A Discrete Kinetic Approximation of Entropy Solutions to Multidimensional Scalar Conservation Laws

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Abstract

We present a new relaxation approximation to scalar conservation laws in several space variables by means of semilinear hyperbolic systems of equations with a finite number of velocities. Under a suitable multidimensional generalization of the Whitham relaxation subcharacteristic condition, we show the convergence of the approximated solutions to the unique entropy solution of the equilibrium Cauchy problem.

1 Introduction

Consider a multidimensional conservation law. Let $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ be a global weak solution to the Cauchy problem

$$(1.1) \qquad \partial_t u + \sum_{j=1}^{d} \partial_{x_j} A_j(u) = 0 \ , \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+ \ ,$$

$$(1.2) \qquad u(x,0) = u_0(x) \ , \quad x \in \mathbb{R}^d \ ,$$

where $\mathbf{A} = (A_1, \ldots, A_d) \in (\mathrm{Lip}_{loc}(\mathbb{R}))^d$ and $u_0 \in L^\infty(\mathbb{R}^d)$. In this paper we shall restrict our consideration to the entropy solutions of (1.1)–(1.2) in the sense of Kružkov, which are known to be unique and having good mathematical properties, see [Kr] and Section 3 below. Consider also the following class of discrete kinetic systems. Let $N \geq d+1$ be fixed and let $\mathbf{f}^\varepsilon = (f_1^\varepsilon, \ldots, f_N^\varepsilon) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^N$ be a solution of the Cauchy problem:

$$(1.3) \qquad \partial_t f_i^\varepsilon + \sum_{j=1}^{d} \lambda_{ij} \partial_{x_j} f_i^\varepsilon = \frac{1}{\varepsilon} \left(M_i(u^\varepsilon) - f_i^\varepsilon \right) \ , \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+, \ i = 1, \ldots, N,$$

$$(1.4) \qquad f_i^\varepsilon(x,0) = f_{0i}(x) \ , \quad x \in \mathbb{R}^d \ , \quad i = 1, \ldots, N \ .$$

Here $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, $\lambda_{ij} \in \mathbb{R}$ $(i = 1, \ldots, N; j = 1, \ldots, d)$, the function u^ε is defined by

$$(1.5) \qquad u^\varepsilon := \sum_{i=1}^{N} f_i^\varepsilon \ ,$$

$\mathbf{M} = (M_1, \ldots, M_N) : \mathbb{R} \to \mathbb{R}^N$ is a Lipschitz continuous function with $\mathbf{M}(0) = 0$ and $\mathbf{f}_0 = (f_{01}, \ldots, f_{0N}) \in (L^\infty(\mathbb{R}^d))^N$.

In this paper we want to show that the sequence $\{u^\varepsilon\}$ converges, as $\varepsilon \downarrow 0$, to the (unique) entropy solution u of problem (1.1)–(1.2). Hence the function $\mathbf{M}$ and the velocities $\{\lambda_{ij}\}$ must satisfy some compatibility conditions with respect to the flux function $\mathbf{A}$. More precisely we shall assume that $\mathbf{M}$ is a (local) monotone maxwellian function for system (1.3) according to the following definition.

Definition 1.1 *Let $I \subseteq \mathbb{R}$ be a fixed interval. A Lipschitz continuous function $\mathbf{M} = \mathbf{M}(u) : I \to \mathbb{R}^N$ is a Monotone Maxwellian Function* (**MMF**) *respect to the interval I if the following conditions are verified:*

$$\sum_{i=1}^{N} M_i(u) = u \ , \quad \text{for any } u \in I \ ; \tag{1.6}$$

$$\sum_{i=1}^{N} \lambda_{ij} M_i(u) = A_j(u) \ , \ j = 1, \ldots, d \ , \quad \text{for any } u \in I \ ; \tag{1.7}$$

(1.8) $\quad M_i$ is a monotone (increasing) function on I, for any $i = 1, \ldots, N$.

If the function $\mathbf{M}$ verifies (1.6)–(1.7), the system (1.3) forms a standard BGK approximation for equation (1.1). Actually, it easy to see that if

$$\mathbf{f} = \lim_{\varepsilon \to 0} \mathbf{f}^{\varepsilon}$$

in a suitable (strong) topology and

$$u(x,t) = \sum_{i=1}^{N} f_i(x,t) = \lim_{\varepsilon \to 0} u^{\varepsilon} \ ,$$

we have that $\mathbf{f} = \mathbf{M}(u)$ and u is a weak solution of (1.1).

The plan of the paper is as follows: in Section 2 we prove that, if $\mathbf{M}$ is a **MMF**, then problem (1.3)–(1.4) has a unique uniformly bounded solution for any $\varepsilon > 0$. The monotonicity condition (1.8) will be our key argument. In fact it implies special comparison and stability properties of system (1.3) and in particular the evolution operator associated to problem (1.3)–(1.4) is contractive in the L^1 norm and the system is quasimonotone, in the spirit of [HN] and [Na]. Section 3 is devoted to the proof of the convergence of the sequence $\{u^{\varepsilon}\}$ to the entropy solution u of problem (1.1)–(1.2). Finally, in Section 4, we present various examples of (local) monotone maxwellian functions $\mathbf{M} = \mathbf{M}(u)$ related to the conservation law (1.1).

Let us first recall that the general theory of discrete Boltzmann equations is far to be complete, see [PI], [BG]. In one space dimension global existence of solutions to systems with particular quadratic interaction terms and satisfying a H-Theorem has been established in [Ta] for arbitrarily large data. In the multidimensional case global existence is known only for small initial data [Bo].

Recall also that the kinetic approximation of fluid dynamic equations is a classical problem in mathematical physics. In particular Euler equations can be formally obtained as the fluid dynamical limit of Boltzmann equations, see [Ce], [CIP] and references therein. Actually this limit has been rigorously established only as long as the limit solutions are regular.

