On the Chapman-Jouguet Limit for a Combustion Model

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Abstract

We study the limiting behaviour of solutions to a simple model for combustion waves when the reaction rate tends to infinity. First we establish strong convergence for locally uniformly bounded sequences of solutions. Next we show the uniform boundedness of piecewise smooth detonations and deflagrations with respect to the reaction rate.

Key words and phrases: combustion, detonations, deflagrations, Chapman-Jouguet limit, conservation laws with source, shock waves.

1 Introduction

In this paper we investigate the qualitative behaviour of solutions to the following simple model for combustion waves:

$$\partial_t(u+v) + \partial_x f(u) = 0 \ , \tag{1.1}$$

$$\partial_t v = -k\varphi(u)v \qquad (k>0) \ , \tag{1.2}$$

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for $(x,t) \in \mathbb{R} \times (0,\infty)$, with the initial conditions

$$u(x,0) = u_0(x) \ , \qquad v(x,0) = v_0(x) \quad (x \in \mathbb{R}) \tag{1.3}$$

when the reaction rate k goes to infinity.

Throughout the paper the functions f and φ will be some given smooth functions. We also assume that

(1.4) $\quad \varphi$ is monotone increasing, with $\varphi(u) \equiv 0$ for $u \leq 0$.

The model (1.1)–(1.3) has been proposed independently by Fickett [Fi] and Majda [Ma1] as a simple mathematical analogous for the equations describing one-dimensional compressible flow in a chemically reacting fluid.

Let us recall that at least two different mathematical theories are used to describe the propagation of combustion waves in reacting flows: the Chapman-Jouguet (CJ) theory and the Zeldovich-Von Neumann-Döring (ZND) theory (see [CF], [Wi], [Ma2]). Both are formulated starting from the classical Euler equations for gas-dynamics. The CJ-theory assumes that the reaction region is *infinitely thin* or, equivalently, that the reaction rate is *infinitely large*; moreover the effects of viscosity and heat conduction are disregarded. The corresponding equations for a simplified (one–dimensional) reactant/product dynamics have the following form

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \ , \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0 \ , \\ \partial_t(\rho E) + \partial_x(\rho u E + uP) = 0 \ . \end{cases} \tag{1.5}$$

Here ρ is the density, u is the fluid velocity, P the pressure and E is the total specific energy, namely

$$E = e + q_0 z + \frac{1}{2}u^2 \ ,$$

the quantity e being the specific internal energy, while q_0 denotes the amount of heat released by the chemical reaction and z is the mass fraction of the reactant.

The internal energy e and the temperature T, are given through equations of state

$$e = e(\rho, P),\ T = T(\rho, P)\ ,$$

which depend on the gas mixture under consideration. To specify the variable z the CJ-theory introduces a further relation, namely

$$z(x,t) = \begin{cases} 0 & \text{if } \sup\limits_{0\le s\le t} T(x,s) > T_i\ , \\ z(x,0) & \text{if } \sup\limits_{0\le s\le t} T(x,s) \le T_i\ , \end{cases} \tag{1.6}$$

where T_i is a given ignition temperature.

Actually the above model is too crude to determine in a unique way the complete structure of combustion waves. It is shown in [CF] that, even considering just the Riemann problem, the corresponding flows present some original features with respect to the non reactive case; a minimal speed for the burnt gas (the CJ-state) exists, such that two different combustion waves (the strong and the weak detonation waves) move at the same speed s, for any given s larger than the minimal speed. Observe that the general Riemann problem (in Lagrangian coordinates) has been solved in [ZZ], where the authors showed the possible existence of even nine distinct waves for some fixed initial data.

In order to describe more precisely the internal structure of combustion waves, the ZND-theory assumes that the reaction rate is *finite* – although the effects of viscosity and heat conduction are disregarded.

This corresponds to replace the equation (1.6) by the following supplementary equation for the mass fraction of unburnt gas:

$$\partial_t(\rho z) + \partial_x(\rho u z) = -k\rho\varphi(T)z\ . \tag{1.7}$$

The rate function $\varphi(T)$ has typically the following form:

$$\varphi(T) = \begin{cases} 0 & \text{if } T < T_i\ , \\ T^\alpha \exp\left\{-\frac{A}{T-T_i}\right\} & \text{if } T \ge T_i\ , \end{cases} \tag{1.8}$$

where T_i is again the ignition temperature, A is the activation energy and k the reaction rate ($T_i, A, k > 0$); as for α, this is a dimensionless parameter in the range $(-1, 2]$.

The system (1.5)-(1.7) is a hyperbolic quasilinear system of balance laws, whose local existence of solutions is established by the results in [DH], at least for initial data of small bounded variation (see also [YW]). Global existence of solutions (even for the Riemann problem) seems to be still an open problem.

In view of the complexity of the above equations, which exhibit different phenomena due to the strongly nonlinear interaction of convective and reaction waves, simpler qualitative models have been developed.

In fact, the system (1.1)-(1.2) can be regarded as a simplified version of the ZND system (1.5)-(1.7). The unknown u plays a rôle similar to that of density, velocity and temperature, in (1.5)-(1.7), while v represents the mass fraction of the unburnt gas. The function φ can be chosen as in (1.8), with $T = u$ and $T_i = 0$. The counterpart of the CJ model (1.5)-(1.6) is now given by equation (1.1), supplemented by the equalities

$$v(x,t) = \begin{cases} 0 & \text{if } \sup\limits_{0\le s\le t} u(x,s) > 0 \ , \\ v(x,0) & \text{if } \sup\limits_{0\le s\le t} u(x,s) \le 0 \ . \end{cases} \tag{1.9}$$

Let us remark that the relation (1.9) is formally obtained if we take the limit as $k \to \infty$ of the solutions to problem (1.1)-(1.2). In fact, let (u^k, v^k) solve to the Cauchy problem (1.1)–(1.3) for any fixed $k > 0$. Since there holds:

$$v^k(x,t) = v_0(x) e^{-k \int_0^t \varphi(u^k(x,s)) \mathrm{d}s} \ ,$$

the relation (1.9) follows by assuming e.g. the *uniform* convergence of the whole sequence $\{(u^k, v^k)\}$ to some limit functions (u, v).

Existence, uniqueness and stability of the (entropy) solutions to the Cauchy problem (1.1)–(1.3) have been first established in [TY1]. Further results can be found in [TY2], [Le], [HN]. A slightly different model, yet with the same viscous travelling waves of model (1.1)–(1.3), was considered in [MR] and [L1,2].

The Riemann problem and some perturbation arguments for system (1.1)-(1.9) have been investigated in [LZ] (see also [Zh], where an interesting survey on the whole theory is given). In particular in [LZ] the authors found up to four distinct configurations for some given initial data to exist. Therefore they proposed to consider as physically eligible those (piecewise smooth, self–similar) solutions for which the number of combustion waves is as small as possible. It was proved in [TY1] that these admissible solutions are the limit of solutions to the Riemann problem (1.1)–(1.3) as the reaction rate $k \to \infty$; the limit for $k \to \infty$ was taken by using the explicit structure of the combustion waves (both detonations and deflagrations) for the Riemann problem, under suitable assumptions on the function φ. It is the purpose of this note to give some further insight in the comprehension of this limit, considering more general initial data and making weaker assumptions on the function $\varphi(u)$.

Observe that the formal arguments used to derive the equality (1.9) (or (1.6)) can hardly be employed to give a rigorous mathematical justification of the Chapman-Jouguet assumptions. Moreover, even the mathematical formulation of the limit problem (1.1)-(1.9) (or (1.5)-(1.6)) appears to be quite meaningless for general discontinuous solutions.

Let us outline the main idea of our approach. Consider the following Chapman-Enskog type expansion with respect to the small parameter $1/k$:

$$\begin{cases} u^k = u^\infty + \frac{1}{k}\tilde{u} + 0\left(\frac{1}{k}\right) \ , \\ v^k = v^\infty + \frac{1}{k}\tilde{v} + 0\left(\frac{1}{k}\right) \ . \end{cases} \tag{1.10}$$

Introducing (1.10) into the system (1.1)-(1.2), gives, up to terms of order $\frac{1}{k}$,

$$\partial_t(u^\infty + v^\infty) + \partial_x f(u^\infty) = 0 \ , \tag{1.11}$$

$$\partial_t v^\infty \le 0 \ , \qquad v^\infty \ge 0 \ , \tag{1.12}$$

$$\varphi(u^\infty) v^\infty = 0 \ . \tag{1.13}$$

In Section 3 we give a rigorous justification of this formal argument, assuming that the sequence $\{u^k\}$ is locally uniformly bounded in the supremum norm. Under this assumption , by compensated compactness arguments (see [Ta], [Mu]) we prove that any sequence $\{(u^k, v^k)\}$ of entropy solutions of (1.1)–(1.3) contains a subsequence, which converges to a weak solution (u^∞, v^∞) of problem (1.11)-(1.13) (see Section 3, Theorem 3.1 for a precise statement).

