L^1 Nonlinear Stability of Travelling Waves for a Hyperbolic System with Relaxation

Corrado MASCIA

Dipartimento di Matematica "G. Castelnuovo"
Università degli Studi di Roma "La Sapienza"
P.le Aldo Moro 5, I–00185 Roma, Italy

Roberto NATALINI

Istituto per le Applicazioni del Calcolo "M. Picone"
Consiglio Nazionale delle Ricerche
Viale del Policlinico 137, I–00161 Roma, Italy

Abstract. We investigate the existence and the asymptotic stability of travelling wave solutions for a hyperbolic 2×2 system with a relaxation source term. Using the subcharacteristic stability condition, which implies special monotonicity properties of the solutions, we are able to establish the L^1 asymptotic attractivity of these solutions.

Key words and phrases: Stability of travelling waves, relaxation approximation of conservation laws, subcharacteristic condition.

1. Introduction

In this paper we investigate nonlinear stability of travelling wave solutions to the following (semilinear) hyperbolic 2×2 system with a relaxation source term:

$$(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + \alpha \dfrac{\partial u}{\partial x} = \dfrac{1}{\varepsilon}(f(u) - v), \qquad (\varepsilon > 0) \end{cases}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. The unknowns u, v belong to $\mathbb{R}$, the function $f = f(u)$ is in $C^1(\mathbb{R})$ and $\alpha > 0$ is a constant to be fixed later.

Relaxation systems often arise in many physical situations. Let us recall, for example, gases not in thermodynamic equilibrium [VK], kinetic theory [Ce], [CIP], [PI], chromatography [RAA], river flows, traffic flows and more general waves [Wh]. More recently the study of a special class of hyperbolic systems with relaxation was developed in view of the numerical approximation of discontinuous solutions of conservation laws [JX], [AN].

The 2×2 relaxation hyperbolic systems of conservation laws were first analyzed by Liu [Li2], who justified some nonlinear stability criteria for diffusion waves, expansion waves and travelling waves.

The main stability criterion in [Li2] (see also [Wh]) was the so–called subcharacteristic condition, which for system (1.1) reads:

$$|f'(u)| < \alpha \quad \text{for } |u| \leq M, \tag{1.2}$$

for some fixed $M > 0$.

The rigorous investigation of the relaxation limit, as $\varepsilon \to 0^+$, was started in [CL] and [CLL], where the authors considered some special models, including the system (1.1), by using the methods of compensated compactness [Ta]. They proved that uniformly bounded sequences of solutions to relaxation systems converge almost everywhere, as $\varepsilon \to 0^+$, to some weak equilibrium solutions of the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{1.3}$$

when the subcharacteristic condition (1.2) is verified, for initial data close to the equilibrium $v = f(u)$. The convergence of solutions to the Cauchy problem for system (1.1) to the unique entropy solution of the Cauchy problem for equation (1.3) and for arbitrary initial data was first given in [Na] by using some of the methods we shall use in the present paper (see Section 2).

Existence and nonlinear stability of travelling wave solutions for nonlinear viscous conservation laws were intensively investigated; see for instance [FS], [Go2], [IO], [JGK], [MN1], [OR], [Se], [We] for the scalar case and [Go1], [KM], [MN2], [Li1], [XS] for $N \times N$ systems.

The same problem was also considered for inviscid hyperbolic equations with a source term; see [Ma] for a rather general result in the the scalar case. The case of 2×2 systems with relaxation were considered in [Li2], [CLL], [HL]; these papers present results of L^2 nonlinear stability by using suitable energy methods. In [Ch], the author obtains the L^1 stability in the case of a quasilinear version of system (1.1), but only for small perturbations of constant equilibrium states.

In the following we shall assume $\varepsilon = 1$. This is not restrictive since we can reduce to this case by changing variables. The system (1.1) takes the form:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + \alpha \dfrac{\partial u}{\partial x} = f(u) - v, \end{cases} \tag{1.4}$$

We consider the Cauchy problem for (1.4) given by the initial data

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x) \qquad \forall x \in \mathbb{R}. \tag{1.5}$$

Under some natural conditions (see Section 2), system (1.4) possesses smooth travelling wave solutions, i.e. solutions of the form

$$(u(x,t), v(x,t)) = (U(x - ct), V(x - ct)), \qquad (c \in \mathbb{R}). \tag{1.6}$$

Here we consider heteroclinic waves. These waves connect some constant equilibrium states $(U_\pm, V_\pm) \in \mathbb{R} \times \mathbb{R}$ of system (1.4), which verify $V_\pm = f(U_\pm)$, in the sense that we have the following conditions at $\pm\infty$:

$$\lim_{\xi \to \pm\infty} U(\xi) = U_\pm, \quad \lim_{\xi \to \pm\infty} V(\xi) = f(U_\pm) \equiv V_\pm. \tag{1.7}$$

In order to have existence of travelling waves, the constant states $(U_\pm, V_\pm)$ and the wave speed c must be related by the so–called Rankine–Hugoniot condition

$$c = \frac{f(U_+) - f(U_-)}{U_+ - U_-}$$

and the Oleinik entropy conditions [Ol]

$$(U_+ - U_-)\big(f(s) - f(U_+) - \frac{f(U_+) - f(U_-)}{U_+ - U_-}(s - U_+)\big) < 0 \quad \forall s \in \mathcal{I}(U_+, U_-),$$

where $\mathcal{I} := (a, b) := (\min\{a, b\}, \max\{a, b\})$.

In this paper we obtain a rather general result of L^1 nonlinear stability under very weak assumptions. Our main result is the following.

