) \in L^1_{loc}(\mathbb{R}_x)$ for all n, we use $f^{l,n}$ to denote the cell average of f in the cell $[x_l, x_{l+1}]$ at time t_n,

$$f^{l,n} = \frac{1}{h} \int_{x_l}^{x_{l+1}} f(\xi, t_n)\ d\xi.$$

Of course, when $g : \mathbb{R}_x \longrightarrow \mathbb{R}^N$, the cell average of g will be denoted by g^l.

Before giving the scheme for the problem (1.1)-(1.2), we need to introduce suitable functional spaces to deal with the interaction products $u_j u_k$.

Given an interval $[a, b] \subset \mathbb{R}_x$ and a time interval $[0, T] \subset \mathbb{R}_t^+$, let us consider the domain

$$\mathcal{R} = \left\{ (x,t) \in \mathbb{R}_x \times \mathbb{R}_t^+ :\ t \in [0,T],\ x - c_j t \in [a,b],\ j \in \Gamma \right\}.$$

Recall that, when we assign a function $f_j \in L^1(\mathcal{R})$ and a function $g_j \in L^1(a,b)$, if we consider the Cauchy problem

$$\begin{cases} \partial_t v + c_j \partial_x v = f_j, & (x,t) \in \mathcal{R}, \\ v(x,0) = g_j(x), & x \in (a,b), \end{cases} \tag{2.1}$$

the corresponding solution is given by

$$v(x,t) = g_j(x - c_j t) + \int_0^t f_j(x - c_j s, t - s)\ ds \qquad a.e.\ (x,t) \in \mathcal{R}. \tag{2.2}$$

Our idea is first of all to give a good discretization of this problem.

We define

$$J = \{l \in \mathbb{Z} :\ x_l \in (a,b)\}$$

and

$$D = \{(l,n) \in \mathbb{Z} \times \mathbb{N} :\ x_l - c_j t_n \in (a,b),\ j = 1, \ldots, N\}.$$

Consider $a \in l^1(J)$ and $b \in l^1(D)$ where

$$l^1(J) = \left\{a :\ a = \{a^l\}_{l \in J}\right\}, \qquad l^1(D) = \left\{b :\ b = \{b^{l,n}\}_{(l,n)\in D}\right\} \tag{2.3}$$

respectively with the norms

$$||a||_J = \sum_{l \in J} \left|a^l\right| h, \qquad ||b||_D = \sum_{(l,n)\in D} \left|b^{l,n}\right| hk. \tag{2.4}$$

We can now define an approximate solution $\left\{v_j^{l,n}\right\}$ for the Cauchy problem (2.1) as

$$v_j^{l,n} = g_j^{l - c_j \lambda n} + k \sum_{m=0}^{n-1} f_j^{l - c_j \lambda m, n-m}. \tag{2.5}$$

Finally, we define the functional spaces

$$\begin{aligned} V_j = \Big\{ v_j = \left\{v_j^{l,n}\right\}_{(l,n)\in D} &: \exists f_j \in l^1(D), \exists g_j \in l^1(J) \\ &\text{and } v_j^{l,n} \text{ is defined by (2.5)} \Big\}, \end{aligned} \tag{2.6}$$

with the norm

$$[v_j]_{V_j} = ||g_j||_J + ||f_j||_D \tag{2.7}$$

and $\mathcal{V} = V_1 \times, \ldots, \times V_N$ with the norm $[v]_{\mathcal{V}} = \sum_{j \in \Gamma} [v_j]_{V_j}$.

It is easy to prove that V_j is isometric to $l^1(D) \times l^1(J)$.

Lemma 2.1 *For all $v_j \in V_j$ there exists $w_j \in l^1(J)$ such that*

$$\begin{cases} \left|v_j^{l,n}\right| \le w_j^{l-c_j\lambda n} & \text{for all } (l,n) \in D \\ \|w_j\|_J = [v_j]_D . \end{cases} \tag{2.8}$$

Proof. Let us consider $\alpha \in J$. Set

$$\begin{aligned} I_j^\alpha &= \{m \in \mathbb{Z} : \ (\alpha + c_j\lambda m, m) \in D\} \\ &= \{m \in \mathbb{Z} : \ \alpha + (c_j - c_k)\lambda m \in J, \ k \in \Gamma\} . \end{aligned}$$

We define $w_j \in l^1(J)$ as

$$w_j^\alpha = \left|g_j^\alpha\right| + k \sum_{m \in I_j^\alpha} \left|f_j^{\alpha + c_j\lambda m, m}\right| .$$

In fact we have

$$\begin{aligned} \left|v_j^{l,n}\right| &\le \left|g_j^{l-c_j\lambda n}\right| + k \sum_{r=0}^{n-1} \left|f_j^{l-c_j\lambda(n-r),r}\right| \\ &\le \left|g_j^{l-c_j\lambda n}\right| + k \sum_{m \in I_j^{l-c_j\lambda n}} \left|f_j^{l-c_j\lambda(n-m),m}\right| = w_j^{l-c_j\lambda n} \end{aligned}$$

and

$$\begin{aligned} \|w_j\|_J &= h \sum_{\alpha \in J} \left|w_j^\alpha\right| = h \sum_{\alpha \in J} \left|g_j^\alpha\right| + kh \sum_{(l,m) \in D} \left|f_j^{l,m}\right| \\ &= \|g_j\|_J + \|f_j\|_D = [v_j]_D . \end{aligned}$$

This concludes the proof. ∎

The following lemma is at the heart of the results under conditions (1.4).