The remainder of the paper is devoted to investigate the uniform boundedness of the sequence $\{u^k\}$, at least for some classes of initial data. In Section 4 we deal with simple detonation waves. Because of the well-known lack of monotonicity structure of these waves, the usual comparison arguments are not expedient in this case; hence we use some original arguments involving characteristics. On the other hand, in Section 5 to study deflagration waves we use a fixed point approach, in the spirit of [TY1]. Then we can generalize previous results in [TY1], both for more general data and for unbounded reaction rates. Also note that the uniform bounds we provide strongly depend on the nonlinear form of the flux function $f = f(u)$ (see Section 4 for a counter-example in the *semilinear* case). Actually, only the dissipative nature of the nonlinear convection term is effective to balance the reaction mechanism.

The limit for $k \to \infty$ was previously considered in [Lu] by a truncation argument concerning the reaction function $\varphi(u)$ and a viscosity regularization. However the weak solutions found by this approximation procedure are qualitatively different from the solutions observed in [LZ] and [TY1]. In particular, the limit function u^∞ in [Lu] satisfies the estimate

$$\|u^\infty\|_\infty \le \|u_0\|_\infty \ ,$$

which in general is not the case for strong detonations considered here and in the above referred papers.

Finally, we would like to stress the strong connection between our singular perturbation problem and the relaxation phenomena, including fluid-dynamical limits (see [CP], [CLL], [Li], [Na], [Xi], [Yo]). In fact, the technical tools of [CLL] have largely inspired our approach in Section 3, although an important difference is given by the intrinsic instability of the present problem (see [LZ], [Yo]).

2 Preliminaries

In this section we shall recall some basic definitions and results concerning weak (admissible) solutions of the Cauchy problem (1.1)–(1.3). First let us fix some notations. Throughout the paper Σ_T denotes the strip $\mathbb{R} \times (0, T)$ $(T > 0)$, $\Sigma = \Sigma_\infty$. For any open set $\Omega \subseteq \mathbb{R}^n$, $C_0^\infty(\Omega)$ is the space of the $C^\infty(\Omega)$ functions with compact support, while $\mathcal{D}'(\Omega)$ denotes the space of distributions. The spaces $L^p(\Omega)$ $(1 \leq p \leq \infty)$ are defined in the usual way; their norm is denoted by $\|\cdot\|_p$. The spaces $L^p_{loc}(\Omega)$ are the spaces of functions u such that for any $\psi \in C_0^\infty(\Omega)$, $\psi u \in L^p(\Omega)$.

Let us recast the problem (1.1)-(1.2) in a more general way. Consider the weakly coupled first order quasilinear hyperbolic 2×2 system

$$\partial_t u_i + \partial_x f_i(u_i) = g_i(U) \ , \qquad i = 1, 2 \ , \tag{2.1}$$

$$U(x, 0) = U^0(x) \ , \tag{2.2}$$

where $U = (u_1, u_2)$ and $U^0 = (u_1^0, u_2^0)$. Here $F(U) = (f_1(u_1), f_2(u_2))$, $G(U) = (g_1(U), g_2(U))$ are given smooth functions.

In the present case we have

$$(u_1, u_2) = (u, v),$$

$$(f_1, f_2) = (f, 0)$$

and

$$(g_1, g_2) = (k\varphi(u)v, -k\varphi(u)v).$$

Following [HN] we define the entropy solutions of (2.1)-(2.2) (see [Kr] for the scalar case).

Definition 2.1 *A vector-valued function $U \in L^\infty(\Sigma_T)^2$ is said to be an entropy solution of* (2.1)-(2.2) *in Σ_T if, for any $\Psi \in C_0^\infty(\Sigma_T)^2$, $\Psi = (\psi_1, \psi_2)$, $\psi_i \geq 0$ $(i = 1, 2)$, and any l_1, $l_2 \in \mathbb{R}$ we have*

$$\text{(2.3)} \qquad \sum_{i=1}^{2} \iint_{\Sigma_T} \{|u_i - l_i\,|\,\partial_t\psi_i + \operatorname{sgn}(u_i - l_i)(f(u_i) - f(l_i))\partial_x\psi_i + g_i(U)\psi\}\,\mathrm{d}x\ \mathrm{d}t \geq 0$$

and, for any interval I,

$$\text{(2.4)} \qquad \lim_{T\to 0^+} \frac{1}{T}\int_0^T \int_I |u_i(x,t) - u^0(x)|\,\mathrm{d}x = 0\ .$$

Remark 2.1 Observe that U verifies the condition (2.3) if and only if for any convex function η, setting $q_i := \int^u \eta'(s)f_i'(s)\,\mathrm{d}s$, we have

$$\partial_t\eta(u^i) + \partial_x q_i(u^i) \leq \eta'(u^i)g_i(U)\ ,$$

in $\mathcal{D}'$. The pair (η, q_i) is called an entropy pair for the i-th equation in (2.1) $(i = 1, 2)$.

The following results were proved in [HN].

Theorem 2.1 (Uniqueness) *For all $U^\circ \in L^\infty(\mathbb{R})^2$ and $T > 0$, there exists at the most one entropy solution U to the Cauchy problem* (2.1)-(2.2) *in Σ_T.*

Theorem 2.2 (Local existence and continuation) *For any $U^0 \in L^\infty(\mathbb{R})^2$, there is $T > 0$ (depending only on $\|U_0\|_\infty$) such that in Σ_T there exists an entropy solution U of* (2.1)-(2.2) *and $U \in C([0,T], L^1_{loc}(\mathbb{R})^2)$.*

Moreover only two possibilities occur: either $U \in L^\infty(\Sigma_T)^2$ *for all* $T > 0$ *or there exists* $T^* < +\infty$ *such that* U *is defined for all* $T < T^*$ *and*

$$\lim_{T \to T^*_-} \|U\|_{L^\infty(\Sigma_T)^2} = +\infty \ .$$

Concerning our problem (1.1)-(1.2), the following statement can be proved by using the comparison tools of [HN].

Theorem 2.3 *Let* φ *verify (1.4); assume that* φ *is Lipschitz continuous and sublinear, namely there is a constant* $C > 0$ *such that*

$$0 \le \varphi(u) \le Cu \qquad \text{for all } u \ge 0 \ . \tag{2.5}$$

Then for all u_0, $v_0 \in L^\infty(\mathbb{R})$, $v_0 \ge 0$, *there is a unique entropy solution* (u, v) *to the Cauchy problem* (1.1)-(1.2). *Moreover, the following global estimates hold true for almost every* $(x,t) \in \Sigma$*:*

$$0 \le v(x,t) \le v_0(x) \ . \tag{2.6}$$

$$-\|u_0\|_\infty \le u(x,t) \le \|u_0\|_\infty e^{Ck\|v_0\|_\infty t} \ . \tag{2.7}$$

In addition, suppose that for some $M > 0$

$$0 \le \varphi(u) \le M \qquad \text{for all } u \ge 0 \ . \tag{2.5$'$}$$

Then instead of (2.7) *we have for almost every* $(x,t) \in \Sigma$*:*

$$-\|u_0\|_\infty \le u(x,t) \le \|u_0\|_\infty + kM\|v_0\|_\infty t \ . \tag{2.7$'$}$$

Observe that existence and uniqueness of entropy solutions were first established in [TY1], by using the Lax-Friedrichs scheme, in case of assumption (2.5′). The convergence of the viscosity approximation, under the same assumption, was later proved in [TY2] and [Le].

3 The Chapman-Jouguet limit

In this section we shall be concerned with the limit as $k \to \infty$ of globally defined entropy solutions (u^k, v^k) to problem (1.1)-(1.2) (see Theorem 2.3).

We assume the sequence $\{u^k\}$ to be uniformly locally bounded in L^∞, namely:

$$\begin{cases} \text{for any compact set } B \subseteq \overline{\Sigma} \text{ there exists} \\ \text{a constant } C_B > 0 \text{ such that, for any } k > 0 \ , \\ \qquad \|u^k\|_{L^\infty(B)} \leq C_B \ . \end{cases} \tag{3.1}$$

Then we have the following result.

Theorem 3.1 *Assume that the function f is not affine on any interval. Let the hypotheses of Theorem 2.3 and (3.1) hold. Then there exist a subsequence (also denoted $\{(u^k, v^k)\}$) of solutions to (1.1)-(1.2) and a limit couple $(\overline{u}, \overline{v}) \in L^\infty_{loc}(\Sigma)$, such that*

$$u^k \longrightarrow \overline{u} \ , \qquad \text{in } L^p_{loc}(\Sigma) \quad (1 \leq p < +\infty) \ , \tag{3.2}$$

$$v^k \rightharpoonup \overline{v} \ , \qquad \text{in } L^\infty\text{-weak}^* \ . \tag{3.3}$$

Moreover, for any $\psi \in C_0^\infty(\overline{\Sigma})$ the following holds:

$$\iint (\overline{u} + \overline{v})\partial_t \psi \ + \ f(\overline{u})\partial_x \psi \ \mathrm{d}x \ \mathrm{d}t \ + \ \int (u_0(x) + v_0(x))\psi(x,0) \, \mathrm{d}x = 0 \ ; \tag{3.4}$$

$$\overline{v} \geq 0 \ ; \tag{3.5}$$

$$\varphi(\overline{u})\overline{v} = 0 \quad \text{for almost every } (x,t) \in \Sigma; \tag{3.6}$$

$$\partial_t \overline{v} \leq 0 \qquad \text{in } \mathcal{D}' \ . \tag{3.7}$$

To prove the above theorem we shall use the following results.