Theorem 1.1. *Let (U, V) be a travelling wave solution to (1.4) satisfying (1.7). Let $u = u(x, t), v = v(x, t)$ be the unique solution of (1.4), (1.5) with*

$$u_0(x) = U(x) + z(x), \qquad v_0(x) = V(x) + w(x) \qquad \forall x \in \mathbb{R}.$$

Suppose
(i) $z, w \in L^1(\mathbb{R})$;
(ii) for any $x \in \mathbb{R}$

$$f(U_-) - \sqrt{\alpha}U_- \leq (v_0 - \sqrt{\alpha}u_0)(x) \leq f(U_+) - \sqrt{\alpha}U_+,$$
$$-f(U_-) - \sqrt{\alpha}U_- \leq (-v_0 - \sqrt{\alpha}u_0)(x) \leq -f(U_+) - \sqrt{\alpha}U_+.$$

Then

$$\lim_{t \to +\infty} \|u(\cdot, t) - U(\cdot - ct + k)\|_1 + \|v(\cdot, t) - V(\cdot - ct + k)\|_1 = 0$$

where k is given by

$$k = \frac{1}{U_+ - U_-} \int_{\mathbb{R}} (u_0(x) - U(x))\, dx. \tag{1.8}$$

Remarks 1.2. (a) The condition (ii) can be interpreted geometrically as follows: the set defined by $(X,Y) = (u_0(x), v_0(x))$ $(x \in \mathbb{R})$ in the (X,Y)-plane must lay inside the region Q bounded by the four straight lines $(X, \sqrt{\alpha}X + f(U_\pm) - \sqrt{\alpha}U_\pm)$ and $(X, -\sqrt{\alpha}X + f(U_\pm) + \sqrt{\alpha}U_\pm)$. By the convexity of Q, we immediately deduce that the set of initial data satisfying (ii) is convex too.

(b) Having in mind the approximation properties of the system (1.1) for the scalar conservation law (1.3), it is natural to consider the initial data satisfying:

$$U_+ \leq u_0(x) \leq U_-, \qquad f(u_0(x)) = v_0(x).$$

These data represent in (X,Y)-plane sets belonging to the curve $(X, f(X))$, $(U_+ \leq X \leq U_-)$. Since the slopes of the straight lines defining the region Q are greater than $\|f'\|_\infty$, this curve stays in Q for any $X \in [U_+, U_-]$. By (a), the condition (ii) of Theorem 1.1 is satisfied and the convergence to travelling wave holds.

The main tool in the proof of Theorem 1.1 will be the monotonicity and contracting properties of the solutions of system (1.4), under the subcharacteristic condition (1.2) (see [Na] and Section 2). To use these properties we follow the approach of [OR] and [Se].

The plan of the paper is as follows. In Section 2 we recall some background results about system (1.4) and establish the existence of travelling wave solutions. The proof of Theorem 1.1 will be given in Section 3.

2. MAIN PROPERTIES OF THE RELAXATION SYSTEMS AND EXISTENCE OF TRAVELLING WAVES

Let us recall some results obtained in [Na] about the analytical properties of the Cauchy problem for (1.4), which will be of use in the sequel.

Let us consider the following assumptions:

$(\mathcal{H}_1)$ f is a C^1 function;

$(\mathcal{H}_2)$ the functions (u_0, v_0) are bounded in $\left[L^\infty(\mathbb{R})\right]^2$ by $N_0 := \max\left(\|u_0\|_\infty, \|v_0\|_\infty\right)$.

Finally let us denote, for any $N > 0$,

$$B(N) := 2N + \sup_{|\xi| \leq 2N} |f(\xi)|,$$
$$M(N) := \sup_{|\xi| \leq B(N)} |f'(\xi)|.$$

First we state the main global existence and boundedness result for the Cauchy problem (1.4)–(1.5).

Theorem 2.1 ([Na], Theorem 3.1). *Assume $(\mathcal{H}_1)$–$(\mathcal{H}_2)$. For any $N_0 > 0$, if*

$$\sqrt{\alpha} > M(N_0),$$

then there exists a unique globally bounded solution (u,v) to (1.4)–(1.5) in $C([0,\infty);\ L^1_{\text{loc}}(\mathbb{R})^2)$ and we have

$$\|v \pm \sqrt{\alpha}\, u\|_{L^\infty(\mathbb{R}\times(0,\infty))} \leq \sqrt{\alpha}\, B(N_0). \tag{2.1}$$

Moreover there holds the subcharacteristic inequality

$$|f'(u(x,t))| < \sqrt{\alpha}$$

for almost every $(x,t) \in \mathbb{R} \times (0,\infty)$.

To consider the qualitative behaviour of solution to system (1.4), it is useful to introduce the so-called *Riemann invariants.* They are defined as follows

$$r_\pm(x,t) := \pm v(x,t) - \sqrt{\alpha}\, u(x,t). \tag{2.2}$$

The inverse of (2.2) are given by

$$\begin{cases} v(x,t) = \dfrac{r_+(x,t) - r_-(x,t)}{2} \\ u(x,t) = -\dfrac{r_+(x,t) + r_-(x,t)}{2\sqrt{\alpha}}. \end{cases} \tag{2.3}$$

Substituting (2.3) in (1.4) we get the equations for $r_\pm$

$$\begin{cases} \dfrac{\partial r_+}{\partial t} - \sqrt{\alpha}\dfrac{\partial r_+}{\partial x} = -G(r_+, r_-), \\ \dfrac{\partial r_-}{\partial t} + \sqrt{\alpha}\dfrac{\partial r_-}{\partial x} = G(r_+, r_-), \end{cases} \tag{2.4}$$

where

$$G(r_+, r_-) = \frac{r_+ - r_-}{2} - f\Big(-\frac{r_+ + r_-}{2\sqrt{\alpha}}\Big). \tag{2.5}$$