Lemma 2.2 *Let us consider $p, q \in \Gamma$ such that $c_p \neq c_q$. When $v_p \in V_p$ and $v_q \in V_q$, we have*

$$\|v_p v_q\|_D \le \frac{1}{|c_p - c_q|} [v_p]_{V_p} [v_q]_{V_q} . \tag{2.9}$$

Proof. In fact, there exist $w_p, w_q \in l^1(J)$ as in Lemma 2.1, such that

$$\begin{aligned} \|v_p v_q\|_D &= \sum_{(l,n)\in D} \left| v_p^{l,n} v_q^{l,n} \right| hk \\ &\leq \lambda \sum_{(l,n)\in D} w_p^{l-c_p\lambda n} w_q^{l-c_q\lambda n} h^2. \end{aligned}$$

If we now consider the change of variables $\gamma : \ (l,n) \mapsto (\varepsilon, \delta)$,

$$\begin{cases} \varepsilon = l - c_p \lambda n, \\ \delta = l - c_q \lambda n, \end{cases}$$

we obtain

$$\|v_p v_q\|_D \leq \frac{1}{|c_p - c_q|} \sum_{(\varepsilon,\delta)\in\gamma(D)} \left| h w_p^{\varepsilon} \right| \left| h w_q^{\delta} \right| \leq \frac{1}{c_p - c_q} \left[v_p \right]_{V_p} \left[v_q \right]_{V_q}.$$

The proof is now complete. ■

We can now give an implicit first order scheme for the problem (1.1)-(1.2). We denote by $u^{l,n}$ the approximating solution in the cell $[x_l, x_{l+1}]$ at time t_n. We define

$$u_i^{l,n} = \varphi_i^{l-c_i\lambda n} + k \sum_{m=0}^{n-1} q_i \left(u^{l-c_i\lambda m, n-m} \right) \tag{2.10}$$

for all $n \geq 1$, $i \in \Gamma$.

As we will see later, it is useful to put the difference scheme (2.10) in conservation form,

$$u_i^{l,n+1} = u_i^{l,n} - \lambda \left(\Phi_i^{l+1/2,n} - \Phi_i^{l-1/2,n} \right) + k q_i \left(u^{l,n+1} \right) \tag{2.11}$$

when for all i the numerical flux Φ_i is defined by

$$\Phi_i^{l+1/2,n} = \frac{1}{\lambda} \sum_{k=0}^{c_i\lambda - 1} u_i^{l-k,n}.$$

In order to analyze the convergence of the solution $u^{l,n}$ of the difference scheme (2.10) we introduce the piecewise constant function u_{Δ_h} defined a.e. in $\mathbb{R} \times \mathbb{R}^+$ by

$$u_{\Delta_h}(x,t) = u^{l,n}, \qquad x_{l-1/2} < x < x_{l+1/2}, \quad t_n \leq t < t_{n+1} \tag{2.12}$$

where $x_{l+1/2} = (x_j + x_{j+1})/2$. Then we study the convergence of u_{Δ_h} towards a weak solution of (1.1)-(1.2).

3 Results for small data

The first result in this section is the stability of the scheme (2.10) for small data in $L^1(\mathbb{R}_x)$.

Theorem 3.1 **(l^1–stability)** *Let us consider the problem (1.1)-(1.2) and assume the hypothesis (1.4). Set*

$$\gamma = \max_{j,k\in\Gamma} \sum_{i\in\Gamma:\ c_j\neq c_k} \frac{\left|A^i_{j,k}\right|}{|c_j - c_k|}.$$

If the initial datum φ satisfies

$$\sum_{i\in\Gamma} ||\varphi_i||_J \leq \varepsilon_0 \tag{3.1}$$

with $4\gamma\varepsilon_0 < 1$, the approximating sequence of solution $u = \left\{u^{l,n}\right\}$ for our problem is uniformly bounded in l^1 and we have

$$\sum_{i,j,k\in\Gamma} \left|\left|A^i_{j,k} u_j u_k\right|\right|_D \leq \sum_{i\in\Gamma} ||\varphi_i||_J\,. \tag{3.2}$$

Moreover the scheme is l^1–stable, that is, if $\bar{\varphi}_i \in l^1(J)$ is an other initial datum such that

$$\sum_{i\in\Gamma} ||\bar{\varphi}_i||_J \leq \varepsilon_0$$

and we denote by $\bar{u} = \left\{\bar{u}^{l,n}\right\}$ the corresponding approximating sequence, then

$$\sup_n \sum_{j\in\Gamma} \left|\left|u^{n,\cdot}_j - \bar{u}^{n,\cdot}_j\right|\right|_J \leq \frac{4\gamma\varepsilon_0}{1-4\gamma\varepsilon_0} \sum_{j\in\Gamma} ||\varphi_j - \bar{\varphi}_j||_J\,.$$

Proof. The idea is to find a solution for (1.1)-(1.2) on D by applying Banach's contraction mapping principle. Let us introduce the operator $\mathcal{C}$ given by $C\left(u^{(\alpha)}\right) = u^{(\alpha+1)}$ where

$$\left(u^{(\alpha+1)}\right)^{l,n}_i = \varphi_i^{l-c_i\lambda n} + k\sum_{m=0}^{n}\sum_{j,k\in\Gamma} A^i_{j,k}\left(u^{(\alpha)}\right)_j^{l-c_i\lambda m,n-m}\left(u^{(\alpha)}\right)_k^{l-c_i\lambda m,n-m}.$$

We shall prove that

$$C:\ \mathcal{V} \longrightarrow \mathcal{V}$$

and $\mathcal{C}$ is a contraction on $\mathcal{V}$. Let us define

$$\varepsilon_j = ||\varphi_j||_J\,,\ \alpha_i = \left\|\sum_{j,k\in\Gamma} A^i_{j,k} u_j^{(\alpha-1)} u_k^{(\alpha-1)}\right\|_D\,,\ \beta_i = \left\|\sum_{j,k\in\Gamma} A^i_{j,k} u_j^{(\alpha)} u_k^{(\alpha)}\right\|_D$$

and denote by ε, α, β the corresponding sums when $i \in \Gamma$.