Proposition 3.2 [Ta]. *Let $f \in C^1$ and $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Let $\{u^k\}$ be a sequence of functions satisfying $u^k \rightharpoonup u$ in $L^\infty(\Omega)$–weak*. Suppose that for any convex function η*

$$\partial_t \eta(u^k) + \partial_x q(u^k) \in (\text{compact set of } H^{-1}(\Omega)) \tag{3.8}$$

where $q' = \eta' f'$. Then, if no interval exists on which f is affine,

$$u^k \longrightarrow u \qquad \text{in } L^p(\Omega) \tag{3.9}$$

for any $p \in [1, \infty)$.

Lemma 3.3 [Mu]. *Let Ω be a bounded open set of $\mathbb{R}^n$ and $p \in (1, \infty)$. Let $\{u^k\}$ be any sequence such that*

$$\{u^k\} \in (\text{bounded set of } W^{-1,p}(\Omega)) \ , \tag{3.10}$$

$$u^k \geq 0 \ . \tag{3.11}$$

Then

$$\{u^k\} \in (\text{compact set of } W^{-1,q}(\Omega)) \tag{3.12}$$

for all $1 < q < p$.

Lemma 3.4 [DCL]. *Let Ω be a bounded open set of $\mathbb{R}^n$ and $1 < p \leq q < r < +\infty$. Let $\{u^k\}$ be a sequence such that*

$$\{u^k\} \in (\text{compact set of } W^{-1,p}(\Omega)) \cap (\text{bounded set of } W^{-1,r}(\Omega)) \ . \tag{3.13}$$

Then

$$\{u^k\} \in (\text{compact set of } W^{-1,q}(\Omega)) \ . \tag{3.14}$$

Proof of Theorem 3.1. Let $\psi \in C_0^\infty(\mathbb{R})$, $\psi \geq 0$. From the second equation in (1.1) we have for all $t \geq 0$ and $k > 0$:

$$\partial_t \int v^k(x,t)\psi(x)\,\mathrm{d}x + k \int \varphi(u^k(x,t)) v^k(x,t)\psi(x)\,\mathrm{d}x = 0 \ .$$

Then for any $T > 0$

$$0 \leq \int_0^T \int \varphi(u^k(x,t))v^k(x,t)\psi(x)\,\mathrm{d}x\;\mathrm{d}t \leq \frac{\|v_0\|_\infty \|\psi\|_{L^1}}{k}\;. \tag{3.15}$$

Let (η, q) be now an entropy pair, namely η is convex and $q' = \eta' f'$. Then, in view of Remark 2.1 we have for all $k > 0$:

$$\partial_t \eta(u^k) + \partial_x q(u^k) \leq k\eta'(u^k)\varphi(u^k)v^k \quad \text{ in } \mathcal{D}'\;.$$

Setting

$$I^k := k\eta'(u^k)\varphi(u^k)v^k$$

we have by (3.1) and (3.15):

$$\|I^k\psi\|_{L^1(\Sigma_T)} \leq C \sup_{(0,T)\times(\operatorname{supp}\psi)} |\eta'(u^k)| \leq \overline{C}\;,$$

where $\overline{C}$ depends only on $\|v_0\|_\infty$, η, ψ and T.

Then the function I^k is in a bounded set of L^1_{loc} for any $k > 0$. Hence by Sobolev embedding results (see for instance [Ad])

$$I^k \in \;(\text{compact set in } W^{-1,p_1}_{loc}) \tag{3.16}$$

with $p_1 < 2$. Moreover,

$$\Lambda^k := \partial_t \eta(u^k) + \partial_x q(u^k) - I^k \leq 0,$$

Λ^k being in a bounded subset in W^{-1,p_1}_{loc}. In view of Lemma 3.3 we have:

$$\Lambda^k \in (\text{compact set in } W^{-1,p}_{loc}) \tag{3.17}$$

for all $1 \leq p < p_1$. Hence by (3.16)–(3.17), there holds

$$\partial_t \eta(u^k) + \partial_x q(u^k) \in \;(\text{compact set of } W^{-1,p}_{loc})\;, \tag{3.18}$$

which by assumption (3.1) implies

$$\partial_t \eta(u^k) + \partial_x q(u^k) \in \;(\text{bounded set of } W^{-1,\infty}_{loc})\;.$$

Then using Lemma 3.4 we get

$$\partial_t \eta(u^k) + \partial_x q(u^k) \in \text{ (compact set of } H^{-1}_{loc}) \ .$$

By Proposition 3.2 there exists $\overline{u} \in L^\infty_{loc}(\Sigma)$ and a subsequence (also denoted u^k) such that

$$u^k \longrightarrow \overline{u} \qquad \text{in } L^p_{loc}(\Sigma) \ .$$

Moreover, there exists $\overline{v} \in L^\infty(\Omega)$ and a further subsequence v^k such that

$$v^k \rightharpoonup \overline{v} \ ,$$

$$\varphi(u^k) v^k \rightharpoonup \varphi(\overline{u}) \overline{v} \ ,$$

both convergences being in the L^∞–weak* sense On the other hand, using (3.10) we obtain

$$\varphi(u^k) v^k \longrightarrow 0 \ , \qquad \text{in } L^1_{loc} \ ,$$

which implies (3.6).

The proof of (3.4), (3.5) and (3.7) is now easy. Hence the conclusion follows. □

Remark 3.1 Observe that we could somewhat relax the assumptions concerning the $\{u^k\}$, by assuming in (3.1) only a uniform control in $L^p_{loc}(\Sigma)$ and using the correspondent L^p theory of compensated compactness (as developed in [Sc]) if the flux function f grows as follows:

$$f(u) = 0(|u|^q) \qquad \text{as } |u| \to \infty \ , \ q < p \ .$$

4 Simple detonations

In this section we establish a uniform bound in the supremum norm for the solutions of (1.1)-(1.2), in the case where the flow is piecewise smooth with finitely many noninteracting detonation waves. According to the results of Section 3, this estimate is of fundamental importance to prove the convergence of the sequence $\{u^k, v^k\}$ as $k \to \infty$.

For simplicity we concentrate on the case where the solution contains only a single detonation; the general case easily follows by similar arguments.

Let us make the following definition.

Definition 4.1 *An entropy solution* (u, v) *of problem* (1.1)-(1.2) *is a* simple detonation *if:*

i) *the initial data* $(u_0, v_0) \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$*;*

ii) $v \in C^1(\mathbb{R}\backslash\{0\})$*;*

iii) *u is C^1 function away from a shock curve $t = \theta(x)$, which has the following properties: θ is defined for $x > 0$ and is monotonically increasing; $\theta(0) = 0$, $\lim_{x\to\infty} \theta(x) = +\infty$;*

iv) *for $x < 0$ and any $t > 0$, u is bounded, uniformly with respect to k. For $x \geq 0$, $t \geq \theta(x)$, u is decreasing in time and nonnegative. For $x > 0$, $0 < t < \theta(x)$, u is nonpositive.*

Notice that, due to the entropy conditions, the function θ satisfies both the Rankine-Hugoniot condition and the Oleinik entropy condition.

Examples of simple detonations which exhibits the above structure are given in [TY1, thm. 3] (see also [Le]); there the function φ is increasing,such that $\varphi(u) \equiv 1$ if $u \geq \beta > 0$.

Let us assume the following:

(H$_1$) the function f is C^2 and convex, with $f(0) = f'(0) = 0$. Moreover, there exist $\delta \in (1/2, 1)$ and $M > 0$ such that

$$\frac{uf'(\delta u)}{f(u)} \geq M + 1 \qquad \text{for all } u > 0\ . \tag{4.1}$$

If $f(u) = Au^\beta$ for $u > 0$ ($A > 0$, $\beta > 1$), assumption (H$_1$) is satisfied with $\delta^{\beta-1} \geq \frac{M+1}{\beta}$;

(H$_2$) the function φ is Lipschitz continuous, satisfies (1.4) and there holds

$$\varphi(u) > 0 \quad \text{for } u > 0 \ .$$

Moreover, there exists $\alpha \geq 0$ such that

$$\varphi(u) = 0(u^\alpha) \quad \text{as } u \to +\infty \ .$$

Among the above assumptions the convexity of f is important, as the following example shows.

Example 4.1 Consider the following semilinear problem ($f(u) \equiv u$ in (1.1)):

$$\begin{cases} \partial_t u + \partial_x u = k\varphi(u)v \ , \\ \partial_t v = -k\varphi(u)v \ , \end{cases}$$

with $u_0(x) = H(-x)$ and $v_0(x) = H(x)$, where H is the usual Heaviside function. It is easily seen that

$$v(x,t) = v_0(x) e^{-k\int_0^t \varphi(u(x,\tau))\mathrm{d}\tau} \ ;$$

in particular,

$$v(x,t) \equiv 1 \qquad \text{for } 0 \leq t \leq x.$$

On the other hand, on the characteristic $x = t$ the function u verifies

$$u(t,t) = k\int_0^t \varphi(u(t,\tau))\,\mathrm{d}\tau + 1 \geq k\varphi(1)t + 1 \ .$$

Hence in general it is impossible to have a uniform bound with respect to k without further assumptions on f. Nevertheless the following holds.