The limiting behavior of discrete velocities models has been also studied by many people, see for instance again the review paper [PI]. In particular we mention the studies on the fluid dynamical limit for the Broadwell model [Ca], [Xi]. Let us also point out that a lot of computational work has been done recently in the very closed framework of lattice Boltzmann and lattice BGK models, see [QSO] and references therein.

The rigorous theory of kinetic approximations for solutions with shocks is more recent and mainly developed when the limit equation is scalar. The first result of convergence of a fractional step continuous Boltzmann approximation, with entropy conditions for the limit (weak) solution, was proved in [Br1] (see also [GM]). Another convergence result was given later in [PT], using a continuous velocities BGK model. A related kinetic formulation can be found in [LPT1]. Let us also recall that some results have been established for special systems or partially kinetic approximations [LPT2], [JPP], [BC], [LPS].

Let us also observe that there is a very strict connection between our kinetic approximation and the relaxation approximation of conservation laws proposed in [JX], at least in the one dimensional case. Consider in fact the case with two velocities ($d = 1, N = 2$). Fix $\lambda_2 = -\lambda_1 = \lambda > 0$. Then the (unique) maxwellian function is given by

$$M_1(u) = \frac{1}{2}\left(u - \frac{A(u)}{\lambda}\right) \ , \quad M_2 = \frac{1}{2}\left(u + \frac{A(u)}{\lambda}\right)$$

and the condition (1.8) is equivalent to the subcharacteristic condition by [Wh] and [Li]:

$$|A'(u)| \le \lambda \quad \text{for any } u \in I \ ,$$

for a suitable interval $I \subseteq \mathbb{R}$. Moreover, setting $u^\varepsilon = f_1^\varepsilon + f_2^\varepsilon$ and $v^\varepsilon = \lambda(f_2^\varepsilon - f_1^\varepsilon)$, the system (1.3) has the following relaxation formulation

$$\text{(1.9)} \qquad \begin{cases} \partial_t u^\varepsilon + \partial_x v^\varepsilon = 0 \\ \partial_t v^\varepsilon + \lambda^2 \partial_x u^\varepsilon = \frac{1}{\varepsilon}(A(u^\varepsilon) - v^\varepsilon) \quad (\varepsilon > 0) \ . \end{cases}$$

We recall that uniform bounds, stability and convergence to the unique entropy solution of the related one dimensional conservation law for this approximation were first given in [Na]. Let us also mention that the L^1 stability of travelling wave solutions to problem (1.9) has been investigated in [MN].

Here we extend the monotonicity arguments of [Na] to cover the multidimensional case. Observe that the approximation proposed in [JX] in the multidimensional case does not enter in our framework (see Remark 4.5 below) and its stability and convergence remain to be proved. On the other hand we present here an example (see Section 4) of a multidimensional discrete velocities approximation, stable and convergent, which possesses a slightly different relaxation formulation.

Finally, let us mention the monotone schemes of [Br2], which are a numerical (relaxed, i.e.: $\varepsilon = 0$) counterpart of our construction. Other investigations on second order relaxation and kinetic schemes, with the same monotone properties of the continuous model (1.3), as well as some numerical applications to systems in several space variables could be found in [AN1,2].

2 Global existence and uniform estimates

In this section we shall prove that, if $\mathbf{M}$ is a **MMF** respect to a suitable interval I, which only depends on the L^∞ norm of the initial data, the Cauchy problem (1.3)–(1.4) possesses a globally bounded solution. Let us observe that in this section our arguments are completely independent from condition (1.7). In fact, for any fixed function $\mathbf{M}$, which satisfies (1.6) and (1.8), we can construct a flux-function $\mathbf{A} = (A_1, \ldots, A_N)$ such that (1.7) holds, just by setting $A_j(u) = \sum_{i=1}^N \lambda_{ij} M_i(u)$ $(j = 1, \ldots, d)$. Our main results are the following.

Theorem 2.1 *Let $u_0 \in L^\infty(\mathbb{R}^d)$ be fixed and set $\mathbf{f}_0 = \mathbf{M}(u_0)$. Assume that $\mathbf{M}$ is a* **MMF** *on the interval $I := \{u \in \mathbb{R} \mid |u| \leq N\|u_0\|_\infty\}$. Then, for any $\varepsilon > 0$, there exists a (unique) global solution $\mathbf{f}^\varepsilon \in C([0,\infty); (L^1_{loc}(\mathbb{R}^d))^N) \cap (L^\infty(\mathbb{R}^d \times \mathbb{R}_+))^N$ to the Cauchy problem (1.3)–(1.4). Moreover the following estimates hold:*

$$(2.1) \qquad \|f_i^\varepsilon\|_\infty \leq N\|u_0\|_\infty \qquad i = 1, \ldots, N \ ;$$

$$(2.2) \qquad \|u^\varepsilon\|_\infty \leq N\|u_0\|_\infty \ .$$

Set $\lambda_i = (\lambda_{ij}, \ldots, \lambda_{id}) \in \mathbb{R}^d$, $i = 1, \ldots, N$. For any open set Ω and $t > 0$, let us denote

$$\Omega_t^\lambda(s) := \{y \in \mathbb{R}^d \mid \exists\, x \in \Omega\ ,\ \exists\, j \in \{1, \ldots, N\} \text{ s.t. } : y = x + (t-s)\lambda_j\}$$

for any $0 \leq s \leq t$. Clearly, $\Omega = \Omega_t^\lambda(0)$.