Then we have, by using first Lemma 2.2 and then Lemma 2.1

$$\begin{aligned}
\beta_i &\leq \sum_{j,k\in\Gamma:\ c_j\neq c_k} \frac{\left|A^i_{j,k}\right|}{|c_j - c_k|} \left[u_j^{(\alpha)}\right]_{V_j} \left[u_k^{(\alpha)}\right]_{V_k} \\
&\leq \sum_{j,k\in\Gamma:\ c_j\neq c_k} \frac{\left|A^i_{j,k}\right|}{|c_j - c_k|} \left(\varepsilon_j + \alpha_j\right)\left(\varepsilon_k + \alpha_k\right).
\end{aligned}$$

Summing the previous inequality for $i \in \Gamma$, we have

$$\sum_{i\in\Gamma} \beta_i \leq \gamma \left(\sum_{j\in\Gamma} \varepsilon_j + \sum_{j\in\Gamma} \alpha_j\right)^2,$$

that is $\beta \leq \gamma\left(\varepsilon_0 + \alpha\right)^2$.

Let us consider

$$\bar{\alpha}_i = \left\|\sum_{j,k\in\Gamma} A^i_{j,k} v_j^{(\alpha-1)} v_k^{(\alpha-1)}\right\|_D\,, \qquad \bar{\beta}_i = \left\|\sum_{j,k\in\Gamma} A^i_{j,k} v_j^{(\alpha)} v_k^{(\alpha)}\right\|_D\,. \tag{3.3}$$

Suppose $\alpha \leq \varepsilon$ and $\bar{\alpha} \leq \varepsilon$. Then we have

$$\begin{aligned}
\left[v_i^{(\alpha)} - u_i^{(\alpha)}\right]_{V_i} &= \left\|\sum_{j,k\in\Gamma} A^i_{j,k}\left(v_j^{(\alpha-1)} v_k^{(\alpha-1)} - u_j^{(\alpha-1)} u_k^{(\alpha-1)}\right)\right\|_D \\
&\leq \sum_{j,k\in\Gamma:\ c_j\neq c_k} \frac{\left|A^i_{j,k}\right|}{|c_j - c_k|}\left\{\left(\varepsilon_j + \alpha_j\right)\left[u_k^{(\alpha-1)} - v_k^{(\alpha-1)}\right]_{V_k}\right. \\
&+ \left.\left[u_j^{(\alpha-1)} - v_j^{(\alpha-1)}\right]_{V_j}\left(\varepsilon_k + \alpha_k\right)\right\}
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{i\in\Gamma}\left[v_i^{(\alpha)} - u_i^{(\alpha)}\right]_{V_i} &\leq 2\gamma\left(\varepsilon_0 + \alpha\right)\sum_{i\in\Gamma}\left[v_i^{(\alpha-1)} - u_i^{(\alpha-1)}\right]_{V_i} \\
&\leq 4\varepsilon_0\gamma\sum_{i\in\Gamma}\left[v_i^{(\alpha-1)} - u_i^{(\alpha-1)}\right]_{V_i}.
\end{aligned}$$

Since $4\gamma\varepsilon_0 < 1$, we obtain $\mathcal{C}$ is a contraction on

$$B_{\varphi,\varepsilon_0} = \{w \in \mathcal{V} : \quad \varepsilon \le \varepsilon_0, \quad \alpha \le \varepsilon_0\}.$$

Finally, let us consider $\varphi_j, \bar{\varphi}_j \in l^1(J)$, with $\sum_{j\in\Gamma} ||\varphi_j||_J \le \varepsilon_0$, $\sum_{j\in\Gamma} ||\bar{\varphi}_j||_J \le \varepsilon_0$ and $u^{(\alpha)}$, $v^{(\alpha)}$ the corresponding approximating sequences. With a calculation analogous to the previous one, we have

$$\begin{aligned}
&\left|\left|\sum_{j,k\in\Gamma} A^i_{j,k}\left(u_j^{(\alpha-1)}u_k^{(\alpha-1)} - v_j^{(\alpha-1)}v_k^{(\alpha-1)}\right)\right|\right|_D \\
&\le \sum_{j,k\in\Gamma:\ c_j\ne c_k} \frac{\left|A^i_{j,k}\right|}{|c_j - c_k|}\left\{(\varepsilon_j + \alpha_j)\left[u_k^{(\alpha-1)} - v_k^{(\alpha-1)}\right]_{V_k}\right. \\
&+ \left.\left[u_j^{(\alpha-1)} - v_j^{(\alpha-1)}\right]_{V_j}(\varepsilon_k + \alpha_k)\right\}
\end{aligned}$$

which implies

$$\left[u^{(\alpha)} - v^{(\alpha)}\right]_{\mathcal{V}} \le 4\gamma\varepsilon_0\left\{\sum_{i\in\Gamma} ||\varphi_i - \bar{\varphi}_i||_J + \left[u^{(\alpha-1)} - v^{(\alpha-1)}\right]_V\right\}$$

so that, when we pass to the limit,

$$[u - v]_{\mathcal{V}} \le \frac{4\gamma\varepsilon_0}{1 - 4\gamma\varepsilon_0}\sum_{i\in\Gamma} ||\varphi_i - \bar{\varphi}_i||_J$$

that is the stability on $B_{\varphi,\varepsilon_0}$. The proof is thus complete. ■

Corollary 3.2 *Let us consider the problem (1.1)-(1.2) and assume the hypotheses (1.5) and (1.6). Then there exists $\varepsilon_0 > 0$ and $c > 0$ such that when the initial datum satisfies $\varphi \ge 0$ for any $i \in \Gamma$ and (3.1), the approximating sequence of solution $u = \{u^{l,n}\}$, when is stable, satisfies*

$$\sum_{i,j,k\in\Gamma} \left|\left|A^i_{j,k}u_j u_k\right|\right|_D \le c\sum_{i\in\Gamma} ||\varphi_i||_J .$$

Proof. It is not restrictive to assume that the initial data have compact support.