Theorem 4.1 *Let (u^k, v^k) be a simple detonation; let assumptions (H$_1$)-(H$_2$) be satisfied. Then there exists a constant $C > 0$ (only depending on f, φ, $\|u_0\|_\infty$, $\|v_0\|_\infty$) such that*

$$u^k(x,t) \leq C \ ,$$

for all $x > 0$, $t > \theta^k(x)$.

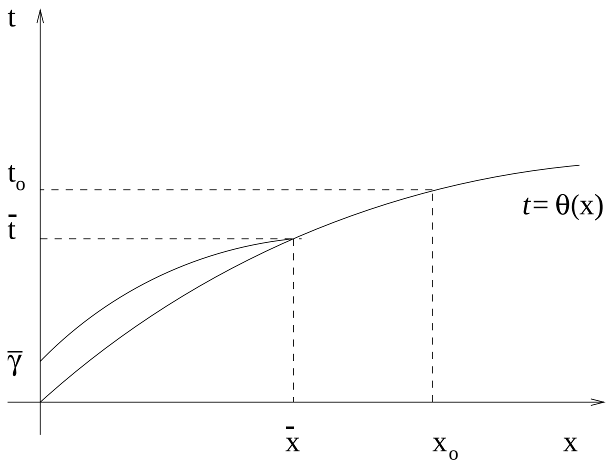

Fig. 1

Proof. By the change of variable $\tilde{x} = \frac{x}{k}$, $\tilde{t} = \frac{t}{k}$ it suffices to consider the case $k = 1$, since all the involved quantities do not depend on the scaling. We can omit the index in the following. Assume $x \geq 0$, $t \geq \theta(x)$ (see Fig.1).

Fix $x_0 > 0$ and $t_0 = \theta(x_0)$; also set

$$\overline{u} = \max_{x \in [0,x_0]} u(x^-, \theta(x)) \ .$$

Let $\overline{x}$ be such that $\overline{u} = u(\overline{x}^-, \theta(\overline{x}))$; set $\overline{t} = \theta(\overline{x})$.

Observe that by Definition 4.1 (iv) we have

$$\overline{u} = \max_{\substack{\theta(x) \leq t \leq t_0 \\ 0 \leq x \leq x_0}} u(x,t) \ .$$

Consider now the backward characteristic $(X(t), U(t))$ issued from the point $(\overline{x}, \overline{t})$. We have

$$\begin{cases} \dot{X} = f'(U) \ , \\ \dot{U} = v_0(X(t))\varphi(U) e^{-\int_{\theta(X(t))}^{t} \varphi(u(X(t),\tau)) \mathrm{d}\tau} \end{cases} , \tag{4.2}$$

with

$$x(\overline{t}) = \overline{x},\ U(\overline{t}) = \overline{u}\ .$$

Let $\overline{\gamma} \in [0, \overline{t}]$ be such that

$$x(\overline{\gamma}) = 0,\ \overline{\alpha} := u(0, \overline{\gamma})$$

and set

$$I(t) := \int_t^{\overline{t}} v_0(x(s))\varphi(u(s))e^{-\int_{\theta(x(s))}^{s} \varphi(u(x(s),\tau))\mathrm{d}\tau}\, \mathrm{d}s\ , \tag{4.3}$$

for all $t \in [\overline{\gamma}, \overline{t}]$. Then we have

$$U(t) + I(t) = \overline{u}\ , \tag{4.4}$$

for all $t \in [\overline{\gamma}, \overline{t}]$. In particular,

$$\overline{\alpha} + I(\overline{\gamma}) = \overline{u}\ . \tag{4.5}$$

If $I(\overline{\gamma}) \leq \frac{1}{2}\overline{u}$, then

$$\overline{u} \leq 2\overline{\alpha}.$$

Otherwise we have $\overline{u} > 2u(0, \overline{\gamma})$. For any fixed $\delta \in \left(\frac{1}{2}, 1\right)$ there exists $t_\delta \in (\overline{\gamma}, \overline{t})$ such that

$$U(t_\delta) = \delta\overline{u}\ ,$$

whence

$$I(t_\delta) = (1 - \delta)\overline{u}\ .$$

Therefore the following estimate holds:

$$\overline{u} \leq \frac{1}{1-\delta}\|v_0\|_\infty \varphi(\overline{u}) \int_{t_\delta}^{\overline{t}} e^{-\int_{\theta(x(s))}^{s} \varphi(u(x(s),\tau))\mathrm{d}\tau}\, \mathrm{d}s\ . \tag{4.6}$$

To estimate the integral on the right-hand side we observe that, since $u_t \leq 0$ for $t \geq \theta(x)$, there holds:

$$\varphi(u(x(s), \tau)) \geq \varphi(U(s)) \geq \varphi(U(t_\delta)) = \varphi(\delta\overline{u})\ .$$

Then

$$-\int_{\theta(x(s))}^{s} \varphi(u(x(s),\tau))\,\mathrm{d}\tau \le -\varphi(\delta\overline{u})(s-\theta(x(s)))\ , \tag{4.7}$$

for $s \in [t_\delta, \overline{t}]$.

From the Mean Value Theorem, there exists $\xi \in (x(s), \overline{x})$ and $\tau \in (s, \overline{t})$ such that

$$\begin{aligned} s-\theta(x(s)) &= s-\overline{t}-\frac{\mathrm{d}\theta}{\mathrm{d}x}(\xi)(x(s)\overline{x}) \\ &= (s-\overline{t})\left(1-\frac{\mathrm{d}\theta}{\mathrm{d}x}(\xi)f'(U(\tau))\right)\ . \end{aligned} \tag{4.8}$$

On the other hand, by the Rankine-Hugoniot condition we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}(\xi) = \frac{u(\xi^-,\theta(\xi))-u(\xi^+,\theta(\xi))}{f(u(\xi^-,\theta(\xi)))-f(u(\xi^+,\theta(\xi)))}\ ,$$

where $u(\xi^-,\theta(\xi))$ and $u(\xi^+,\theta(\xi))$ are the left, respectively the right limit of u on the shock curve. Since θ is monotone increasing, we have $\frac{\mathrm{d}\theta}{\mathrm{d}x} \ge 0$. Moreover, the function f is convex, $f(0)=f'(0)=0$ and

$$u(\xi^+,\theta(\xi)) \le 0 \le u(\xi^-,\theta(\xi)) \le \overline{u}\ .$$

It follows that

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}(\xi) \ge \frac{\overline{u}}{f(\overline{u})}$$

for all $\xi \ge 0$.

On the other hand

$$U(\tau) \ge \delta\overline{u}\ .$$

Combined with (4.8) this gives

$$\begin{aligned} s-\theta(x(s)) &\ge (\overline{t}-s)\left(\frac{\overline{u}}{f(\overline{u})}f'(\delta\overline{u})-1\right) \\ &\ge (\overline{t}-s)M \end{aligned} \tag{4.9}$$

by assumption (H_1). Now the inequality (4.6) yields:

$$\begin{aligned} \overline{u} &\leq \frac{1}{1-\delta}\|v_0\|_\infty \varphi(\overline{u}) \int_{t_\delta}^{\overline{t}} e^{-\varphi(\delta\overline{u})M(\overline{t}-s)}\,\mathrm{d}s \\ &\leq \frac{1}{1-\delta}\|v_0\|_\infty \frac{\varphi(\overline{u})}{\varphi(\delta\overline{u})M}\ . \end{aligned} \tag{4.10}$$

Then by assumption (H_2) there exist $\alpha \geq 0$, $\tilde{C} > 0$ such that

$$\overline{u} \leq \tilde{C}\frac{\delta^{-\alpha}}{1-\delta}\|v_0\|_\infty\ ,$$

where $\tilde{C}$ depends only on φ, f.

The conclusion follows by choosing

$$C \geq \max\left(2\overline{\alpha}, \tilde{C}\frac{\delta^{-\alpha}}{1-\delta}\|v_0\|_\infty\right)\ . \quad \square$$

5 Deflagrations

This section is devoted to a detailed investigation of the structure of deflagration waves. Let (u, v) be a solution of problem (1.1)–(1.3) with initial data

$$(u_0(x), v_0(x)) = \begin{cases} (\alpha_1, 0) & \text{if } x < 0\ , \\ (\alpha_2, \beta_2) & \text{if } x > 0\ . \end{cases} \tag{5.1}$$

As usual φ is a given increasing function, $\varphi(u) \equiv 0$ for $u \leq 0$. The function f satisfies the following:

(H_3) $f(0) = f'(0) = 0$; there exists a constant $\nu > 0$ such that

$$f''(u) \geq \nu > 0 \quad \text{for all } u \geq 0\ .$$

Due to the convexity of f we say that a deflagration wave is generated if

$$0 < \alpha_1 < \alpha_2\ , \qquad \beta_2 > 0\ . \tag{5.2}$$

Deflagration waves are the analogous, in the reactive case, of the rarefaction waves in the nonreactive one.

The main result of this section is given in the following theorem, which provides a global uniform L^∞ bound (independent on k) for the solution (u,v), as required by Theorem 3.1.

Theorem 5.1 *Let the assumption (H_3) hold and the initial data satisfy* (5.1)-(5.2). *Then the problem* (1.1)–(1.3) *has a unique global entropy solution* (u,v) *such that*

$$\alpha_1 \le u(x,t) \le \alpha_2 + \beta_2 \ , \tag{5.3}$$

$$0 \le v(x,t) \le \beta_2 \tag{5.4}$$

for almost every $(x,t) \in \mathbb{R} \times (0,\infty)$.