Theorem 2.2 *Under the assumptions of Theorem 2.1, let $\tilde{\mathbf{f}}^\varepsilon$ be another global solution with the same properties, associated to the initial condition $\tilde{\mathbf{f}}_0$. Then*

(a) *if* $\mathbf{f}_0(x) \le \tilde{\mathbf{f}}_0(x)$ *for almost every* $x \in \mathbb{R}^d$, *then* $\mathbf{f}^\varepsilon(x,t) \le \tilde{\mathbf{f}}^\varepsilon(x,t)$ *for almost every* $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$;

(b) *for any* $\Omega \subseteq \mathbb{R}^d$ *and* $0 \le s \le t$

$$\sum_{i=1}^{N} \int_\Omega |f_i^\varepsilon(x,t) - \tilde{f}_i^\varepsilon(x,t)|\,\mathrm{d}x \le \sum_{i=1}^{N} \int_{\Omega_t^\lambda(s)} |f_i^\varepsilon(x,s) - \tilde{f}_i^\varepsilon(x,s)|\,\mathrm{d}x \ . \tag{2.3}$$

Let us observe that it is easy to extend the present results, as well as the results of the next section, to the case of initial data not in equilibrium, i.e.: $\mathbf{f}_0 \ne \mathbf{M}(u_0)$, just by arguing as in [Na].

To prove Theorems 2.1 and 2.2 we need to recall some general results concerning semilinear (diagonal) hyperbolic systems. The proofs can be found in [HN] (in the general quasilinear weakly coupled case). Consider the Cauchy problem

$$\partial_t f_i + \sum_{j=1}^{N} \lambda_{ij} \partial_{x_j} f_i = g_i(\mathbf{f}) \ , \quad (x,t) \in R^d \times \mathbb{R}_+ \ , \quad i = 1, \dots, N \ , \tag{2.4}$$

$$f_i(x,0) = f_{0i}(x) \ , \quad \lambda_{ij} \in \mathbb{R}^d, \ i = 1, \dots, N \ , \tag{2.5}$$

where $\mathbf{f} := (f_1, \dots, f_N) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$, $\lambda_{ij} \in \mathbb{R}$ $(i = 1, \dots, N, \ j = 1, \dots, d)$ and $\mathbf{g}(\mathbf{f}) = (g_1(\mathbf{f}), \dots, g_N(\mathbf{f}))$ is a given Lipschitz continuous function; $\mathbf{f}_0 = (f_{01}, \dots, f_{0N}) \in (L^\infty(\mathbb{R}^d))^N$.

Definition 2.3 *A function* $\mathbf{f} \in (L^\infty(\mathbb{R}^d \times (0,T)))^N$ $(T > 0)$ *is a (weak) solution of the Cauchy problem* (2.4)–(2.5) *if, for all* $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,T))$ *and every* $i = 1, \dots, N$, *it holds*

$$\iint \left[f_i \left(\partial_t \varphi + \sum_{j=1}^{N} \lambda_{ij} \partial_{x_j} \varphi \right) + g_i(\mathbf{f}) \varphi \right] \mathrm{d}x\,\mathrm{d}t = 0 \tag{2.6}$$

and, for any open set $\Omega \subseteq \mathbb{R}^d$,

$$\lim_{T \to 0+} \frac{1}{T} \int_0^T \int_\Omega |f_i(x,t) - f_{0i}(x)|\,\mathrm{d}x\,\mathrm{d}t = 0 \ . \tag{2.7}$$

Proposition 2.4 *For any* $\mathbf{f}_0 \in (L^\infty(\mathbb{R}^d))^N$*, there is* $T > 0$ *(only depending on* $\|\mathbf{f}_0\|_\infty$*) such that there exists a unique (weak) solution* $\mathbf{f}$ *of* (2.4)–(2.5) *in* $\mathbb{R}^d \times (0,T)$ *and* $\mathbf{f} \in C([0,T);(L^1_{loc}(\mathbb{R}^d))^N)$*. Moreover, there are only two possibilities: either* $\mathbf{f} \in (L^\infty(\mathbb{R}^d \times (0,T)))^N$ *for any* $T > 0$*, or there exists* $T^* < +\infty$ *such that, for any* $T < T^*$*,* $\mathbf{f}$ *is defined on* $\mathbb{R}^d \times (0,T)$ *and*

$$\lim_{T \to T^*_-} \|\mathbf{f}\|_{L^\infty(\mathbb{R}^d \times (0,T))} = +\infty \ .$$

One of the main tools in this paper are the monotonicity properties of some special systems. These properties hold under the so–called quasimonotonicity of the source term $\mathbf{g}$ (see [HN], [Sm], [PW]).

Proposition 2.5 *Let* $\mathbf{f}$ *and* $\tilde{\mathbf{f}}$ *be two weak solutions of problem* (2.4)–(2.5) *in* $\mathbb{R}^d \times (0,T)$ *for the initial data* $\mathbf{f}_0$ *and* $\tilde{\mathbf{f}}_0$*, respectively. Let* $Q \subseteq \mathbb{R}^N$ *be an interval (with non empty interior) such that*

(2.8) each component g_i of $\mathbf{g}$ is nondecreasing in f_j, for $i \neq j$, for any $f \in Q$;

(2.9) $$\mathbf{f},\ \tilde{\mathbf{f}} \in Q \quad \text{a.e. in } \mathbb{R}^d \times (0,T) \ .$$

If $\mathbf{f}_0 \leq \tilde{\mathbf{f}}_0$ *for almost every* $x \in \mathbb{R}^d$*, then* $\mathbf{f} \leq \tilde{\mathbf{f}}$ *for almost every* $(x,t) \in \mathbb{R}^d \times (0,T)$*.*

Proof of Theorem 2.1. Set

(2.10) $$M_i^\#(u) = \begin{cases} M_i(-N\|u_0\|_\infty) \ , & u < -N\|u_0\|_\infty \ , \\ M_i(u) \ , & |u| \leq N\|u_0\|_\infty \ , \\ M_i(N\|u_0\|)_\infty \ , & u > N\|u_0\|_\infty \ , \end{cases}$$

for $i = 1,\ldots,N$. Assume also $u_0 \in (\mathrm{Lip}_{loc}(\mathbb{R}^d))$ and let $\mathbf{f}^\# \in (\mathrm{Lip}_{loc}(\mathbb{R}^d \times (0,\infty)))$ the solution of problem (1.3)–(1.4) with $\mathbf{M}$ replaced by $\mathbf{M}^\#$ and $\mathbf{f}_0 = \mathbf{M}(u_0)$. For simplicity in this proof we shall omit the index ε. Let

$$T^\delta = \sup\{T \geq 0 \mid |u^\#(x,t)| \leq N\|u_0\|_\infty + \delta/2 \ , \quad \text{a.e. in } \mathbb{R}^d \times (0,T)\} \ ,$$

where $u^{\#} = \sum_{i=1}^{N} f_i^{\#}$. Clearly $T^\delta > 0$. Moreover, according to Definition 1.1 and thanks to present assumptions, the functions $M_i^{\#}$ are monotone (increasing) on $\mathbb{R}$. Then the system (1.3), with $\mathbf{M}^{\#}$, verifies the assumptions of Proposition 2.5.