Since $\varphi_i^l \ge 0$ for all i, l, by assuming existence from the fixed-point argument we can easily prove the solution satisfies $u_i^{l,n} \ge 0$ for all (l, n).

From (2.10), multiplying the i-th equation by λ_i and summing on i we obtain, by applying (1.6),

$$\begin{aligned} & k \sum_{i,j,k\in\Gamma:\ c_j=c_k} \left|A^i_{jk}\right| u_j^{l,n} u_k^{l,n} \\ & \leq \sum_{i\in\Gamma} \lambda_i \varphi_i^{l-c_i\lambda n} - k \sum_{i,j,k\in\Gamma:\ c_j\neq c_k} \sum_{m=1}^{n-1} \lambda_i A^i_{jk} u_j^{l-c_i\lambda m, n-m} u_k^{l-c_i\lambda m, n-m}. \end{aligned} \tag{3.4}$$

By applying first Lemma 2.1 and then Lemma 2.2 we have

$$\begin{aligned} \alpha_i(D) =: & \left|\left| \sum_{j,k\in\Gamma:\ c_j\neq c_k} A^i_{jk} u_j u_k \right|\right|_D \\ & \leq \sum_{j,k\in\Gamma:\ c_j\neq c_k} \frac{\left|A^i_{j,k}\right|}{|c_j-c_k|} \left(||\varphi_j||_J + \alpha_j(D)\right)\left(||\varphi_k||_J + \alpha_k(D)\right). \end{aligned} \tag{3.5}$$

From (3.4) we then deduce there exist $c_1 > 0$, $c_2 > 0$

$$\sum_{j,k\in\Gamma} ||u_j u_k||_D \leq c_1 \sum_{i\in\Gamma} ||\varphi_i||_J + c_2 \left(\sum_{i\in\Gamma} ||\varphi_i||_J + \sum_{i\in\Gamma} \alpha_i(D) \right)^2 \tag{3.6}$$

and then, by using (3.6) in (3.5), there exist $c_3 > 0$, $c_4 > 0$ such that

$$\sum_{i\in\Gamma} \alpha_i(D) \leq c_3 \sum_{i\in\Gamma} ||\varphi_i||_J + c_4 \left(\sum_{i\in\Gamma} ||\varphi_i||_J + \sum_{i\in\Gamma} \alpha_i(D) \right)^2. \tag{3.7}$$

It is now sufficient to notice that the inequality

$$x \leq c_3\varepsilon + c_4 (\varepsilon + x)^2 \tag{3.8}$$

is verified for $\varepsilon \leq \varepsilon_0 =: \frac{1}{4c_4(1+c_3)}$, when

$$0 \leq x \leq \frac{(1-2c_4\varepsilon) - \sqrt{1-4(1+c_3)c_4\varepsilon}}{2c_4}$$

and in particular there exists a constant $c > 0$ such that (3.8) is verified for $0 \leq x \leq c\varepsilon$.

The corollary is thus proved. ∎

Theorem 3.3 **(l^∞–boundedness)** *Let us assume(1.4). When we define*

$$\Theta = \max_{i\in\Gamma} \sum_{j,k\in\Gamma:\ c_j\neq c_i} \frac{\left|A^i_{j,k}\right|}{|c_i - c_j|},$$

if there exists $0 < \varepsilon_1 < \varepsilon_0$ *with*

$$2\Theta\varepsilon_1 < 1 \tag{3.9}$$

and $\varphi_i \in l^\infty(J) \cap l^1(J)$,

$$\sum_{i\in\Gamma} ||\varphi_i||_J \leq \varepsilon_1, \tag{3.10}$$

then the solution $u \in l^\infty(D)$ *and we have*

$$\max_{i\in\Gamma} ||u_i||_{\infty,D} := \max_{i\in\Gamma}\left(\sum_n ||u_i^{\cdot,n}||_{\infty,J}\right) \leq \frac{1}{1-2\Theta\varepsilon_1}\max_{i\in\Gamma}||\varphi_i||_{\infty,J}\,. \tag{3.11}$$

Proof. We define, for all $m \in \mathbb{N}$, $D_m = \{(l,n) \in D:\ n \leq m\}$,

$$M_m = \max_{j\in\Gamma}\sup_{n\leq m,(l,n)\in D}\left|u_j^{l,n}\right| = \max_{j\in\Gamma}||u_j||_{\infty,D_m}\,, M_0 = \max_{j\in\Gamma}||u_j||_{\infty,J}\,.$$

The aim is to control M_m with M_0. We have

$$u_i^{l,m} = \varphi_i^{l-c_i\lambda m} + k\sum_{\alpha=1}^{m}\sum_{j,k\in\Gamma}A_{j,k}^i u_j^{l-c_i\lambda\alpha,m-\alpha}u_k^{l-c_i\lambda\alpha,m-\alpha}. \tag{3.12}$$