Given any couple $r^1_\pm, r^2_\pm$ of solutions to (2.4), it holds (see [Na], Theorem 4.1)

$$\begin{aligned} &\|r^1_+(\cdot,t) - r^2_+(\cdot,t)\|_1 + \|r^1_-(\cdot,t) - r^2_-(\cdot,t)\|_1 \\ &\qquad \le \|r^1_+(\cdot,0) - r^2_+(\cdot,0)\|_1 + \|r^1_-(\cdot,0) - r^2_-(\cdot,0)\|_1 + \int_0^t \int_{\mathbb{R}} I(x,s)\, dx\, ds, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} I(x,s) := &\Big[\operatorname{sgn}(r^1_+(x,s) - r^2_+(x,s)) - \operatorname{sgn}(r^1_-(x,s) - r^2_-(x,s))\Big] \\ &\times \Big[\Big(\int_0^1 \frac{\partial G}{\partial r_+}(\theta r^1_+(x,s) + (1-\theta) r^2_+(x,s), r^1_-(x,s))\, d\theta\Big)\big(r^1_+(x,s) - r^2_+(x,s)\big) \\ &\quad + \Big(\int_0^1 \frac{\partial G}{\partial r_-}(r^2_+(x,s), \theta r^1_-(x,s) + (1-\theta) r^2_-(x,s))\, d\theta\Big)\big(r^1_-(x,s) - r^2_-(x,s)\big)\Big]. \end{aligned}$$

This estimate shows that the semigroup generated by the system (2.4) is contractive in $\left[L^1(\mathbb{R})\right]^2$ since $I \leq 0$. Formula (2.6) turns to be very useful in considering the stability character of travelling waves.

Finally let us recall the monotonicity property for (2.4) under the assumptions of Theorem 2.1. Let $r^1_\pm, r^2_\pm$ be solutions to (2.4) such that $r^1_\pm(x,0) \leq r^2_\pm(x,0)$ for almost every $x \in \mathbb{R}$, then

$$r^1_\pm(x,t) \leq r^2_\pm(x,t) \qquad \text{a.e. in } \mathbb{R} \times \mathbb{R}_+. \tag{2.7}$$

This property is consequence of the quasimonotonicity of the source term in (2.4) under the subcharacteristic condition (see [Na] and [HN]).

Now we turn our attention to the problem of existence of travelling waves of (1.4).

Theorem 2.2. *Let $U_+ \neq U_-$. Assume that f satisfies*

$$(U_+ - U_-)\big(f(s) - f(U_+) - \frac{f(U_+) - f(U_-)}{U_+ - U_-}(s - U_+)\big) < 0 \quad \forall s \in \mathcal{I}(U_+, U_-), \tag{2.8}$$

where $\mathcal{I}(a,b) := (\min\{a,b\}, \max\{a,b\})$.

Then there exists a unique (up to translations) travelling wave solution of (1.4) satisfying (1.7), with speed c given by

$$c = \frac{f(U_+) - f(U_-)}{U_+ - U_-}. \tag{2.9}$$

Moreover the function U is strictly monotone.

Proof of Theorem 2.2. If u and v are solutions of (1.4) of the form (1.6), we get the following ordinary differential equation for the unknown functions U and V

$$\begin{cases} -cU' + V' = 0 \\ -cV' + \alpha U' = f(U) - V. \end{cases}$$

Then, integrating the first equation,

$$\begin{cases} V = cU + a, \\ (\alpha - c^2)U' = f(U) - cU - a, \end{cases}$$

where $a \in \mathbb{R}$.

By the asymptotic conditions (1.7), we obtain

$$a = f(U_\pm) - cU_\pm \quad \text{and} \quad c = \frac{f(U_+) - f(U_-)}{U_+ - U_-} \equiv \frac{V_+ - V_-}{U_+ - U_-}.$$

Therefore we get a first order equation for U

$$U' = \frac{1}{\alpha - c^2}[f(U) - f(U_\pm) - c(U - U_\pm)] \tag{2.10}$$

Since (2.8) holds, the right hand side of (2.10) is not zero for any value in between U_- and U_+. Hence the profile of U can be given implicitly by

$$\int_{U(\xi_0)}^{U(\xi)} \frac{dU}{f(U) - f(U_\pm) - c(U - U_\pm)} = \frac{\xi - \xi_0}{\alpha - c^2} \qquad (\xi_0 \in \mathbb{R}),$$

and, consequently, V is determined by

$$V(\xi) = cU(\xi) + f(U_\pm) - cU_\pm.$$

Thus the proof is complete. □

3. L^1-Attractivity

In this section we prove the stability result of travelling waves stated in Theorem 1.1. Since we need some regularity result on the $\omega-$limit set of the solution, let us first present the following lemma which will be useful to obtain uniform estimate on the derivatives of solutions.

Lemma 3.1. *Let $(w(x,t), z(x,t))$ be the solution of*

$$\begin{cases} \dfrac{\partial w}{\partial t} + \lambda \dfrac{\partial w}{\partial x} = -A(x,t)w + B(x,t)z, \\ \dfrac{\partial z}{\partial t} + \mu \dfrac{\partial z}{\partial x} = A(x,t)w - B(x,t)z, \end{cases} \tag{3.1}$$

where $\mu < \lambda$, $A(x,t), B(x,t) \in [0, C)$ for some $C > 0$, with initial conditions given by

$$\begin{cases} w(x,0) = w_0(x) \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \\ z(x,0) = z_0(x) \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}). \end{cases} \tag{3.2}$$

Then there exist a positive constant M, independent of t, such that

$$\|w\|_\infty + \|z\|_\infty \leq M(\|w_0\|_\infty + \|z_0\|_\infty + \|w_0\|_1 + \|z_0\|_1). \tag{3.3}$$