In particular its right-hand side is quasimonotone in the strip $\mathbb{R}^d \times (0, T)$, i.e. it verifies condition (2.8).

Consider now the associated system of ordinary differential equations

$$\begin{cases} \dot{p}_i = \frac{1}{\varepsilon}(M_i^{\#}(v) - p_i) \ , \\ p_i(0) = p_{0i} \ , \qquad\qquad i = 1, \ldots, N, \end{cases} \tag{2.11}$$

for $v = \sum_{i=1}^{N} p_i$. If $|v(0)| \le N\|u_0\|_\infty$, then the global solution v is explicitely given by

$$v(t) = v(0) = \sum_{i=1}^{N} p_{0i} \ , \quad p_i = e^{-t/\varepsilon} p_{0i} + (1 - e^{-t/\varepsilon}) M_i(v(0)) \ . \tag{2.12}$$

Let $(p_i^{\pm})$ be the solution corresponding to the initial data

$$p_{0i}^{\pm} = \pm \|f_{0i}\|_\infty \ .$$

First we have that

$$|v(t)| = |v(0)| \le \sum_{i=1}^{N} \|f_{0i}\|_\infty = \sum_{i=1}^{N} \|M_i^{\#}(u_0)\|_\infty \le N\|u_0\|_\infty \ ,$$

since $|M_i^{\#}(u)| \le |u|$, for any $i = 1, \ldots, N$.

Then we can apply Proposition 2.5 to obtain

$$p_i^-(t) \le f_i^{\#}(x,t) \le p_i^+(t) \ , \quad i = 1, \ldots, N \tag{2.13}$$

and

$$|u^{\#}(x,t)| \le N\|u_0\|_\infty \ , \tag{2.14}$$

for every $(x,t) \in \mathbb{R}^d \times (0, T^\delta)$.

On the other hand

$$|p_i^{\pm}(t)| \le e^{-t/\varepsilon}\|f_{0i}\|_\infty + (1-e^{-t/\varepsilon})\sum_{j=1}^{N}\|f_{oj}\|_\infty \le N\|u_0\|_\infty \ .$$

Therefore, by standard continuation arguments, $T^\delta = +\infty$ and (2.13), (2.14) hold in $\mathbb{R}^d \times (0,\infty)$.

Then, on the range of $u^{\#}$, $\mathbf{M} = \mathbf{M}^{\#}$, and we obtain estimates (2.1)–(2.2) for our original problem. Finally it is easy to extend our result to general initial data, by density arguments. □

Proof of Theorem 2.2. Let us omit in this proof the index ε. The part (a) is an easy consequence of the monotonicity arguments used in the previous proof and Proposition 2.5. To prove (b) we use the Duhamel formula

$$f_i(x,t) = e^{-\frac{t-s}{\varepsilon}} f_i(x-(t-s)\lambda_i, s) + \int_s^t \frac{1}{\varepsilon} e^{-\frac{t-\tau}{\varepsilon}} M_i(u(x-(t-s-\tau)\lambda_i,\tau))\,\mathrm{d}\tau \tag{2.15}$$

for $i = 1,\dots,N$, $x \in \mathbb{R}^d$, $t \ge s \ge 0$. A similar formula holds for $\tilde{\mathbf{f}}$. Then, integrating both the identities against the function $\operatorname{sgn}(f_i - \tilde f_i)$ over a fixed open region $\Omega \subseteq \mathbb{R}^d$ and taking the sum on the indexes $i = 1,\dots,N$, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega |f_i(x,t)-\tilde f_i(x,t)|\,\mathrm{d}x &\le e^{-\frac{t-s}{\varepsilon}}\sum_{i=1}^{N}\int_{\Omega_t^\lambda(s)} |f_i(x,s)-\tilde f_i(x,s)|\,\mathrm{d}x \\ &\quad + \int_s^t \frac{1}{\varepsilon}\, e^{-\frac{t-\tau}{\varepsilon}} \int_{\Omega_t^\lambda(\tau)} \sum_{i=1}^{N} |M_i(u(x,\tau)) - M_i(\tilde u(x,\tau))|\,\mathrm{d}x\,\mathrm{d}\tau \ .\end{aligned}$$

Let us observe now that, since u, $\tilde u \in I$, we can apply the monotonicity properties of the function $\mathbf{M}$:

$$\begin{aligned}\sum_{i=1}^{N}|M_i(u)-M_i(\tilde u)| &= \left(\sum_{i=1}^{N}\int_0^1 M_i'(\alpha u + (1-\alpha)\tilde u)\,\mathrm{d}\alpha\right)|u-\tilde u| \\ &= |u-\tilde u| \le \sum_{i=1}^{N}|f_i-\tilde f_i| \ ,\end{aligned}$$

since $\sum_{i=1}^N M_i'(u) = 1$, for almost every $u \in I$.