Also in this case it is enough to consider the case $k = 1$. Observe that, by contrast with Theorem 2.3, no growth assumptions are made here on the function φ. In fact, according to the estimate (5.3), the result is independent from the values of $u > \alpha_2 + \beta_2$. Hence in the following we can assume without loss of generality:

$$\varphi(u) = \varphi(\alpha_2 + \beta_2) \quad \text{for } u \ge \alpha_2 + \beta_2 \ . \tag{$\mathbf{H_4}$}$$

Observe that the main step in the proof of Theorem 5.1 is given by the inequality

$$u \le \alpha_2 + \beta_2 \ .$$

In fact, since

$$v(x,t) = v_0(x) e^{-\int_0^t \varphi(u(x,s))\mathrm{d}s} \ , \tag{5.5}$$

estimates (5.4) are obvious. Moreover, by the comparison properties of the scalar conservation laws (see for instance [HN] and references therein) also the inequality $u \geq \alpha_1$ follows directly from the assumptions. To prove the second inequality in (5.3), we consider the integro-differential problem

$$\partial_t u + \partial_x f(u) = \varphi(u) v_0(x) e^{-\int_0^t \varphi(u(x,s))\mathrm{d}s} \ , \tag{5.6}$$

$$u(x,0) = u_0(x) \ , \tag{5.7}$$

where u_0 and v_0 are given by (5.1). To describe the structure of the solution (u,v), let us introduce, for any $\alpha, \beta \geq 0$, the solution $(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v})(\alpha, \beta)$ of the ordinary differential equation

$$\begin{cases} \dot{\tilde{u}} = \varphi(\tilde{u})\tilde{v} \\ \dot{\tilde{v}} = -\varphi(\tilde{u})\tilde{v} \end{cases} \tag{5.8}$$

with Cauchy data $\tilde{u}(0) = \alpha, \ \tilde{v}(0) = \beta$. It is easily seen that:

(i) $$\tilde{u} + \tilde{v} = \alpha + \beta \ ;$$

(ii) $$0 \leq \tilde{v} \leq \beta \ ;$$

(iii) $$\alpha \leq \tilde{u} \leq \alpha + \beta \ .$$

Set

$$\tilde{x}(t) = \int_0^t f'(\tilde{u}(\tau))\,\mathrm{d}\tau \qquad (t > 0) \ . \tag{5.9}$$

The half-plane Σ is the union of the following regions (see Fig. 2):

$$\begin{aligned} I &:= \{(x,t) \,|\, x \leq 0 \ , \ t > 0\} \ ; \\ II &:= \{(x,t) \,|\, 0 < x \leq \tilde{x}(t) \ , \ t > 0\} \ ; \\ III &:= \{(x,t) \,|\, x > \tilde{x}(t) \ , \ t > 0\} \ . \end{aligned}$$

Since

$$\dot{\tilde{x}} = f'(\tilde{u}) \geq 0$$

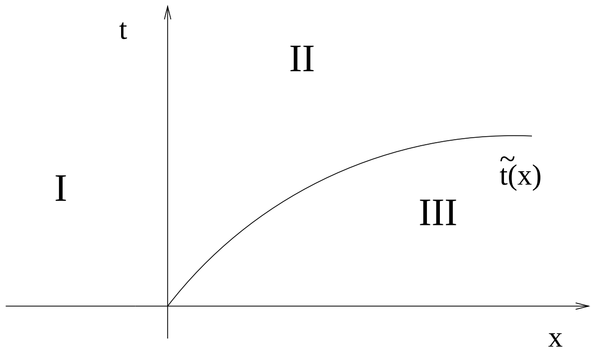

Fig. 2

and

$$\ddot{\tilde{x}} = f''(\tilde{u})\dot{\tilde{u}} = f''(\tilde{u})\varphi(\tilde{u})\tilde{v} \geq 0 \ ,$$

the curve $x = \tilde{x}(t)$ is strictly increasing and convex for $t > 0$. Denote by $\tilde{t}(x)$ its inverse function for $x > 0$. The following statement describes the complete structure of deflagration waves.

Theorem 5.2 *Let the assumption* (H_3) *and* (H_4) *hold. If the initial data satisfy* (5.1)-(5.2) *there exists a unique global solution* (u, v) *to problem* (1.1)–(1.3) *such that* $u \in \mathrm{Lip}_{loc}(\overline{\Sigma} \setminus (0,0))$. *Moreover,*

$$(u, v) = (\alpha_1, 0) \quad \text{in region } I \ ; \tag{5.10}$$

$$(u, v) = (\tilde{u}, \tilde{v})(\alpha_2, \beta_2) \quad \text{in region } III \ ; \tag{5.11}$$

there exists $\overline{M} > 0$ *such that:*

$$o \leq \partial_x u \leq \overline{M} max(\frac{1}{t}, 1) \ , \tag{5.12}$$

$$-\frac{\overline{M}}{t} \leq \partial_t u \leq 0 \quad \text{in region } II \ ; \tag{5.13}$$

The remainder of the paper is devoted to the proof of the above theorem. In fact, it is easily seen that Theorem 5.1 follows directly from the above result.

Proof of Theorem 5.2. To solve the integro-differential problem (5.6)-(5.7), we introduce the space $\mathcal{A}^M(\Sigma)$ as the space of $w \in \mathrm{Lip}_{loc}(\overline{\Sigma} \setminus (0,0))$ such that:

$$\text{(i)} \qquad \alpha_1 \leq w \leq \alpha_2 + \beta_2 \ ;$$

$$\text{(ii)} \qquad w = \alpha_1 \quad \text{in region } I \ ;$$

$$\text{(iii)} \qquad w = \tilde{u}(\alpha_2, \beta_2) \quad \text{in region } III \ ;$$

there exists $M > 0$ such that

$$\text{(iv)} \qquad 0 \leq \partial_x w \leq M max(\frac{1}{t}, 1) \ , \quad \text{in } \Sigma \ ;$$

$$\text{(v)} \qquad -\frac{M}{t} \leq \partial_t w \leq 0 \quad \text{in region } II \ .$$

For any $w \in \mathcal{A}^M(\Sigma)$ we define a map $u = \Lambda(w)$ by considering the (unique) entropy solution of the problem:

$$\partial_t u + \partial_x f(u) = \varphi(u) v_0(x) e^{-\int_0^t \varphi(w(x,s))\mathrm{d}s} \ , \tag{5.14}$$

$$u(x,0) = u_0(x) \ . \tag{5.15}$$

This solution exists by classical results (see Section 2), also considering the assumption (H_4). The first step of the proof is contained in the following statement.

Proposition 5.3 *There exists $M > 0$ such that, for any $w \in \mathcal{A}^M(\Sigma)$, then $\Lambda(w) \in \mathcal{A}^M(\Sigma)$.*

The proof of Proposition 5.3 will be postponed after some preliminary lemmas (Lemma 5.4-5.7). Set

$$v(x,t) = v_0(x) e^{-\int_0^t \varphi(w(x,s))\mathrm{d}s} \ ,$$

where $v_0 \equiv 0$ for $x \leq 0$, $v_0 \equiv \beta_2$ for $x > 0$. Then $u \equiv \alpha_1$ for $x \leq 0$. To deal with the case $x > 0$ we solve the following system for the characteristics of equation (5.14):

$$\begin{cases} \dot{X} = f'(U) \ , \\ \dot{U} = \varphi(U)V \ , \end{cases} \tag{5.16}$$

where

$$X = X(t), \ U = U(t), \ V(t) = v(X(t), t).$$

Since $\dot{X} > 0$, the function X is invertible; denote by $T = T(x)$ its inverse function for $x > 0$. Then the system (5.16) is equivalent to

$$\begin{cases} \dfrac{\mathrm{d}T}{\mathrm{d}x} = \dfrac{1}{f'(U)} \\ \dfrac{\mathrm{d}U}{\mathrm{d}x} = \dfrac{\varphi(U)}{f'(U)} V \ , \end{cases} \tag{5.17}$$

where $U(x) = U(T(x))$ and $V(x) = V(T(x))$.

We supplement (5.17) with two different sets of initial data. The first is

$$(T(0), U(0)) = (\gamma, \alpha_1) \ , \qquad \text{for } \gamma \geq 0 \ , \tag{5.18}$$

which gives the characteristic curve $(T(x,\gamma), U(x,\gamma))$.

The second is

$$(T(0), U(0)) = (0, \alpha) \ , \qquad \text{for } \alpha \in [\alpha_1, \alpha_2] \ , \tag{5.19}$$

which gives the characteristic curve $(T(x,\alpha), U(x,\alpha))$ (see Fig. 3).

Due to the above assumptions (in particular (H_4)) it is easily seen that the solutions of system (5.17) with initial conditions (5.18) or (5.19) are globally defined for $x > 0$. Then we can prove the next result.