Proof of Lemma 3.1. The solution of (3.1), (3.2) satisfies

$$\begin{aligned} w(x,t) = w_0(x - \lambda t) \exp\bigl(-\int_0^t A(x - \lambda(t-\tau), \tau)\, d\tau\bigr) \\ + \int_0^t (Bz)(x - \lambda(t-\theta), \theta) \exp\bigl(-\int_\theta^t A(x - \lambda(t-\tau), \tau)\, d\tau\bigr)\, d\theta, \end{aligned} \tag{3.4}$$

$$\begin{aligned} z(x,t) = z_0(x - \mu t) \exp\bigl(-\int_0^t B(x - \mu(t-\tau), \tau)\, d\tau\bigr) \\ + \int_0^t (Az)(x - \mu(t-\theta), \theta) \exp\bigl(-\int_\theta^t B(x - \mu(t-\tau), \tau)\, d\tau\bigr)\, d\theta. \end{aligned} \tag{3.5}$$

Multiplying the first equation in (3.1) by $\operatorname{sgn} w$ and the second by $\operatorname{sgn} z$, we get

$$\begin{cases} \dfrac{\partial|w|}{\partial t} + \lambda\dfrac{\partial|w|}{\partial x} + A|w| = Bz \operatorname{sgn} w, \\ \dfrac{\partial|z|}{\partial t} + \mu\dfrac{\partial|z|}{\partial x} + B|z| = Aw \operatorname{sgn} z. \end{cases}$$

Adding the two equations

$$\frac{\partial\big(|w|+|z|\big)}{\partial t} + \frac{\partial\big(\lambda|w|+\mu|z|\big)}{\partial x} = (Bz - Aw)(\operatorname{sgn} w - \operatorname{sgn} z) \leq 0. \tag{3.6}$$

Given any $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, let T be the triangle bounded by the three following segments

$$\begin{aligned} \Gamma_0 &:= \{(y,0) : x - \mu t \leq y \leq x - \lambda t\}, \\ \Gamma_1 &:= \{(x - \lambda(t-\theta), \theta) : 0 \leq \theta \leq t\}, \\ \Gamma_2 &:= \{(x - \mu(t-\theta), \theta) : 0 \leq \theta \leq t\}. \end{aligned}$$

Then, by Green formula and (3.6), we obtain

$$\int_{\partial T^+} \big((\lambda|w| + \mu|z|)\,dt - (|w|+|z|)\,dx\big) \leq 0.$$

Computing explicitly the integral on the left hand side gives

$$\int_0^t \big(|z(x-\lambda(t-\theta),\theta)| + |w(x-\mu(t-\theta),\theta)|\big)\,dt \leq \frac{1}{\mu-\lambda}\int_{x-\mu t}^{x-\lambda t} (|z_0(y)| + |w_0(y)|)\,dy.$$

Using this estimate in (3.4) and (3.5), it follows

$$\begin{aligned} |w(x,t)| + |z(x,t)| \leq \big(|w_0(x-\lambda t)| + |z_0(x-\mu t)|\big) \\ + \frac{C}{\mu-\lambda}\int_{x-\mu t}^{x-\lambda t} (|z_0(y)| + |w_0(y)|)\,dy, \end{aligned}$$

hence

$$|w(x,t)| + |z(x,t)| \leq (\|w_0\|_\infty + \|z_0\|_\infty) + \frac{C}{\mu-\lambda}(\|z_0\|_1 + \|w_0\|_1),$$

that gives (3.3). □

Now we turn our attention to the proof of Theorem 1.1. We consider the change of variables $(x,t) \mapsto (x-ct,t)$. Then the system (2.4) takes the form

$$\begin{cases} \dfrac{\partial r_+}{\partial t} - (c+\sqrt{\alpha})\dfrac{\partial r_+}{\partial x} = -G(r_+,r_-), \\ \dfrac{\partial r_-}{\partial t} - (c-\sqrt{\alpha})\dfrac{\partial r_-}{\partial x} = G(r_+,r_-). \end{cases} \tag{3.7}$$

By the existence result given by Theorem 2.2, we have that

$$R_\pm(x) := -(\sqrt{\alpha} \mp c)U(x) \pm [f(U_\pm) - cU_\pm] \tag{3.8}$$

is a stationary solution of (3.7). Moreover

$$R'_\pm(x) = -(\sqrt{\alpha} \mp c)U'(x),$$

thus $R_\pm$ are monotone. In what follows we will assume $U_+ < U_-$, hence $R_\pm$ are increasing functions. The other case can be dealt with similarly.

In order to study the stability of the solutions (3.8), we consider the Cauchy problem for (3.7) determined by the initial data:

$$r_\pm(x,0) = R_\pm(x) + \psi^0_\pm(x) \qquad \forall x \in \mathbb{R}. \tag{3.9}$$

First of all, we restrict our attention to initial data satisfying some additional assumptions. Then we shall extend our result by density to a larger class of data.

We suppose $\psi^0_\pm \in C^2(\mathbb{R})$ and that there exist $\beta, \gamma \in \mathbb{R}$ such that

$$R_\pm(x+\beta) \le R_\pm(x) + \psi^0_\pm(x) \le R_\pm(x+\gamma) \qquad \forall x \in \mathbb{R}. \tag{3.10}$$

Moreover we assume that

$$\psi^0_\pm(x) = 0 \qquad \forall |x| > M, \tag{3.11}$$

for some $M > 0$.