If $c_j \neq c_k$ we have (by (1.4) we can assume for example $c_j \neq c_i$)

$$\begin{aligned}&\sum_{\alpha=1}^{m}\sum_{j,k\in\Gamma}\left|A_{j,k}^i\right|\left|u_j^{l-c_i\lambda\alpha,m-\alpha}\right|\left|u_k^{l-c_i\lambda\alpha,m-\alpha}\right|\\ &\quad\leq\ M_m\sum_{j,k\in\Gamma}\left|A_{j,k}^i\right|\sum_{\alpha=1}^{m}\left|u_j^{l-c_i\lambda\alpha,m-\alpha}\right| \leq M_m\sum_{j,k\in\Gamma}\frac{\left|A_{j,k}^i\right|}{|c_i-c_j|}[u_j]_{V_j}\,,\end{aligned}$$

which implies

$$\left|u_i^{l,m}\right| \leq ||\varphi_i||_{\infty,J} + \Theta M_m\sum_{j\in\Gamma}[u_j]_{V_j} = ||\varphi_i||_{\infty,J} + \Theta M_m[u]_V\,.$$

But we have from (3.10) that $[u]_{\mathcal{V}} \leq 2\varepsilon_1$ so that, from (3.9) we have

$$M_m \leq \frac{1}{1-2\Theta\varepsilon_1}M_0$$

for all $m \in \mathbb{N}$, which concludes the proof. ■

Corollary 3.4 *Let us assume(1.5), (1.6) and (1.7). Assume $\varphi_i \geq 0$ and $\varphi_i \in l^\infty(J) \cap l^1(J)$ for any $i \in \Gamma$. With the symbols of the previous Theorem 3.3, if (3.10) holds, then the approximating sequence of solution is stable in $l^\infty(D)$ and we have (3.11).*

Proof. As in the previous Theorem 3.3 we have to bound from above the quantity (3.12). By (1.7) we can delete the terms for which $c_i = c_j = c_k$ because $A^i_{j,k} \geq 0$ and $u_j u_k \geq 0$. The remaining terms are treated as in Theorem 3.3. ∎

Theorem 3.5 (Asymptotic behavior) *Under condition (1.4), there exists an $\varepsilon_0 > 0$ such that, if (3.1) is valid, there exists a sequence $\left\{\bar{u}^l_i\right\}_{l\in J}$, $\bar{u}_i \in l^1(J)$ such that*

$$\sum_{l\in J}\left|u_i^{l+c_i\lambda n,n} - \bar{u}^l_i\right| \longrightarrow 0$$

when $n \to \infty$.

If the approximating solution $u = \left\{u^{l,n}\right\}$ is also bounded in $l^\infty(D)$, then $\bar{u}_i \in l^\infty(J)$ and the convergence is also in $l^\infty(J)$.

Proof. By the validity of the hypothesis (3.1) we obtain (3.2). But

$$\begin{aligned} u_i^{l+c_i\lambda n,n} &= \varphi^l_i + k\sum_{m=0}^{n-1} q_i\left(u^{l+c_i\lambda(n-m),n-m}\right) \\ &= \varphi^l_i + k\sum_{r=1}^{n} A^i_{j,k} u_j^{l+c_i\lambda r,r} u_k^{l+c_i\lambda r,r} \end{aligned}$$

and $u_i^{l+c_i\lambda n,n} \in l^1(D)$. If we set

$$\bar{u}^l_i = \varphi^l_i + k\sum_{r=1}^{\infty} A^i_{j,k} u_j^{l+c_i\lambda r,r} u_k^{l+c_i\lambda r,r},$$

by applying Lemma 2.2 we obtain

$$\sum_{l\in J}\left|u_i^{l+c_i\lambda n,n} - \bar{u}^l_i\right| \leq \sum_{l\in J} k \sum_{r=n+1}^{\infty}\left|A^i_{j,k} u_j^{l+c_i\lambda r,r} u_k^{l+c_i\lambda r,r}\right| \longrightarrow 0$$

as $n \to \infty$, which completes the first part of the theorem.

Moreover, when $u_i \in l^\infty(D)$, we have for any $\tau \in \mathbb{N}$

$$u_i^{l+c_i\lambda(n+\tau),(n+\tau)} - u_i^{l+c_i\lambda\tau,\tau} = k\sum_{m=0}^{n} q_i\left(u^{l+c_i\lambda(m+\tau),m+\tau}\right)$$

which implies

$$\left|u_i^{l+c_i\lambda(n+\tau),(n+\tau)} - u_i^{l+c_i\lambda\tau,\tau}\right| \leq kM\sum_{j,k\in\Gamma}\sum_{m=\tau}^{n+\tau}\left|A_{j,k}^i u^{l+c_i\lambda m,m}\right| \leq$$

by assuming $c_i \neq c_j$ and using Lemma 2.1

$$\leq kM\sum_{j,k\in\Gamma}\sum_{m=\tau}^{n+\tau}\left|A_{j,k}^i\right|\omega_j^{l+(c_i-c_j)\lambda m} \leq kM\sum_{j,k\in\Gamma}\sum_{m=\tau}^{\infty}\left|A_{j,k}^i\right|\omega_j^{l+(c_i-c_j)\lambda m} \longrightarrow 0$$

as $\tau \to \infty$ since $\omega_j \in l^1(J)$. ■

Let us now give the main convergence result of this section.