Lemma 5.4 *The characteristic curves (T, U), which correspond to different values of the parameter γ or α, do not intersect for all $x > 0$.*

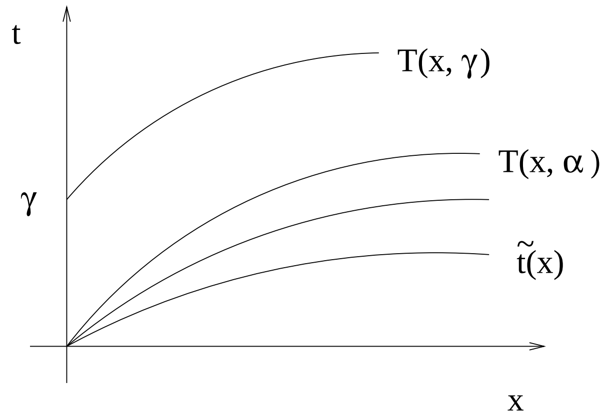

Fig. 3

Proof. First consider (5.17)-(5.18). Let

$$0 \leq \gamma_1 < \gamma_2 \ ;$$

set

$$T_i(x) = T(x, \gamma_i) \ , \ U_i(x) = U(x, \gamma_i) \quad (i = 1, 2) \ ,$$
$$\tilde{T}(x) = T_2(x) - T_1(x) \ , \ \tilde{U}(x) = U_2(x) - U_1(x) \ .$$

We have:

$$\begin{aligned} \frac{\mathrm{d}\tilde{T}}{\mathrm{d}x} &= \frac{1}{f'(U_2)} - \frac{1}{f'(U_1)} \\ &= -\frac{(\int_0^1 f''(\theta U_1 + (1-\theta)U_2)\,\mathrm{d}\theta)\tilde{U}}{f'(U_1)f'(U_2)} \ , \end{aligned} \tag{5.20}$$

$$\begin{aligned} \frac{d^2\tilde{T}}{\mathrm{d}x^2} &= -\frac{f''(U_2)}{(f'(U_2))^2}\frac{\mathrm{d}U_2}{\mathrm{d}x} + \frac{f''(U_1)}{(f'(U_1))^2}\frac{\mathrm{d}U_1}{\mathrm{d}x} \\ &= -\frac{f''(U_2)\varphi(U_2)}{(f'(U_2))^3}V(x, \gamma_2) + \frac{f''(U_1\varphi(U_1)}{(f'(U_1))^3}V(x, \gamma_1) \ . \end{aligned} \tag{5.21}$$

It follows that:

$$\tilde{T}(0) > 0 \ , \qquad \frac{\mathrm{d}\tilde{T}}{\mathrm{d}x}(0) = 0 \ ,$$

$$\frac{\mathrm{d}^2\tilde{T}}{\mathrm{d}x^2}(0) = \frac{f''(\alpha_1)\varphi(\alpha_1)}{(f'(\alpha_1))^3}\,[V(0,\gamma_1) - V(0,\gamma_2)] > 0 \ .$$

Therefore there is a right neighbourhood of $x = 0$ where $\frac{\mathrm{d}\tilde{T}}{\mathrm{d}x} > 0$. If the characteristics intersect, let $x_1 > 0$ such that $\tilde{T}(x) > 0$ for $x \in (0, x_1)$, $\tilde{T}(x_1) = 0$.

Then there is $x_0 \in (0, x_1)$ such that

$$\frac{\mathrm{d}\tilde{T}}{\mathrm{d}x} > 0 \quad \text{in } (0, x_0) \ , \qquad \frac{\mathrm{d}\tilde{T}}{\mathrm{d}x}(x_0) = 0 \ .$$

Hence by (5.20),

$$\tilde{U} < 0 \text{ in } (0, x_0),\ \tilde{U}(x_0) = 0,$$

which gives

$$\frac{\mathrm{d}\tilde{U}}{\mathrm{d}x}(x_0) \geq 0.$$

On the other hand,

$$\frac{\mathrm{d}\tilde{U}}{\mathrm{d}x} = \frac{\varphi(U_2)V(x,\gamma_2)}{f'(U_2)} - \frac{\varphi(U_1)V(x,\gamma_1)}{f'(U_1)} \ . \tag{5.22}$$

Since $\tilde{T}(x_0) > 0$, we have

$$\begin{aligned} &V(x_0,\gamma_2) - V(x_0,\gamma_1) \\ &\quad = \beta_2 \left\{ e^{-\int_0^{T(x,\gamma_2)} \varphi(w(x_0,s))\mathrm{d}s} - e^{-\int_0^{T(x,\gamma_1)} \varphi(w(x_0,s))\mathrm{d}s} \right\} < 0 \ . \end{aligned}$$

It follows that

$$\frac{\mathrm{d}\tilde{U}}{\mathrm{d}x}(x_0) < 0 \ ,$$

hence a contradiction.

(ii) Consider now the problem (5.17)-(5.19). Let

$$\alpha_1 \leq \alpha' < \alpha'' \leq \alpha_2 \ ;$$

define $T_i(x)$, $U_i(x)$ $(i=1,2)$ and $\tilde{T}(x)$, $\tilde{U}(x)$ as in (i). Then there holds:

$$\tilde{T}(0) = 0\ , \qquad \frac{\mathrm{d}\tilde{T}}{\mathrm{d}x}(0) = -\frac{\tilde{U}(0)}{U_1(0)U_2(0)} = \frac{\alpha'\alpha''}{\alpha'\alpha''} < 0\ .$$

It follows that $\tilde{T}(x) < 0$ in a right neighbourhood of $x = 0$. Let $x_1 > 0$ be such that $\tilde{T}(x_1) = 0$. Then there exists $x_0 \in (0, x_1)$ such that

$$\frac{\mathrm{d}\tilde{T}}{\mathrm{d}x} < 0 \text{ in } (0, x_0),\ \ \frac{\mathrm{d}\tilde{T}}{\mathrm{d}x}(x_0) = 0\ .$$

Hence by (5.20)

$$\tilde{U} > 0 \text{ in } (0, x_0),\ \ \tilde{U}(x_0) = 0,$$

which implies

$$\frac{\mathrm{d}\tilde{U}}{\mathrm{d}x}(x_0) \le 0\ .$$

Since $\tilde{T}(x_0) > 0$, it follows hence by (5.22) that

$$\frac{\mathrm{d}\tilde{U}}{\mathrm{d}x}(x_0) > 0\ .$$

This leads again to a contradiction; hence the conclusion follows. □

Lemma 5.5 *The following inequalities hold:*

(i) $\frac{\partial U}{\partial \gamma}(x,\gamma) < 0,\ \frac{\partial T}{\partial \gamma}(x,\gamma) \ge 1\ (x > 0, \gamma \ge 0)$;

(ii) $\frac{\partial U}{\partial \alpha}(x,\alpha) > 0,\ \frac{\partial T}{\partial \alpha}(x,\alpha) < 0\ (x > 0, \alpha \in [\alpha_1, \alpha_2])$.

Proof. (i) By (5.17)-(5.18) we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial T}{\partial \gamma}\right) = -\frac{f''(U)}{(f'(U))^2}\ \frac{\partial U}{\partial \gamma}\ , \tag{5.23}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial U}{\partial \gamma}\right) = {} & \frac{\varphi(U)}{f'(U)}\ \frac{\partial V}{\partial \gamma} \\ & + \frac{\varphi'(U)f'(U) - \varphi(U)f''(U)}{(f'(U))^2}\ V\frac{\partial U}{\partial \gamma}\ ,\end{aligned} \tag{5.24}$$

with

$$\frac{\partial T}{\partial \gamma}(0,\gamma) = 1 \ , \qquad \frac{\partial U}{\partial \gamma}(0,\gamma) = 0 \quad (\gamma > 0) \ .$$

Observe that:

$$\frac{\partial V}{\partial \gamma}(x,\gamma) = -V(x,\gamma)\varphi(U(x,\gamma))\frac{\partial T}{\partial \gamma} \ . \tag{5.25}$$

Since

$$\frac{\partial U}{\partial \gamma}(0,\gamma) = 0, \ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial U}{\partial \gamma}\right)(0,\gamma) < 0$$

we have

$$\frac{\partial U}{\partial \gamma} < 0$$

in a right neighborhood of $x = 0$. Let $x_1 > 0$ be such that

$$\frac{\partial U}{\partial \gamma} < 0 \quad \text{in } (0, x_1) \ , \qquad \frac{\partial U}{\partial \gamma}(x_1,\gamma) = 0 \ . \tag{5.26}$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial U}{\partial \gamma}\right)(x_1,\gamma) \geq 0 \ .$$

On the other hand, in view of (5.18), (5.23), (5.26) we have

$$\frac{\partial T}{\partial \gamma}(x_1,\gamma) \geq 1 \ .$$

Then by (5.24), (5.25),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial U}{\partial \gamma}\right)(x_1,\gamma) < 0 \ ,$$

which leads to a contradiction. Therefore

$$\frac{\partial U}{\partial \gamma}(x,\gamma) < 0$$

and (i) follows. The inequalities in (ii) follow by arguing in the same way; the proof is omitted. □

Lemma 5.6 *The characteristics* $\{T(x,\gamma), \gamma \geq 0\}$ *and* $\{T(x,\alpha), \alpha \in [\alpha_1, \alpha_2]\}$ *cover the entire region II.*

Proof. First observe that

$$T(x,\gamma)|_{\gamma=0} = T(x,\alpha)|_{\alpha=\alpha_1}\ , \quad \text{for all } x \geq 0\ .$$