Proposition 3.2. *Let $r_\pm = r_\pm(x,t)$ be the unique solution of (3.7)–(3.9) with $\psi^0_\pm \in C^2(\mathbb{R})$, satisfying (3.10) and (3.11) for some $\beta, \gamma \in \mathbb{R}$ and $M > 0$.*

Then there exists $k \in \mathbb{R}$ such that

$$\lim_{t \to +\infty} \|r_+(\cdot,t) - R_+(\cdot + k)\|_1 + \|r_-(\cdot,t) - R_-(\cdot + k)\|_1 = 0. \tag{3.12}$$

Proof. By the comparison properties of solutions of (3.7), we have

$$R_\pm(x+\beta) \le r_\pm(x,t) \le R_\pm(x+\gamma) \qquad \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{3.13}$$

Let $\psi_\pm(x,t) := r_\pm(x,t) - R_\pm(x)$. We want to show that the family of functions $\left\{ (\psi_+(\cdot,t), \psi_-(\cdot,t)) \right\}_{t>0}$ is relatively compact in $\left[L^1(\mathbb{R})\right]^2$.

As a consequence of (3.13), we obtain

$$R_\pm(x+\beta) - R_\pm(x) \le \psi_\pm(x,t) \le R_\pm(x+\gamma) - R_\pm(x) \qquad \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

thus $\{(\psi_+(\cdot,t),\psi_-(\cdot,t))\}_{t>0}$ is uniformly bounded by a function belonging to $L^1(\mathbb{R})$. In order to have compactness, we need L^1-equicontinuity. We claim that this property holds. Indeed, by (2.6), we get

$$\begin{aligned}(3.14)\quad \|\psi_+(\cdot+h,t)-\psi_+(\cdot,t)\|_1 &+ \|\psi_-(\cdot+h,t)-\psi_-(\cdot,t)\|_1 \\ &\le \|r_+(\cdot+h,t)-r_+(\cdot,t)\|_1+\|R_+(\cdot+h)-R_+(\cdot)\|_1 \\ &+\|r_-(\cdot+h,t)-r_-(\cdot,t)\|_1+\|R_-(\cdot+h)-R_-(\cdot)\|_1 \\ &\le \|r_+(\cdot+h,0)-r_+(\cdot,0)\|_1+\|R_+(\cdot+h)-R_+(\cdot)\|_1 \\ &\qquad +\|r_-(\cdot+h,0)-r_-(\cdot,0)\|_1+\|R_-(\cdot+h)-R_-(\cdot)\|_1,\end{aligned}$$

and, since the estimate in (3.14) is independent of t, the claim is proved.

Hence $\{(\psi_+(\cdot,t),\psi_-(\cdot,t))\}_{t>0}$ is relatively compact in $[L^1(\mathbb{R})]^2$.

Thus, if we call B_s the set of accumulation points of $\{(\psi_+(\cdot,t),\psi_-(\cdot,t)) : t\ge s\}$, then B_s is not empty by compactness. Hence

$$\mathcal{A} := (R_+(x),R_-(x)) + \bigcap_{s\ge 0} B_s \ne \varnothing,$$

since it is the intersection of a decreasing family of not empty compact sets. We call $\mathcal{A}$ the *attractor* of the solution $(r_+(x,t),r_-(x,t))$, because it represents the sets of all the possible L^1-limits as $t\to+\infty$. It is enough to prove that $\mathcal{A}\equiv\{(R_+(x+k),R_-(x+k))\}$ for some $k\in\mathbb{R}$.

Let $(a_+^0(x),a_-^0(x))\in\mathcal{A}$. There exists $\{t_n\}_{n\in}$ such that $t_n\to+\infty$ as $n\to\infty$ and

$$(3.15)\qquad \lim_{n\to\infty}\|r_+(\cdot,t_n)-a_+^0(\cdot)\|_1+\|r_-(\cdot,t_n)-a_-^0(\cdot)\|_1=0$$

Given any $h\in\mathbb{R}$, by the contractivity properties of solution of (3.7), the function

$$(3.16)\qquad t\longmapsto \|r_+(\cdot,t)-R_+(\cdot+h)\|_1+\|r_-(\cdot,t)-R_-(\cdot+h)\|_1$$

is decreasing and thus admits limits $c_h\ge 0$ as $t\to+\infty$.

Considering the above sequence $\{t_n\}$ in (3.16) and letting $n\to\infty$, we obtain, taking in account (3.5),

$$\|a_+^0(\cdot)-R_+(\cdot+h)\|_1+\|a_-^0(\cdot)-R_-(\cdot+h)\|_1=c_h.$$

Let $(a_+(x,t),a_-(x,t))$ be the solution of (3.7) with initial data

$$(3.17)\qquad r_\pm(x,0)=a_\pm^0(x)\qquad \forall x\in\mathbb{R}.$$

Then, since $(a_+(\cdot,t),a_-(\cdot,t))\in\mathcal{A}$ ($\mathcal{A}$ is invariant under the flow defined by (3.7)),

$$\|a_+(\cdot,t)-R_+(\cdot+h)\|_1+\|a_-(\cdot,t)-R_-(\cdot+h)\|_1=c_h\qquad \forall t>0,\ h\in\mathbb{R}.$$

Now we claim that the function $(a_+^0(x), a_-^0(x))$ is Lipschitz continuous. The family $\{(r_+(\cdot,t), r_-(\cdot,t))\}_{t>0}$ is uniformly bounded with respect to t, thus we need some estimates on the derivatives to prove the claim via Ascoli-Arzelá theorem. Set

$$p_\pm := \frac{\partial r_\pm}{\partial x}.$$

Deriving system (3.7) with respect to x, we obtain the following equations for $p_\pm$

$$\frac{\partial p_\pm}{\partial t} - (c \pm \sqrt{\alpha})\frac{\partial p_+}{\partial x} = \mp\Big(\frac{\partial G}{\partial r_+}(r_+, r_-)p_+ + \frac{\partial G}{\partial r_-}(r_+, r_-)p_-\Big), \tag{3.18}$$

with initial data

$$p_\pm(x,0) = R'_\pm(x) + \psi'_\pm(x). \tag{3.19}$$

We want to estimate the solution of (3.18), (3.19).