Theorem 3.6 (Convergence) *Let us consider the difference scheme (2.10) with continuous and bounded initial data $\varphi_i \in C_b(\mathbb{R})$. Under the assumptions of Theorems 3.1 and 3.3, there exists a weak solution u of (1.1)-(1.2) and the sequence $\{u_{\Delta_h}\}$ converges to u in $L^\infty(0,T;L^\infty(\mathbb{R}))$ for any $T>0$ as $h\to 0$.*

Proof. From Theorem 3.3 there exists $M = M(T)$ such that $\left|u_j^{l,n}\right| \leq M$ for all $j \in \Gamma$, for all (l,n). Let us define

$$J^n = \sup_{j\in\Gamma}\sup_{l}\left|u_j^{l+1,n} - u_j^{l,n}\right|$$

for all $n > 0$ and

$$J^0 = \sup_{j\in\Gamma}\sup_{l}\left|\varphi_j^{l+1} - \varphi_j^l\right| = o(1).$$

Since from (2.11)

$$u_j^{l,n+1} = u_j^{l-c_i\lambda,n} + kq_j\left(u^{l,n+1}\right),$$

we obtain

$$u_j^{l+1,n+1} - u_j^{l,n+1} = u_j^{l+1-c_j\lambda,n} - u_j^{l-c_j\lambda,n} + k\left(q_j\left(u^{l+1,n+1}\right) - q_j\left(u^{l,n+1}\right)\right)$$

so that

$$J^{n+1} \leq J^n + kNMJ^{n+1}.$$

Since $k\downarrow 0$, we can assume $kNM < 1$ and we obtain that there exists a constant $C>0$ such that for any $n>0$ $J^n \leq e^{CT}J^0$. In particular $J^n \leq o(1)$. On the other hand, from (2.11) we have

$$\begin{aligned}\left|u_j^{l,n+1} - u_j^{l,n}\right| &= \left|\lambda\left(\Phi_j^{l+1/2,n} - \Phi_j^{l-1/2,n}\right) - kq_j\left(u^{l,n+1}\right)\right| \\ &\leq const\, J^n + kM^2N^2 \leq o(1).\end{aligned} \tag{3.13}$$

Let us consider u_{Δ_h} as in Definition 2.12. Then $||u_{\Delta_h}||_\infty \leq M$ uniformly with respect to $h > 0$.

Let us fix $R > 0$ and consider a space interval $[-R, +R]$. For almost every fixed $(\bar{x}, \bar{t}) \in [-R, +R] \times [0, T]$ there exist a sequence $\{h_\nu\}_{\nu \in \mathbb{N}}$, $h_\nu \downarrow 0$ as $\nu \to \infty$, and a number $u(\bar{x}, \bar{t})$ such that

$$u_{\Delta_{h_\nu}}(\bar{x}, \bar{t}) \longrightarrow u(\bar{x}, \bar{t})$$

as $\nu \to \infty$. Let us consider a countable dense subset $\{\bar{x}_\xi\}_{\xi \in \mathbb{N}}$ of $[-R, +R]$. For each $\bar{x}_\xi$ there exists a sequence $u_{\Delta_{h_\nu^{(\xi)}}}(\bar{x}_\xi, \bar{t})$ which converges to a number $u(\bar{x}_\xi, \bar{t})$. By a diagonalization argument we can conclude there exists a subsequence, still denoted by $u_{\Delta_{h_\nu}}(\bar{x}_\xi, \bar{t})$, which converges to $u(\cdot, \bar{t})$ at every point $\bar{x}_\xi$. In order to reach every point $x \in [-R, +R]$, for any $\varepsilon > 0$ let us partition the interval $[-R, +R]$ into Σ intervals $(x_\alpha, x_{\alpha+1})$ where the x_α are selected from the dense set $\{\bar{x}_\xi\}_{\xi \in \mathbb{N}}$ in such a way that, for any α, $|x_{\alpha+1} - x_\alpha| \leq \varepsilon$. Therefore for any $x \in [-R, +R]$ there exists x_α such that $|x_\alpha - x| \leq \varepsilon$,

$$\begin{aligned}&\left|u_{\Delta_{h_\nu}}(x, \bar{t}) - u_{\Delta_{h_{\nu'}}}(x, \bar{t})\right| \leq \left|u_{\Delta_{h_\nu}}(x_\alpha, \bar{t}) - u_{\Delta_{h_\nu}}(x, \bar{t})\right| \\ &+ \left|u_{\Delta_{h_\nu}}(x_\alpha, \bar{t}) - u_{\Delta_{h_{\nu'}}}(x_\alpha, \bar{t})\right|_\infty + \left|u_{\Delta_{h_{\nu'}}}(x_\alpha, \bar{t}) - u_{\Delta_{h_{\nu'}}}(x, \bar{t})\right|.\end{aligned}$$

The term in the middle comes from a Cauchy sequence, while the first and last terms are bounded by $const\,(o(1) + \varepsilon)$. From the previous results, for almost every fixed $\bar{t} > 0$ there exist a subsequence, still denoted by $\{h_\nu\}_{\nu \in \mathbb{N}}$, $h_\nu \downarrow 0$ as $\nu \to \infty$, and a function u such that

$$u_{\Delta_{h_\nu}}(\cdot, \bar{t}) \longrightarrow u(\cdot, \bar{t}) \qquad \text{in } L^\infty$$