Therefore, it suffices to show that

$$T(x,\alpha_2) = \tilde{t}(x)\ , \qquad \text{for all } x \geq 0\ .$$

First we observe that $(X(t,\alpha_2), U(t,\alpha_2))$ satisfies the system

$$\begin{cases} \dot{X} = U \\ \dot{U} = \varphi(U)V\ , \end{cases}$$

with $X(0) = 0$, $U(0) = \alpha_2$, for

$$V(t) = \beta_2\, e^{-\int_0^t \varphi(w(X(t),s))\mathrm{d}s}\ . \tag{5.27}$$

On the other hand, setting $X(t) = \tilde{x}(t)$ in (5.27), we find $V(t) = \tilde{v}(t)$. Since the couple $(\tilde{u},\tilde{v}) = (\tilde{u},\tilde{v})(\alpha_2,\beta_2)$ verifies

$$\begin{cases} \dot{\tilde{x}} = \tilde{u}\ , \\ \dot{\tilde{u}} = \varphi(\tilde{u})\tilde{v} = \varphi(\tilde{u})V\ , \end{cases}$$

and $\tilde{x}(0) = 0$, $\tilde{u}(0) = \alpha_2$, by the uniqueness theorem for ordinary differential equations we have $X(t,\alpha_2) = \tilde{x}(t)$ and the conclusion follows. □

Set

$$II_a = \{(x,t) \in II \,|\, t > T(x,\alpha_1)\}\ ;$$
$$II_b = \{(x,t) \in II \,|\, T(x,\alpha_2) < t \leq T(x,\alpha_1)\}\ .$$

For any $(x,t) \in II_a$, there exists a unique value $\gamma = \gamma(x,t) > 0$ such that $t = T(x,\gamma)$. In the same way, for any $(x,t) \in II_b$, there exists a unique $\alpha = \alpha(x,t) \in [\alpha_1, \alpha_2)$ such that $t = T(x,\alpha)$.

Hence we can define in II

$$\lambda(x,t) = \begin{cases} \gamma(x,t) \ , & \text{if } (x,t) \in II_a \ , \\ \alpha(x,t) \ , & \text{if } (x,t) \in II_b \ . \end{cases} \tag{5.28}$$

The function λ is invertible and differentiable. Define a function $u : \Sigma \to [\alpha_1, \infty)$ as follows

$$u(x,t) = \begin{cases} \alpha_1 & \text{in region } I \setminus (0,0) \ , \\ U(x, \lambda, (x,t)) & \text{in region } II \ , \\ \tilde{u}(t) & \text{in region } III \ . \end{cases} \tag{5.29}$$

The next result shows that $u = \Lambda w$.

Lemma 5.7 *The function u given by* (5.29) *is in* $C^1(\overline{\Sigma} \setminus \{(0,0)\})$ *and solves problem* (5.6)-(5.7).

Proof. The only non trivial point to check is that the function u given by (5.29) solves problem (5.6)-(5.7) in the region II. Since

$$T(x, \lambda(x,t)) = t \qquad (x \in (0, \tilde{x}(t))) \ ,$$

it follows that

$$\frac{\partial T}{\partial \lambda} \frac{\partial \lambda}{\partial t} = 1 \ , \qquad \frac{\partial T}{\partial x} + \frac{\partial T}{\partial \lambda} \frac{\partial \lambda}{\partial x} = 0 \ .$$

This implies

$$\frac{\partial \lambda}{\partial t} = \frac{1}{\partial T / \partial \lambda} \ , \qquad \frac{\partial \lambda}{\partial x} = - \frac{1}{\partial T / \partial \lambda} \frac{1}{f'(U)} \ .$$

Hence there holds

$$\partial_t u = \frac{\partial U}{\partial \lambda} \cdot \left(\frac{\partial T}{\partial \lambda} \right)^{-1} \ , \qquad \partial_x u = \frac{\partial U}{\partial x} - \frac{1}{f'(U)} \frac{\partial U}{\partial \lambda} \left(\frac{\partial T}{\partial \lambda} \right)^{-1} \ ,$$

whence

$$\partial_t u + \partial_x(f(u)) = f'(u)\frac{\varphi(u)}{f'(u)}v = \varphi(u)v$$

in region II. This proves the result. □

Proof of Proposition 5.3. Let $u = \Lambda w$ as defined by (5.29). First let us show that

$$\partial_x u \geq 0 \quad \text{in } \Sigma, \ \partial_t u < 0 \quad \text{in region } II\ .$$

In region I or region III, $\partial_x u \equiv 0$. On the other hand,

$$u(x,t) = U(x, \lambda(x,t)) \quad \text{in region } II\ ;$$

hence

$$\partial_t u = \frac{\partial U}{\partial \lambda}\ \frac{\partial \lambda}{\partial t} = \frac{\partial U}{\partial \lambda}\left(\frac{\partial T}{\partial \lambda}\right)^{-1} < 0$$

by Lemma 5.5. Also,

$$f'(u)\partial_x u = -\partial_t u + \varphi(u)v > 0\ .$$

Since $\tilde{u}(t) \leq \alpha_2 + \beta_2$ in region III and $\partial_t u < 0$ in region II, we have

$$u(x,t) \leq \alpha_2 + \beta_2 \qquad \text{in region } II\ .$$

Hence properties (i), (ii), (iii) of the space $\mathcal{A}^M(\Sigma)$ are satisfied. We must check that there exists $M > 0$ such that, if $w \in \mathcal{A}^M(\Sigma)$, (iv) and (v) are verified for u. Set $p = \partial_x u$ and differentiate equation (5.6). We have:

$$\partial_t p + f'(u)\partial_x p + p^2 = \varphi'(u)vp + \varphi(u)\partial_x v\ . \tag{5.30}$$

For $x = 0$, it follows from the equation that

$$p(0,\gamma) = (\partial_x u)(0,\gamma) = \frac{\varphi(\alpha_1)}{\alpha_1}\beta_2 e^{-\varphi(\alpha_1)\gamma}\ . \tag{5.31}$$

Let

$$M_1 = \beta_2 \max\left(\sup_{u\in(\alpha_1,\alpha_2+\beta_2)} \varphi'(u), \frac{\varphi(\alpha_1)}{\alpha_1}\right)\ .$$

Then $\overline{p} := M_1$ is a supersolution for problem (5.30)-(5.31) in region II_a. By standard comparison arguments on the characteristic curves we have

$$0 \leq \partial_x u \leq M_1 \tag{5.32}$$

and

$$\begin{aligned} 0 \leq -\partial_t u &= u\partial_x u - \varphi(u)v \\ &\leq (\alpha_2 + \beta_2)M_1 \quad \text{for all } (x,t) \in II_a \ . \end{aligned} \tag{5.33}$$

To estimate the derivatives of u in region II_b we set

$$P(t,\alpha) = p(X(t,\alpha),t) = (\partial_x u)(X(t,\alpha),t) \ .$$

Then, by using (5.17)-(5.19), this yields

$$P(t,\alpha) = \frac{\varphi(U)}{f'(U)}V - \frac{1}{f'(U)} \frac{\partial U/\partial\alpha}{\partial T/\partial\alpha} \ .$$

Therefore

$$\lim_{t\to 0^+} t \ P(t,\alpha) = - \lim_{t\to 0^+} \frac{1}{f'(U)} \frac{\partial U}{\partial \alpha} \frac{t}{\frac{\partial T}{\partial \alpha}(X(t,\alpha),\alpha)} \ .$$

We have

$$\begin{aligned} &\lim_{t\to 0^+} \frac{1}{f'(U(t))} = \frac{1}{f'(\alpha)} \ ; \\ &\lim_{t\to 0^+} \frac{\partial U}{\partial \alpha} = 1 \ . \end{aligned}$$

Moreover, since $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial T}{\partial \alpha}\right) = -\frac{f''(U)}{(f'(U))^2}\frac{\partial U}{\partial \alpha}$ it holds:

$$\frac{\partial T}{\partial \alpha}(X(t,\alpha),\alpha) = -\int_0^{X(t,\alpha)} \left(\frac{f''(U)}{(f'(U))^2}\frac{\partial U}{\partial \alpha}\right)(y,\alpha)\,\mathrm{d}y \ .$$

Then

$$\lim_{t\to 0^+} \frac{t}{\frac{\partial T}{\partial \alpha}(X(t,\alpha),\alpha)} = -\frac{f'(\alpha)}{f''(\alpha)} \ .$$

In conclusion we have

$$\lim_{t\to 0^+} t\ P(t,\alpha) = \frac{1}{f''(\alpha)}\ . \tag{5.34}$$

Since P is a solution of the ordinary differential equation

$$\dot P = P^2 + \varphi'(U)VP + \varphi(U)v_x(X(t,\alpha),t)\ , \tag{5.35}$$

we look for a supersolution of (5.35)-(5.34) of the form

$$\overline{P} = \frac{M_2}{t}\ .$$

A simple calculation proves that we can choose

$$M_2 = \max\left(\frac{1}{f''(\alpha)}, 1 + \left[\sup_{\alpha_1\le u\le\alpha_2+\beta_2}\varphi'(U)\right]\frac{\beta_2}{\varphi(\alpha_1)}\right)\ .$$

This yields:

$$P(t,\alpha) \le \frac{M_2}{t}\ .$$

Then in region II_b we have

$$0 \le \partial_x u \le \frac{M_2}{t}\ , \tag{5.36}$$

$$0 \le -\partial_t u \le \frac{M_2}{t}(\alpha_2+\beta_2)\ . \tag{5.37}$$

By (5.32), (5.29), (5.36) and (5.37) we can choose $M = \max(M_1, M_2)$, thus the conclusion follows. $\square$

The second step in the proof of Theorem 5.2 is the proof of the following fixed point result.