From (2.5) we derive

$$\pm\frac{\partial G}{\partial r_\pm}(r_+, r_-) = \pm\frac{1}{2}\Big[1 \pm \frac{1}{\sqrt{\alpha}} f'\Big(-\frac{r_+ + r_-}{2\sqrt{\alpha}}\Big)\Big] > 0,$$

by the subcharacteristic condition. Thus Lemma 3.1 applies to (3.18), (3.19) and $\mathcal{A} \subset \big[\mathrm{Lip}(\mathbb{R})\big]^2$ by Ascoli-Arzelá theorem.

Applying formula (2.6) to $a_\pm(x,t)$ and $R_\pm(x+h)$, we obtain

$$\begin{aligned} 0 = \; & \|a_+^0(\cdot) - R_+(\cdot + h)\|_1 + \|a_-^0(\cdot) - R_-(\cdot + h)\|_1 \\ & - \|a_+(\cdot,t) - R_+(\cdot + h)\|_1 - \|a_-(\cdot,t) - R_-(\cdot + h)\|_1 \geq \int_0^t \int_{\mathbb{R}} I_h(x,s)\,dx\,ds, \end{aligned}$$

where

$$\begin{aligned} I_h(x,s) := \; & \big[\operatorname{sgn}(a_+(x,s) - R_+(x+h)) - \operatorname{sgn}(a_-(x,s) - R_-(x+h))\big] \\ & \times \Big[\Big(\int_0^1 \frac{\partial G}{\partial r_+}\,d\theta\Big)(a_+(x,s) - R_+(x+h)) + \Big(\int_0^1 \frac{\partial G}{\partial r_-}\,d\theta\Big)(a_-(x,s) - R_-(x+h))\Big] \geq 0. \end{aligned}$$

Since $a_\pm$ and $R_\pm$ are continuous, the functions I_h are continuous too. Thus

$$I_h(x,t) = 0 \qquad \forall (x,t) \in \mathbb{R}\times\mathbb{R}_+, \quad \forall h \in \mathbb{R}.$$

This implies that for any $(x,t) \in \mathbb{R}\times\mathbb{R}_+$

$$\operatorname{sgn}(a_+(x,t) - R_+(x+h)) = \operatorname{sgn}(a_-(x,t) - R_-(x+h)) \text{ or } a_\pm(x,t) = R_\pm(x+h). \tag{3.20}$$

Following [Se], we set $k_\pm(x,t) := R_\pm^{-1}(a_\pm(x,t)) - x \in C(\mathbb{R}\times\mathbb{R}_+)$, then $k_+ = k_-$ over $\mathbb{R}\times\mathbb{R}_+$. Indeed suppose $k_+(x,t) < k_-(x,t)$ for some $(x,t) \in \mathbb{R}\times\mathbb{R}_+$ and let $k_0 \in (k_+(x,t), k_-(x,t))$. Then

$$\begin{aligned} \operatorname{sgn}(a_\pm(x,t) - R_\pm(x+k_0)) &= \operatorname{sgn}(R_\pm(x + k_\pm(x,t)) - R_\pm(x+k_0)) \\ &= \operatorname{sgn}(k_\pm(x,t) - k_0) = \mp 1, \end{aligned}$$

and (3.20) is violated. Hence there exist $k(x,t) \in C(\mathbb{R}\times\mathbb{R}_+)$, which is Lipschitz continuous with respect to x, such that

$$a_\pm(x,t) = R_\pm(x + k(x,t)). \tag{3.21}$$

To conclude the proof, we apply the following result.

Lemma 3.3. *Let $a_\pm(x,t) = R_\pm(x + k(x,t))$ be the solution of (3.7)–(3.17) for some function $k \in C(\mathbb{R} \times \mathbb{R}_+)$ Lipschitz continuous with respect to x. Then $k(x,t) \equiv$constant.*

Let us postpone the proof of this result.

By Lemma 3.3 there exists $k \in \mathbb{R}$ and $\{t_n\}_{n\in}$ such that $t_n \to +\infty$ as $n \to \infty$ with the property

$$\lim_{n\to\infty} \|r_+(\cdot,t_n) - R_+(\cdot + k)\|_1 + \|r_-(\cdot,t_n) - R_-(\cdot + k)\|_1 = 0.$$

Since the function defined in (3.16) decreases as time increase and converges to zero on a diverging sequence, we conclude that the solution $(r_+(x,t), r_-(x,t))$ of (3.7), (3.9) converges in $L^1(\mathbb{R})$ to $(R_+(x+k), R_-(x+k))$ as $t \to +\infty$. $\square$

Proof of Lemma 3.3. The couple $a_\pm(x,t) = R_\pm(x + k(x,t))$ is a solution of (3.7), thus

$$\begin{aligned} R'_\pm(x + k(x,t))\big[\frac{\partial k}{\partial t}(x,t) - (c \pm \sqrt{\alpha})(1 + \frac{\partial k}{\partial x}(x,t))\big] \\ = \mp G(R_+(x + k(x,t)), R_-(x + k(x,t))) \qquad \text{a.e. in } \mathbb{R} \times \mathbb{R}_+. \end{aligned} \tag{3.22}$$

Moreover $R_\pm$ is a stationary solution of (3.7), hence

$$-(c \pm \sqrt{\alpha})R'_\pm(x) = \pm G(R_+(x), R_-(x)) \qquad \forall x \in \mathbb{R}. \tag{3.23}$$