as $\nu \to \infty$. Again, let us consider a countable dense subset $\{\bar{t}_s\}_{s \in \mathbb{N}}$ of $[0, T]$. For each $\bar{t}_s$ there exists a sequence $u_{\Delta_{h_\nu^{(s)}}}(\cdot, \bar{t}_s)$ bounded in $L^\infty(\mathbb{R})$ which converges in $L^\infty(\mathbb{R})$ to a function $u(\cdot, \bar{t}_s)$. By a diagonalization argument we can conclude there exists a subsequence, still denoted by $u_{\Delta_{h_\nu}}(\cdot, \bar{t}_s)$, which converges to $u(\cdot, \bar{t}_s)$ at every point $\bar{t}_s$. For any $\varepsilon > 0$ let us partition the interval $[0, T]$ into S intervals $(t_\alpha, t_{\alpha+1})$ where the t_α are selected from the dense set $\{\bar{t}_s\}_{s \in \mathbb{N}}$ in such a way that, for any α, $|t_{\alpha+1} - t_\alpha| \leq \varepsilon$. Therefore for any $t \in [0, T]$ there exists t_α such that $|t_\alpha - t| \leq \varepsilon$,

$$\begin{aligned}&\left|\left|u_{\Delta_{h_\nu}}(\cdot, t) - u_{\Delta_{h_{\nu'}}}(\cdot, t)\right|\right|_\infty \leq \left|\left|u_{\Delta_{h_\nu}}(\cdot, t_\alpha) - u_{\Delta_{h_\nu}}(\cdot, t)\right|\right|_\infty \\ &+ \left|\left|u_{\Delta_{h_\nu}}(\cdot, t_\alpha) - u_{\Delta_{h_{\nu'}}}(\cdot, t_\alpha)\right|\right|_\infty + \left|\left|u_{\Delta_{h_{\nu'}}}(\cdot, t_\alpha) - u_{\Delta_{h_{\nu'}}}(\cdot, t)\right|\right|_\infty.\end{aligned}$$

The term in the middle comes from a Cauchy sequence, while by (3.13) the first and last terms are bounded by $const\,(o(1) + \varepsilon)$. Thus $u_{\Delta_{h_\nu}}(\cdot, t)$ is a uniform in $t \in [0, T]$ Cauchy sequence in $L^\infty(\mathbb{R})$.

Hence, we can extract a subsequence which converges in $L^\infty(0,T;L^\infty(\mathbb{R}))$. By Lax–Wendroff Theorem [19] the limit is a weak solution of (1.1)-(1.2). Finally, the uniqueness of the weak solutions implies the convergence of the whole sequence. ∎

4 Results for large data

The aim is now to obtain stability for arbitrarily large nonnegative initial data. More precisely, we will prove the following result:

Theorem 4.1 (Uniform l^∞–stability) *Under condition (1.4), if the H-Theorem is valid, that is (1.5) and*

$$\sum_{i,j,k\in\Gamma} A^i_{j,k\in\Gamma}\lambda_j\lambda_k\left(\log\left(\lambda_i\right)+1\right)\geq 0 \quad \textit{for all } \lambda \textit{ such that } \lambda_i>0\ \forall i \tag{4.1}$$

hold, there exists a growth function $F_n(M)$ such that, when the datum satisfy $0\leq\varphi^l_i\leq M$ for all i, then the sequence of approximating solutions is uniformly stable in time and satisfies

$$0\leq u^{l,n}_i\leq F_n(M) \qquad\qquad \textit{for all } i. \tag{4.2}$$

Before proving the previous theorem, it is convenient to establish the following lemma holds.

Lemma 4.2 *Assume the hypotheses (1.4), (1.5) and (4.1). If the initial datum satisfies $0\leq\varphi^l_i\leq M$ with compact support, the corresponding approximating sequence v_i to the Cauchy problem (1.1)-(1.2) satisfies*

$$k\sum_{i\in\Gamma}\sum_{l\in\mathbb{Z}} v^{l,n}_i\log_+ v^{l,n}_i\leq const\left(||\varphi||_\infty\log_+||\varphi||_\infty+\tau\right) \tag{4.3}$$

for all $n\leq\tau$ and there exists $L>0$ such that

$$\sum_{i\in\Gamma}\sum_{l=\bar{l}}^{\bar{l}+L} v^{l,n}_i\leq\sum_{i\in\Gamma}\sum_{l\in J}\varphi^l_i \tag{4.4}$$

uniformly with respect to $\bar{l}$, $n\in[0,\tau]$.

Proof. Since the initial datum φ^l_i has compact support, also the approximating sequence $v^{l,n}_i$ has compact support for any n and we have

$$\#\left\{l:\ v^{l,n}_i\neq 0,\ i\in\Gamma\right\}\leq const\cdot\tau$$

for $n \leq \tau$. So we have

$$\sum_{l\in\mathbb{Z}} v_i^{l,n} = \sum_{l\in\mathbb{Z}} v_i^{l-c_i\lambda,n}. \tag{4.5}$$

By (2.10) we obtain for any $n \geq 0$

$$v_i^{l,n+1} = v_i^{l-c_i\lambda,n} - k \sum_{j,k\in\Gamma} A^i_{j,k} v_j^{l,n+1} v_k^{l,n+1}. \tag{4.6}$$