Proposition 5.8 *Let M be given by Proposition 5.3. Then the map $\Lambda : \mathcal{A}^M(\Sigma) \to \mathcal{A}^M(\Sigma)$ has a fixed point.*

Proof.

Step 1. We shall use the Schauder Theorem in the space

$$\mathcal{A}(Q) = \mathcal{A}^M(\Sigma)|_Q \ ,$$

where $Q = [0, T] \times [-R, R]$ $(T > 0, R > 0)$. Set also

$$B(\Sigma) = t\mathcal{A}^M(\Sigma), \ B(Q) = t\mathcal{A}(Q) \ .$$

First we prove that, for any fixed R, T, the set $B(Q)$ is a convex, compact subset of $L^1(Q)$. $B(Q)$ is obviously convex and moreover, from (i), $B(Q)$ is a bounded subset of $L^1(Q)$.

Observe now that, in view of (iv) and (v), for any $z \in B(Q)$, $\partial_t z, \partial_x z \in L^\infty(Q)$. Next we claim that $B(Q)$ is a closed subset of $L^1(Q)$. In fact, let $\{z_n\} \subset B(Q)$ with

$$z_n \longrightarrow z \ , \qquad \text{in } L^1(Q) \ .$$

Then, for any multi-index $\alpha = (\alpha_1, \alpha_2)$,

$$D^\alpha z_n \rightharpoonup D^\alpha x \qquad \text{in } \mathcal{D}(Q) \ ,$$

where D^α denotes any partial derivative with respect to the (x, t)–variables. Since $D^\alpha z_n$ is in a bounded set of $L^\infty(Q)$ for any α with $|\alpha| = 1$, we conclude that there exist $\xi^\alpha \in L^\infty(Q)$ and a subsequence of $\{z_n\}$, say $\{z_{n_k}\}$, such that

$$D^\alpha z_{n_k} \rightharpoonup \xi^\alpha \qquad \text{in } L^\infty - \text{weak}^* \ .$$

Hence $D^\alpha z = \xi^\alpha \in L^\infty(Q)$ and the claim follows. Finally, the compact embedding of $B(Q)$ in $L^1(Q)$ follows, since $B(Q) \subseteq W^{1,\infty}(Q)$.

Step 2. To define a continuous map from $B(Q)$ in itself, fix $t \leq T$ and $R > \tilde{x}(T)$. For $w_1, w_2 \in \mathcal{A}(Q)$, let $u_1 = \Lambda w_1$ and $u_2 = \Lambda w_2$. The following inequality holds true:

$$\begin{aligned} &\int_{-R}^{R} |u_2(x,t) - u_1(x,t)| \, \mathrm{d}x \\ &\qquad \leq \int_0^t \int_{-R}^{R} |\varphi(u_2)v_2 - \varphi(u_1)v_1| \, \mathrm{d}\tau \, \mathrm{d}x \ . \end{aligned} \tag{5.38}$$

In fact, by the Gauss-Green formula applied to equation (5.6) we have

$$\begin{aligned}\int_{-R}^{R} |u_2(x,t) - u_1(x,t)|\,\mathrm{d}x &= -\int_0^t [f(u_2) - f(u_1)](\tau, R)\,\mathrm{d}\tau \\ &+ \int_0^t [f(u_2) - f(u_1)](\tau, -R)\,\mathrm{d}\tau \\ &+ \int_0^t \int_{-R}^{R} [\varphi(u_2)v_2 - \varphi(u_1)v_1]\,\mathrm{d}x\ \mathrm{d}\tau\ .\end{aligned}$$

The second term on the right-hand side is obviously equal to zero. Also the first term is zero since $R > \tilde{x}(T)$; thus (5.38) follows.

Now we show that there exists a constant $\overline{C} > 0$ (which depends only on T, φ, α_1, α_2, β_2, M) such that

$$\begin{aligned}\int_0^t \int_{-R}^{R} |\varphi(u_1)v_1 - \varphi(u_2)v_2|\,\mathrm{d}x\ \mathrm{d}\tau \\ \le \overline{C} \int_0^t \int_{-R}^{R} [\,|u_1(x,\tau) - u_2(x,\tau)| \\ + |w_1(x,\tau) - w_2(x,\tau)|\,]\,\mathrm{d}x\ \mathrm{d}\tau\ .\end{aligned} \tag{5.39}$$

In fact, since v_i is bounded and φ is Lipschitz continuous, we have that

$$\begin{aligned}&\int_0^t |v_2(x,\tau) - v_1(x,\tau)|\,\mathrm{d}\tau \\ &= \beta_2 \int_0^t |e^{-\int_0^\tau \varphi(w_2(x,s))\mathrm{d}s} - e^{-\int_0^\tau \varphi(w_1(x,s))\mathrm{d}s}|\,\mathrm{d}\tau \\ &\le \overline{C}_1 \int_0^t |w_2(x,\tau) - w_1(x,\tau)|\,\mathrm{d}\tau\ .\end{aligned}$$

Then from the inequality

$$\begin{aligned}|\varphi(u_1)v_1 - \varphi(u_2)v_2| &\le |\varphi(u_2) - \varphi(u_1)|\ |v_2| \\ &+ \varphi(u_1)|v_2 - v_1|\ .\end{aligned}$$

the inequality (5.39) follows.

Hence by Gronwall inequality there exists a constant $C > 0$ (which depends on T, φ, α_1, α_2, β_2, M) such that:

$$\int_{-R}^{R} |u_2(x,t) - u_1(x,t)|\,\mathrm{d}x \tag{5.40}$$

$$\leq C \int_0^t \int_{-R}^R |w_2(x,\tau) - w_1(x,\tau)| \,\mathrm{d}x \;\mathrm{d}\tau \ .$$

Now define a map

$$\mathcal{T} : B(Q) \to B(Q)$$

setting

$$\mathcal{T}(z) = t\Lambda\left(\frac{1}{t}z\right) \qquad \text{for all } z \in B(Q) \ .$$

The following holds.

Lemma 5.9 *The map $\mathcal{T}$ is continuous from $B(Q)$ endowed with the L^1 topology into itself.*

Proof. Set $z_i = tw_i$ $(i = 1, 2)$. Then (5.40) yields

$$\begin{aligned}
\|\mathcal{T}z_1 - \mathcal{T}z_2\|_{L^1(Q)} &= \int_0^T \int_{-R}^R t|u_1(x,\tau) - u_2(x,\tau) \,\mathrm{d}x \;\mathrm{d}\tau \\
&\leq T \int_0^T \int_{-R}^R |u_1(x,\tau) - u_2(x,\tau)| \,\mathrm{d}x \;\mathrm{d}\tau \\
&\leq CT \int_0^T \mathrm{d}t \int_0^t \int_{-R}^R \frac{1}{\tau}|z_1(x,\tau) - z_2(x,\tau)| \,\mathrm{d}x \;\mathrm{d}\tau \\
&\leq C' \int_0^T \int_{-R}^R \frac{1}{\tau}|z_1(x,\tau) - z_1(x,\tau)| \,\mathrm{d}x \;\mathrm{d}\tau \ .
\end{aligned}$$

Fix $\varepsilon > 0$. Since $z_i = tw_i$ and $\alpha_1 \leq w_i \leq \alpha_2 + \beta_2$ $(1 = 1, 2)$, there exists $t_\varepsilon > 0$ such that

$$\int_0^{t_\varepsilon} \int_{-R}^R \frac{1}{\tau}|z_1(x,\tau) - z_2(x,\tau)| \,\mathrm{d}x \;\mathrm{d}\tau \leq \varepsilon \ .$$

On the other hand,

$$\begin{aligned}
&\int_{t_\varepsilon}^T \int_{-R}^R \frac{1}{\tau}|z_1(x,\tau) - z_2(x,\tau)| \,\mathrm{d}x \;\mathrm{d}\tau \\
&\qquad \frac{1}{t_\varepsilon}\|z_1 - z_2\|_{L^1(Q)} \ .
\end{aligned}$$

Therefore, for any $\varepsilon > 0$ there is $\delta > 0$ such that, if $\|z_1 - z_2\|_{L^1(Q)} < \delta$, then $\|\mathcal{T}z_1 - \mathcal{T}z_2\|_{L^1(Q)} < \varepsilon$. Hence the continuity of the map $\mathcal{T}$ is proved. □

To conclude the proof of Theorem 5.2 we only need to apply the Schauder fixed point theorem: for any fixed rectangle Q, there is a fixed point of the map $\mathcal{T}$ in $B(Q)$. Let $\overline{z}$ be this fixed point. Then, the function $\overline{u} = \frac{1}{z}\overline{z}$ is a fixed point of Λ in $B(Q)$ and the conclusion follows. □

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