Then, setting for $0 \le s \le t$

$$\mu^t(s) := \int_{\Omega_t^\lambda(s)} \sum_{i=1}^N |f_i(x,t) - \tilde{f}_i(x,t)| \, \mathrm{d}x \ ,$$

we have

$$\mu^t(t) \le e^{-\frac{t-s}{\varepsilon}} \mu^t(s) + \int_s^t \frac{1}{\varepsilon} e^{-\frac{t-\tau}{\varepsilon}} \mu^t(\tau) \, \mathrm{d}\tau \ , \tag{2.16}$$

which implies

$$\mu^t(t) \le \mu^t(s) \ , \tag{2.17}$$

for any $t \ge s \ge 0$. This concludes the proof. □

3 Convergence to the entropy solution

In this section we establish the convergence, as $\varepsilon \downarrow 0$, of the sequence $\mathbf{f}^\varepsilon$ of solutions of problem (1.3)–(1.4) to a limit function $\mathbf{f}$. Let $u = \sum_{i=1}^N f_i = \lim_{\varepsilon \to 0} u^\varepsilon$ (in a suitable topology). We prove also that $\mathbf{f}$ is a maxwellian distribution, i.e.: $\mathbf{M}(u) = \mathbf{f}$, and u is the (unique) entropy solution of problem (1.1)–(1.2) according to the following classical definition [Kr].

Definition 3.1 *A function $u \in L^\infty(\mathbb{R}^d \times (0,T))$ $(T > 0)$ is an entropy solution of problem* (1.1)–(1.2) *if:*

(i) *for any $k \in \mathbb{R}$ and $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,T))$, $\varphi \ge 0$, we have*

$$\iint \left\{ |u-k| \partial_t \varphi + \operatorname{sgn}(u-k) \sum_{j=1}^d (A_j(u) - A_j(k)) \partial_{x_j} \varphi \right\} \mathrm{d}x \, \mathrm{d}t \ge 0 \ ; \tag{3.1}$$

(ii) *for any interval $I \subseteq \mathbb{R}^d$*

$$\lim_{T \to 0^+} \frac{1}{T} \int_0^T \int_I |u(x,t) - u_0(x)| \, \mathrm{d}x \, \mathrm{d}t = 0 \ . \tag{3.2}$$

Our convergence theorem is the following.

Theorem 3.2 *Let* $u_0 \in L^\infty(\mathbb{R}^d)$ *be fixed and set* $\mathbf{f}_0 = \mathbf{M}(u_0)$. *Assume that* $\mathbf{M}$ *is a* **MMF** *on the interval* $I := \{u \in \mathbb{R} \mid |u| \le N\|u_0\|\}$. *Let* $\mathbf{f}^\varepsilon \in C([0,\infty); (L^1_{loc}(\mathbb{R}^d))^N) \cap (L^\infty(\mathbb{R}^d \times \mathbb{R}_+))^N$ *be the solution of problem* (1.3)–(1.4) ($\varepsilon > 0$) *given by Theorem 2.1. Let* $u \in C((0,\infty); L^1_{loc}(\mathbb{R}^d))$ *be the (unique) entropy solution of problem* (1.1)–(1.2). *Then, as* $\varepsilon \to 0^+$,

$$u^\varepsilon := \sum_{i=1}^N f_i^\varepsilon \to u \quad \text{in } C([0,\infty); L^1_{loc}(\mathbb{R}^d)) \ , \tag{3.3}$$

$$v_j^\varepsilon := \sum_{i=1}^N \lambda_{ij} f_i^\varepsilon \to A_j(u) \quad \text{in } C([0,\infty); L^1_{loc}(\mathbb{R}^d)) \ , \tag{3.4}$$

for $j = 1, \ldots, N$.

The proof of this theorem will follow after a sequence of preliminary results.

Proposition 3.3 *Under the assumptions of Theorem 3.2, let* $\mathbf{f}^\varepsilon$ *be the solution of problem* (1.3)–(1.4). *Then, for any bounded open set* $\Omega \subseteq \mathbb{R}^d$, *there exist a positive constant* $h_0 > 0$ *and a continuous nondecreasing function* $\omega \in C((0,h_0])$, *not depending on* ε *and with* $\omega(0) = 0$, *such that, for every* $t \ge 0$

$$\int_\Omega \sum_{i=1}^N |f_i^\varepsilon(x+h,t) - f_i^\varepsilon(x,t)| \, \mathrm{d}x \le \omega(|h|) \ , \tag{3.5}$$

for any $h \in \mathbb{R}^d$, $|h| \le h_0$.

Proof. The proof is a direct consequence of Theorem 2.2, by taking $\tilde{\mathbf{f}}^\varepsilon(x,t) = \mathbf{f}^\varepsilon(x+h,t)$. $\square$

To establish the equicontinuity in time of the sequences u^ε and $\mathbf{f}^\varepsilon$, we need the following interpolation lemma due to Kružkov [Kr].

Lemma 3.4 *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded convex open set and set $\Omega_{h_0} = \{x \in \mathbb{R}^d \mid d(x,\Omega) < h_0\}$, for some fixed $h_0 > 0$. Let w be a measurable bounded function in $\Omega_{h_0} \times (0,T)$ $(T > 0)$ and let $\omega_\Omega \in C([0,h_0))$ be a nondecreasing function, with $\omega_\Omega(0) = 0$, such that for every $t \in (0,T)$, $|h| \le h_0$*

$$\int_{\Omega_{h_0}} |w(x+h,t) - w(x,t)| \,\mathrm{d}x \le \omega_\Omega(|h|) \ . \tag{3.6}$$

Assume the following condition

$$\left| \int_\Omega (w(x,t+\tau) - w(x,t))\varphi(x) \,\mathrm{d}x \right| \le C_\Omega \tau \|\varphi\|_{C^2} \tag{3.7}$$

for any t, $t+\tau \in (0,T)$ $(\tau > 0)$, for any $\varphi \in C_0^2(\Omega)$ and some constant $C_\Omega > 0$. Then for any $0 \le t \le t+\tau \le T$ there holds

$$\int_\Omega |w(x,t+\tau) - w(x,t)| \,\mathrm{d}x \le \tilde{\omega}_\Omega(\tau) \ , \tag{3.8}$$

where

$$\tilde{\omega}_\Omega(\tau) = C_\Omega \min_{|h| \le h_0} \left[|h| + \omega_\Omega(|h|) + \frac{\tau}{h^2} \right] \ .$$