So we have

$$\begin{aligned}
&\sum_{i\in\Gamma}\sum_{l\in\mathbb{Z}} v_i^{l,n+1}\log v_i^{l,n+1} \\
&= \sum_{i\in\Gamma}\sum_{l\in\mathbb{Z}} \left(v_i^{l-c_i\lambda,n} - k\sum_{j,k\in\Gamma} A^i_{j,k} v_j^{l,n+1} v_k^{l,n+1}\right) \log v_i^{l,n+1} =
\end{aligned}$$

multiplying and using (4.6) inside the logarithm,

$$\begin{aligned}
&= \sum_{i\in\Gamma}\sum_{l\in\mathbb{Z}} v_i^{l-c_i\lambda,n} \log \left[v_i^{l-c_i\lambda,n} \left(1 + \frac{-k\sum_{j,k\in\Gamma} A^i_{j,k} v_j^{l,n+1} v_k^{l,n+1}}{v_i^{l-c_i}} \right)\right] \\
&- k \sum_{l\in\mathbb{Z}} \sum_{i,j,k\in\Gamma} A^i_{j,k} v_j^{l,n+1} v_k^{l,n+1} \log v_i^{l,n+1} \\
&\leq \sum_{i\in\Gamma}\sum_{l\in\mathbb{Z}} v_i^{l-c_i\lambda,n} \log v_i^{l-c_i\lambda,n} - k\sum_{l\in\mathbb{Z}} \sum_{i,j,k\in\Gamma} A^i_{j,k} v_j^{l,n+1} v_k^{l,n+1} \\
&- k \sum_{l\in\mathbb{Z}} \sum_{i,j,k\in\Gamma} A^i_{j,k} v_j^{l,n+1} v_k^{l,n+1} \log v_i^{l,n+1} \\
&\leq \sum_{i\in\Gamma}\sum_{l\in\mathbb{Z}} v_i^{l-c_i\lambda,n} \log v_i^{l-c_i\lambda,n} = \sum_{i\in\Gamma}\sum_{l\in\mathbb{Z}} v_i^{l,n} \log v_i^{l,n}
\end{aligned}$$

by standard property of the logarithm and by using (4.1) and (4.5).

The estimate (4.3) now follows by the previous estimate and elementary logarithmic properties.

To prove (4.4) first we observe that for all $L \geq 1$, $R > 1$ we have by using (4.3) for the first term

$$\begin{aligned}
k\sum_{l=\bar{l}}^{\bar{l}+L} v_i^{l,n} &= k \sum_{v_i^{l,n}\geq R} v_i^{l,n} + k \sum_{v_-^{l,n}<R} v_i^{l,n} \\
&\leq const\left(||\varphi||_\infty \log_+ ||\varphi||_\infty + \tau\right) \log_+ R + kRL.
\end{aligned}$$

We now choose $R > 1$ and $L \geq 1$ such that

$$\begin{cases} const\left(||\varphi||_\infty \log_+ ||\varphi||_\infty + \tau\right) \log_+ R \leq \frac{1}{2N} ||\varphi_i||_J \\ kRL \leq \frac{1}{2N} ||\varphi_i||_J \end{cases}$$

(we can assume in any case to take k such that $kR \leq \frac{1}{2N} ||\varphi_i||_J$) and we obtain

$$k \sum_{l=\bar{l}}^{\bar{l}+L} v_i^{l,n} \leq \frac{||\varphi_i||_J}{N}.$$

The proof of the lemma is thus complete. ∎

Proof of Theorem 4.1. Let us fix $\tau \in \mathbb{N}$, $\tau > 0$, $l_0 \in J$. Denote

$$J_n \left\{ l \in J : \; l \in \left[l_0 - n \max_{i\in\Gamma} c_i, l_0 - n \min_{i\in\Gamma} c_i \right] \right\}$$

and define

$$\hat{\varphi}_i^l = \begin{cases} \varphi_i^l & \text{if } l \in J_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then the sequence of approximating solution $v_i^{l,n}$ for the Cauchy problem (1.1) with datum $\hat{\varphi}_i^l$ will coincide with $u_i^{l,n}$ at (l_0, τ). By applying Lemma 4.2 we obtain there exists $L = L\left(||\varphi||_\infty, \tau\right)$ such that (4.4) holds uniformly with respect to $\bar{l} \in J$, $n \in [0, \tau]$.

It is now sufficient to apply l^∞ boundedness for small data on $l^1 \cap l^\infty$ proved in Theorem 3.3 on any $D_{\bar{l}}$, constructed with basis interval

$$J_{\bar{l}} = \left\{ l : \; \bar{l} \leq l \leq \bar{l} + L \right\}$$

at time $\tilde{n} \in [0, \tau]$. Then on $D_{\bar{l}}$ we have

$$\max_{i\in\Gamma;(l,n)\in D_{\bar{l}}} u_i^{l,n} \leq const \max_{i\in\Gamma; l\in J_{\bar{l}}} u_i^{l,\tilde{n}}.$$

By moving $\bar{l} \in \mathbb{Z}$ we obtain

$$M_n \leq const M_{\tilde{n}} \qquad \text{for } \tilde{n} < n < \tilde{n} + \frac{L}{\max_{i\in\Gamma} c_i - \min_{i\in\Gamma} c_i}. \tag{4.7}$$

To cover the time interval $[0, T]$ it is sufficient to apply (4.7) finitely many times. The proof of Theorem 4.1 is now complete. ∎

With a similar proof we obtain also the following result.

Corollary 4.3 *Under condition (1.5), (1.6), (1.7) and (4.1), there exists a growth function* $F_n(M)$ *such that, when the datum satisfies* $0 \leq \varphi_i^l \leq M$ *for all* i*, then the approximating sequence is uniformly stable in time and satisfies (4.2).*

As for small data, we can now conclude with a convergence result.

Theorem 4.4 (Convergence) *Let us consider the difference scheme (2.10) with continuous and bounded nonnegative initial data* $\varphi_i \in C_b(\mathbb{R})$*. Under assumptions of Theorem 4.1, there exists a weak solution* u *of (1.1)-(1.2) and the sequence* $\{u_{\Delta_h}\}$ *converges to* u *in* $L^\infty(0, T; L^\infty(\mathbb{R}))$ *for any* $T > 0$ *as* $h \to 0$*.*

The proof follows like in Theorem 3.6, then it is omitted.

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