From (3.22) and (3.23), it follows

$$R'_\pm(x + k(x,t))\big[\frac{\partial k}{\partial t}(x,t) - (c \pm \sqrt{\alpha})\frac{\partial k}{\partial x}(x,t)\big] = 0 \qquad \text{a.e. in } \mathbb{R} \times \mathbb{R}_+.$$

Since $R'_\pm(x) \neq 0$ for any $x \in \mathbb{R}$, we get

$$\frac{\partial k}{\partial t}(x,t) - (c \pm \sqrt{\alpha})\frac{\partial k}{\partial x}(x,t) = 0 \qquad \text{a.e. in } \mathbb{R} \times \mathbb{R}_+.$$

Then $\dfrac{\partial k}{\partial t}(x,t) = \dfrac{\partial k}{\partial x}(x,t) = 0$ a.e. in $\mathbb{R} \times \mathbb{R}_+$, and hence k is constant. $\square$

Proof of Theorem 1.1. We follow the density argument of [Se]. Let $\psi^0_\pm \in L^1(\mathbb{R})$ satisfying

$$R_\pm(x) + \psi^0_\pm(x) \in [R_\pm(-\infty), R_\pm(+\infty)], \qquad \forall x \in \mathbb{R}. \tag{3.24}$$

Let $\psi^{0,n}_\pm \in L^1(\mathbb{R})$ ($n \in \mathbb{N}$) be a sequence of functions, satisfying the hypothesis of Proposition 3.2, such that

$$\lim_{n\to\infty} \|\psi^0_+ - \psi^{0,n}_+\|_1 + \|\psi^0_- - \psi^{0,n}_-\|_1 = 0.$$

Let $r_\pm^n = r_\pm^n(x,t)$ be the solution of the Cauchy problem (3.7), $r_\pm(x,0) = R_\pm(x) + \psi_\pm^{0,n}(x)$ for any $x \in \mathbb{R}$. Then, by Proposition 3.2, there exists $\{k_n \in \mathbb{R}\}_{n\in}$ such that

$$\lim_{t\to+\infty} \|r_+^n(\cdot,t) - R_+(\cdot + k_n)\|_1 + \|r_-^n(\cdot,t) - R_-(\cdot + k_n)\|_1 = 0.$$

By contractivity properties of solutions of (3.7), we deduce that $\{(r_+^n(\cdot,t), r_-^n(\cdot,t))\}$ is a Cauchy sequence for any $t > 0$. Hence, for $t \to +\infty$, we obtain that $\{(R_+(\cdot + k_n), R_-(\cdot + k_n))\}$ is a Cauchy sequence too. Since

$$\begin{aligned}\int_{\mathbb{R}} \big(|R_+(x+k_n) - R_+(x+k_m)| + |R_-(x+k_n) - R_-(x+k_m)|\big)\, dx \\ = \big((R_+ + R_-)(+\infty) - (R_+ + R_-)(-\infty)\big)\big)\, |k_n - k_m|,\end{aligned}$$

for any $\eta > 0$ there is n_0 such that for any $n, m > n_0$, it holds $|k_n - k_m| < \eta$. Therefore there exists $k \in \mathbb{R}$ such that

$$\lim_{n\to\infty} k_n = k,$$
$$\lim_{n\to\infty} \|R_+(\cdot + k_n) - R_+(\cdot + k)\|_1 + \|R_-(\cdot + k_n) - R_-(\cdot + k)\|_1 = 0.$$

Finally

$$\begin{aligned}\|r_+(\cdot,t) - R_+(\cdot + k)\|_1 + \|r_-(\cdot,t) - R_-(\cdot + k)\|_1 \le \|r_+(\cdot,t) - r_+^n(\cdot,t)\|_1 \\ + \|r_-(\cdot,t) - r_-^n(\cdot,t)\|_1 + \|r_+^n(\cdot,t) - R_+(\cdot + k_n)\|_1 + \|r_-^n(\cdot,t) - R_-(\cdot + k_n)\|_1 \\ + \|R_+(\cdot + k_n) - R_+(\cdot + k)\|_1 + \|R_-(\cdot + k_n) - R_-(\cdot + k)\|_1.\end{aligned}$$

Hence it follows that for any $\psi_\pm^0 \in L^1(\mathbb{R})$, such that (3.24) is satisfied, there exists $k \in \mathbb{R}$ such that (3.12) holds.

Now we can derive Theorem 3.1, coming back to the original unknown functions (u, v).

Let z, w satisfying the assumptions of Theorem 3.1. Then $\psi_\pm^0(x) := \pm w(x) - \sqrt{\alpha} z(x) \in L^1(\mathbb{R})$ and satisfy (3.24).

Then there exists $k \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} \int_{\mathbb{R}} |r_+(y,t) - R_+(y+k)|\, dy + \int_{\mathbb{R}} |r_-(y,t) - R_-(y+k)|\, dy = 0,$$

where $y = x - ct$. Coming back to variables (x,t), we get:

$$\lim_{t\to+\infty} \|r_+(\cdot,t) - R_+(\cdot - ct + k)\|_1 + \|r_-(\cdot,t) - R_-(\cdot - ct + k)\|_1 = 0.$$

From (3.8), we obtain

$$\begin{aligned}\|u(\cdot,t) - U(x - ct + k)\|_1 + \|v(\cdot,t) - V(x - ct + k)\|_1 \\ \le \frac{1}{2}\left(\frac{1}{\sqrt{\alpha}} + 1\right)\big(\|r_+(\cdot,t) - R_+(\cdot - ct + k)\|_1 + \|r_-(\cdot,t) - R_-(\cdot - ct + k)\|_1\big),\end{aligned}$$

and thus the time-asymptotic convergence follows.