Proposition 3.5 *Under the assumptions of Theorem 3.2, let $u^\varepsilon = \sum_{i=1}^N f_i^\varepsilon$. Then for any bounded convex open set $\Omega \subseteq \mathbb{R}^d$ and for any $\nu > 0$, there exist a positive constant $\tau_0 > 0$ and a continuous nondecreasing function $\tilde{\omega} \in C([0,\tau_0))$, not depending on ε and with $\tilde{\omega}(0) = 0$ such that for every $0 \le t \le t+\tau$ $(\tau \in (0,\tau_0))$ it holds*

$$\int_\Omega |u^\varepsilon(x,t+\tau) - u^\varepsilon(x,t)| \,\mathrm{d}x \le \tilde{\omega}(\tau) \ . \tag{3.9}$$

Proof. Using Lemma 3.4 and thanks to Proposition 3.3, we have just to establish the inequality (3.7) for u^ε. In fact, for any $\varphi \in C_0^2(\Omega)$, we have

$$\left| \int_\Omega (u^\varepsilon(x,t+\tau) - u^\varepsilon(x,t))\varphi(x) \,\mathrm{d}x \right|$$

$$
\begin{aligned}
&= \left| \int_\Omega \left(\int_t^{t+\tau} \partial_t u^\varepsilon(x,s)\,\mathrm{d}s \right) \varphi(x)\,\mathrm{d}x \right| \\
&= \left| \int_\Omega \left(-\int_t^{t+\tau} \sum_{i=1}^N \sum_{j=1}^d \lambda_{ij} \partial_{x_j} f_i^\varepsilon(x,s)\,\mathrm{d}s \right) \varphi(x)\,\mathrm{d}x \right| \\
&= \left| \int_\Omega \sum_{i=1}^N \sum_{j=1}^d \lambda_{ij} \left(\int_t^{t+\tau} f_i^\varepsilon(x,s)\,\mathrm{d}s \right) \partial_{x_j}\varphi(x)\,\mathrm{d}x \right| \\
&= C_\Omega \tau \|\varphi\|_{C^1(\Omega)} \ .\square
\end{aligned}
$$

Proposition 3.6 *Under the assumptions of Theorem 3.2, let* $\mathbf{f}^\varepsilon$ *be the solution of problem* (1.3)–(1.4). *Then, for any bounded convex open set* $\Omega \subseteq \mathbb{R}^d$ *and for any* $\nu > 0$, *there exists a positive constant* $\tau_0 > 0$ *and a continuous nondecreasing function* $\tilde{\omega}^\nu \in C([0,\tau_0))$, *not depending on* ε *and with* $\tilde{\omega}^\nu(0) = 0$, *such that, for every* $\nu \le t \le t+\tau$ ($\tau \in (0,\tau_0)$), *there holds*

$$
\int_\Omega \sum_{i=1}^N |f_i^\varepsilon(x,t+\tau) - f_i^\varepsilon(x)|\,\mathrm{d}x \le \tilde{\omega}^\nu(\tau) \ , \tag{3.10}
$$

for any $0 < \varepsilon < \nu$.

Proof. Thanks to Lemma 3.4 and inequality (3.5), we have only to prove that for any bounded convex open set $\Omega \subseteq \mathbb{R}^d$ there exists constant $C > 0$ such that

$$
\left| \int_\Omega (f_i^\varepsilon(x,t+\tau) - f_i^\varepsilon(x,t))\varphi(x)\,\mathrm{d}x \right| \le C\tau \|\varphi\|_{C^2} \tag{3.11}
$$

for $i = 1, \ldots, N$, $\nu \le t \le t+\tau$ ($t > 0$) and for any $\varphi \in C_0^2(\Omega)$.

From the Duhamel formula (2.15) and for any $\varphi \in C_0^2(\mathbb{R}^d)$ we have, omitting for simplicity the index ε,

$$
\begin{aligned}
\int f_i(x,t)\varphi(x)\,\mathrm{d}x &= \int e^{-t/\varepsilon} f_{0i}(x) \varphi(x+\lambda_i t)\,\mathrm{d}x \\
&\quad + \int_0^t \frac{1}{\varepsilon}\, e^{-\frac{t-s}{\varepsilon}} \int M_i(u(x,s))\varphi(x+\lambda_i(t-s))\,\mathrm{d}x\,\mathrm{d}s \ ,
\end{aligned}
$$

for $i = 1, \ldots, N$. Then

$$
\begin{aligned}
I(t) &= \left| \int (f_i(x, t+\tau) - f_i(x,t))\varphi(x)\,\mathrm{d}x \right| \\
&\leq \left| \int \left(e^{-\frac{t+\tau}{\varepsilon}} \varphi(x + \lambda_i(t+\tau)) - e^{-\frac{t}{\varepsilon}} \varphi(x + \lambda_i t) \right) f_{0i}(x)\,\mathrm{d}x \right| \\
&+ \frac{1}{\varepsilon} \left| \int_0^{t+\tau} e^{-\frac{t+\tau-s}{\varepsilon}} \int M_i(u(x,s))\varphi(x + \lambda_i(t+\tau-s))\,\mathrm{d}x\,\mathrm{d}s \right. \\
&- \left. \int_0^t e^{-\frac{t-s}{\varepsilon}} \int M_i(u(x,s))\varphi(x+\lambda_i(t-s))\,\mathrm{d}x\,\mathrm{d}s \right| \\
&= I_1 + \frac{1}{\varepsilon} I_2 \ .
\end{aligned}
$$

First we have, for $\nu \leq t \leq t+\tau$

$$
\begin{aligned}
I_1 &\leq \left| \int \left(e^{-\frac{t+\tau}{\varepsilon}} - e^{-t/\varepsilon} \right) \varphi(x + \lambda_i(t+\tau)) f_{0i}(x)\,\mathrm{d}x \right| \\
&+ \left| \int e^{-t/\varepsilon} (\varphi(x+\lambda_i(t+\tau)) - \varphi(x+\lambda_i t)) f_{0i}(x)\,\mathrm{d}x \right| \\
&\leq C\ \frac{e^{-\frac{\nu}{\varepsilon}}}{\varepsilon} \tau \|f_{0i}\|_\infty \|\varphi\|_{C^1} \ .
\end{aligned}
$$