Finally we determine the precise value of the shift k. By the first of (1.4), it follows

$$\int_{\mathbb{R}} (u(x,t) - U(x-ct))\,dx = \text{constant}.$$

Hence

$$\int_{\mathbb{R}} (u_0(x) - U(x))\,dx = \int_{\mathbb{R}} (U(x-ct+k) - U(x-ct))\,dx$$
$$= \int_0^k \int_{\mathbb{R}} U'(\xi + x - ct)\,dx\,d\xi = (U_+ - U_-)k,$$

and (1.8) holds. $\square$

Aknowledgements. The authors thank Denis Serre for some useful discussion, which partially motivated this research. We are also grateful to him for the present proof of Lemma 3.3 which simplify a previous longer one.

References

[AN] D. Aregba-Driollet and R. Natalini, *Convergence of relaxation schemes for conservation laws*, to appear in Applicable Anal..

[Ce] C. Cercignani, *The Boltzmann equation and its applications*, Springer–Verlag, New York, 1988.

[Ch] I.-L. Chern, *Long-time effect of relaxation for hyperbolic conservation laws*, Comm. Math. Phys. **172** (1995), 39–55.

[CIP] C. Cercignani, R. Illner and M. Pulvirenti, *The mathematical theory of dilute gases*, Springer–Verlag, New York, 1994.

[CL] G.-Q. Chen and T.-P. Liu, *Zero relaxation and dissipation limits for hyperbolic conservation laws*, Comm. Pure Appl. Math. **46** (1993), 755–781.

[CLL] G.-Q. Chen, C. Levermore and T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math. **47** (1994), 787–830.

[FS] H. Freistühler and D. Serre, *L^1 stability of shock waves in scalar viscous conservation laws*, in preparation.

[Go1] J. Goodman, *Nonlinear Asymptotic stability of viscous shock profiles for conservation laws*, Arch. Rational Mech. Anal. **95** (1986), 325–344.

[Go2] J. Goodman, *Stability of viscous scalar shocks fronts in several dimensions*, Trans. Amer. Math. Soc. **311** (1989), 683–695.

[HL] L. Hsiao and T. Luo, *The stability of travelling wave solutions for a rate-type viscoelastic system*, preprint (1995).

[HN] B. Hanouzet and R. Natalini, *Weakly coupled systems of quasilinear hyperbolic equations*, to appear in Differential Integral Equat..

[IO] A. M. Il'in and O. A. Oleĭnik, *Behaviour of the solutions of the Cauchy problem for certain quasilinear equations for unbounded increase of time*, AMS Translations **42** (1964), 19–23.

[JGK] C. K. R. T. Jones, R. Gardner and T. Kapitula, *Stability of travelling waves for nonconvex scalar viscous conservation laws*, Comm. Pure Appl. Math. **46** (1993), 505–526.

[JX] S. Jin and Z. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math. **48** (1995), 235–277.

[KM] S. Kawashima and A. Matsumura, *Asymptotic stability of travelling wave solutions of systems for one-dimensional gas motion*, Comm. Math. Phys. **101** (1985), 97–127.

[Li1] T.-P. Liu, *Nonlinear stability of shock waves for viscous conservation laws*, Memoirs of Amer. Math. Soc. **328** (1986).

[Li2] T.-P. Liu, *Hyperbolic conservation laws with relaxation*, Comm. Math. Phys. **108** (1987), 153–175.

[Ma] C. Mascia, *Travelling wave solutions for a balance law*, to appear in Proc. Roy. Soc. Edinburgh Sect. A.

[MN1] A. Matsumura and K. Nishiara, *Asymptotic stability of travelling waves for scalar viscous conservation laws with non-convex nonlinearity*, Comm. Math. Phys. **165** (1994), 83–96.

[MN2] A. Matsumura and K. Nishiara, *On a stability of travelling wave solution for a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math. **3** (1986), 1–13.

[Na] R. Natalini, *Convergence to equilibrium for the relaxation approximations of conservation laws*, to appear in Comm. Pure Appl. Math..

[Ol] O. A. Oleinik, *Uniqueness and stability of the generalized solution of the Cauchy problem for a quasilinear equation*, AMS Translations Ser. 2 **33** (1964), 285–290.

[OR] S. Osher and J. Ralston, *L^1-stability of travelling waves with application to convective porous media flow*, Comm. Pure Appl. Math. **35** (1982), 737–749.

[PI] T. Platkowski and R. Illner, *Discrete velocity models of the Boltzmann equation: a survey on the mathematical aspects of the theory*, SIAM Review **30** (1988), 213–255.

[RAA] H. K. Rhee, R. Aris and N. R. Amundsen, *First-order partial differential equations, Theory and Applications of Quasi Linear Equations*, vol. I (1986), vol. II (1989), Prentice Hall International Series.

[Se] D. Serre, *Stabilité des ondes de choc de viscosité qui peuvent être caractéristiques*, preprint (1995).

[Ta] L. Tartar, *Compensated compactness and applications to partial differential equations*, Nonlinear analysis and mechanics: Heriot-Watt symposium (R. J. Knops, eds.), Research notes in mathematics, vol. 4, Pitman Press, New York, 1979, pp. 136–212.

[VK] W. G. Vincenti and C. K. Kruger, *Introduction to physical gas dynamics*, Wiley, New York, 1965.

[We] H. F. Weinberger, *Long-time behavior for regularized scalar conservation law in the absence of genuine nonlinearity*, Ann. Inst. Henri Poincaré **7** (1990), 407–425.

[Wh] J. Whitham, *Linear and nonlinear waves*, Wiley, New York, 1974.

[XS] A. Szepessy and Z. Xin, *Nonlinear stability of viscous shock waves*, Arch. Rational Mech. Anal. **122** (1993), 53–103.