For the next term we have

$$
\begin{aligned}
I_2 &\leq \left| \int_0^t \int [M_i(u(x,s+\tau)) - M_i(u(x,s))] e^{-\frac{t-s}{\varepsilon}} \varphi(x+\lambda_i(t-s))\,\mathrm{d}x\,\mathrm{d}s \right| \\
&+ \left| \int_0^\tau \int M_i(u(x,s)) e^{-\frac{t+\tau-s}{\varepsilon}} \varphi(x+\lambda_i(t-s))\,\mathrm{d}x\,\mathrm{d}s \right| \ .
\end{aligned}
$$

Then, since M_i is Lipschitz continuous and by using Proposition 3.5, it follows

$$
I_2 \leq C\tau \|\varphi\|_{C^1} \left(\varepsilon + \frac{e^{-t/\varepsilon}}{\varepsilon} \right) \ .
$$

Therefore, the conclusion follows by summing up the different estimates. $\square$

Next we can estimate the deviation from the equilibrium in the L^1 norm.

Proposition 3.7 *Under the assumptions of Theorem 3.1, suppose that the initial data u_0 is locally of bounded variation. Then there exists a constant $C > 0$ such that, for any open set $\Omega \subseteq \mathbb{R}^d$ and any $t > 0$*

$$
\int_\Omega \sum_{i=1}^N |f_i^\varepsilon - M_i(u^\varepsilon)|\,\mathrm{d}x \leq C\varepsilon \sum_{i=1}^N \|f_{0i}\|_{BV(\Omega_t^\lambda(0))} \ . \tag{3.12}
$$

Proof. For any $\varepsilon > 0$ and $i = 1, \ldots, N$, take smooth initial data. then we have

$$\partial_t(f_i^\varepsilon - M_i(u^\varepsilon)) + \frac{1}{\varepsilon}(f_i^\varepsilon - M_i(u^\varepsilon)) = -\sum_{j=1}^{d}(\lambda_{ij}\partial_{x_j} f_i^\varepsilon + M_i'(u^\varepsilon)\sum_{k=1}^{N}\lambda_{kj}\partial_{x_j} f_k^\varepsilon) \ .$$

Hence, after integration and by standard methods, we obtain that there exists a constant $C > 0$ such that:

$$\int_\Omega \sum_{i=1}^{N} |f_i^\varepsilon - M_i(u^\varepsilon)| \, \mathrm{d}x \le C \int_0^t e^{-\frac{t-s}{\varepsilon}} \int_\Omega \sum_{i=1}^{N}\sum_{j=1}^{d} |\partial_{x_j} f_i^\varepsilon| \, \mathrm{d}x \, \mathrm{d}s \ .$$

Then, from Proposition 3.3 and using density arguments, we obtain (3.12). $\square$

The previous results were established to prove the compactness of the sequence $\mathbf{f}^\varepsilon$ in $C([0,\infty); (L^1_{loc}(\mathbb{R}^d))^N)$ via the classical Fréchet–Kolmogorov and Ascoli Theorems . Let us study now the consistency of our approximation with the entropy inequalities (3.1). The next result is in the spirit of [Br1], [PT]. Since we need to define the function $\mathbf{M}$ for all $u \in \mathbb{R}$, let us set through the remainder of the paper $\mathbf{M} = \mathbf{M}^\#$, the last function being defined by (2.10).

Proposition 3.8 *For any solution* $\mathbf{f}^\varepsilon \in C([0,\infty); L^1_{loc}(\mathbb{R}^d)^N)$ *of problem* (1.3)–(1.4), *for any* $k \in \mathbb{R}$ *and for any* $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,\infty))$, $\varphi \ge 0$, *we have*

$$\iint \sum_{i=1}^{N} |f_i^\varepsilon - M_i(k)| \left(\partial_t \varphi + \sum_{j=1}^{d} \lambda_{ij}\partial_{x_j}\varphi\right) \mathrm{d}x \, \mathrm{d}t \tag{3.13}$$
$$\ge \frac{1}{\varepsilon}\left(\iint \left[\sum_{i=1}^{N} |f_i^\varepsilon - M_i(k)| - |u^\varepsilon - k|\right] \varphi \, \mathrm{d}x \, \mathrm{d}t\right) \ge 0 \ .$$

Proof. To obtain (3.13), first multiply equation (1.3) by $\operatorname{sgn}(f_i^\varepsilon - M_i(k))\varphi(x,t)$, take the sum for $i = 1, \ldots, N$ and integrate over $\mathbb{R}^d \times (0,\infty)$. Then we have

$$\iint \sum_{i=1}^{N} |f_i^\varepsilon - M_i(k)| \left(\partial_t \varphi + \sum_{j=1}^{d} \lambda_{ij}\partial_{x_j}\varphi\right) \mathrm{d}x \, \mathrm{d}t$$

$$= \frac{1}{\varepsilon} \iint \sum_{i=1}^{N} \operatorname{sgn}(f_i^\varepsilon - M_i(k))(f_i^\varepsilon - M_i(u^\varepsilon))\varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$= \frac{1}{\varepsilon} \iint \sum_{i=1}^{N} [(|f_i^\varepsilon - M_i(k)| - \operatorname{sgn}(f_i^\varepsilon - M_i(k))(M_i(u^\varepsilon) - M_i(k))] \varphi \, \mathrm{d}x \, \mathrm{d}t \ .$$

On the other hand we have

$$\begin{aligned} &\sum_{i=1}^{N} \operatorname{sgn}(f_i^\varepsilon - M_i(k))(M_i(u^\varepsilon) - M_i(k)) \\ &\qquad \le \sum_{i=1}^{N} |M_i(u^\varepsilon) - M_i(k)| \\ &\qquad = |u^\varepsilon - k| \le \sum_{i=1}^{N} |f_i^\varepsilon - M_i(k)| \ . \end{aligned}$$

Then the proof is complete. □

Proof of Theorem 3.2. First let us take the initial data $u_0 \in BV_{loc}(\mathbb{R}^d)$. Then, thanks to Theorem 2.1, 2.2, Propositions 3.3, 3.5, 3.6, 3.7 and by standard compactness arguments, there exists $\mathbf{f} \in L^\infty(\mathbb{R}^d \times (0,\infty))^N$ which is the limit as $\varepsilon \downarrow 0$ in $C([0,\infty);(L^1_{loc}(\mathbb{R}^d))^N)$ of $\mathbf{f}^\varepsilon$ and such that, setting $u = \sum_{i=1}^N f_i$, we have the convergences (3.3) and (3.4). In particular, thanks to Propositions 3.5, 3.7, we can prove the convergence in this topology even for $t = 0$. The case of initial data u_0 which are not of bounded variation is considered just following step by step the arguments given in the proof of Theorem 5.1 in [Na]. Hence we obtain the convergence result.

To show that the limit function u verifies the entropy inequalities (3.1), we first prove the following limits, in the topology of $L^p_{loc}(\mathbb{R}^d \times (0,\infty))$, for any $1 \le p < +\infty$:

$$\sum_{i=1}^{N} |f_i^\varepsilon - M_i(k)| \to |u - k| \ , \quad \varepsilon \downarrow 0 \tag{3.14}$$

and for any $j = 1, \ldots, N$,

$$\sum_{i=1}^{N} \lambda_{ij} |f_i^\varepsilon - M_i(k)| \to \operatorname{sgn}(u-k)(A_j(u) - A_j(k)) \ , \quad \varepsilon \downarrow 0 \ . \tag{3.15}$$

To prove (3.14) we just use (3.13) and the uniform boundedness of $\{\mathbf{f}^\varepsilon\}$. In fact, for any nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,\infty))$, we have:

$$
\begin{aligned}
0 &\le \iint \left[\sum_{i=1}^N |f_i^\varepsilon - M_i(k)| - |u^\varepsilon - k|\right] \varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\le \varepsilon \iint \sum_{i=1}^N |f_i^\varepsilon - M_i(k)| \left(\partial_t \varphi + \sum_{j=1}^d \lambda_{ij} \partial_{x_j} \varphi\right) \,\mathrm{d}x\,\mathrm{d}t \\
&\le \varepsilon C(\varphi)\ .
\end{aligned}
$$

This proves the convergence in L^1_{loc}. Convergence in L^p_{loc} follows easily by the boundedness of the sequence.

Set now, for almost any $(x,t) \in \mathbb{R}^d \times (0,T)$

$$
I(x,t) = \{i \in \{1,\dots,N\} \mid \operatorname{sgn}(f_i^\varepsilon - M_i(k)) \neq \operatorname{sgn}(u^\varepsilon - k)\}\ .
$$

Then, for any nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,\infty))$, we have

$$
\begin{aligned}
&\iint \left|\sum_{i=1}^N \lambda_{ij}[|f_i^\varepsilon - M_i(k)| - \operatorname{sgn}(u^\varepsilon - k)(f_i^\varepsilon - M_i(k))]\right| \varphi \,\mathrm{d}x\,\mathrm{d}t \\
&= \iint \left|\sum_{i=1}^N \lambda_{ij} |f_i^\varepsilon - M_i(k)| \operatorname{sgn}(f_i^\varepsilon - M_i(k))\right. \\
&\left.\cdot(\operatorname{sgn}(f_i^\varepsilon - M_i(k)) - \operatorname{sgn}(u^\varepsilon - k))\right| \varphi \,\mathrm{d}x\,\mathrm{d}t \\
&= \iint \left|\sum_{i\in I(x,t)} \lambda_{ij} 2 |f_i^\varepsilon - M_i(k)|\right| \varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\le \sup_{i=1,\dots,N} |\lambda_{ij}| \iint \left|\sum_{i\in I(x,t)} 2 |f_i^\varepsilon - M_i(k)|\right| \varphi \,\mathrm{d}x\,\mathrm{d}t \\
&= \sup_{i=1,\dots,N} |\lambda_{ij}| \iint \left|\sum_{i=1}^N |f_i^\varepsilon - M_i(k)| - |u^\varepsilon - k|\right| \varphi \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

Therefore the limit (3.15) follows, since the last term tends to zero, as $\varepsilon \downarrow 0$, from (3.14). Finally, we recover the entropy inequalities (3.1), just passing into the limit in inequality (3.13). $\square$

4 Examples and remarks

In this section we present some examples of monotone maxwellian functions $\mathbf{M}$ related to Cauchy problem (1.1)–(1.2).

Example 4.1 First let us consider the one dimensional case with two velocities, $d = 1$, $N = 2$. Consider the conservation law

$$\partial_t u + \partial_x A(u) = 0 \tag{4.1}$$

and the approximating discrete Boltzmann system, for $\lambda_1 < \lambda_2$

$$\begin{cases} \partial_t f_1^\varepsilon + \lambda_1 \partial_x f_1^\varepsilon = \frac{1}{\varepsilon}(M_1(u^\varepsilon) - f_1^\varepsilon) \ , \\ \partial_t f_2^\varepsilon + \lambda_2 \partial_x f_2^\varepsilon = \frac{1}{\varepsilon}(M_2(u^\varepsilon) - f_2^\varepsilon) \ , \end{cases} \tag{4.2}$$

where $u^\varepsilon := f_1^\varepsilon + f_2^\varepsilon$. From (1.6)–(1.7) the **MMF** for this problem is given by

$$M_1(u) = \frac{\lambda_2 u - A(u)}{\lambda_2 - \lambda_1} \ , \quad M_2(u) = \frac{\lambda_1 u - A(u)}{\lambda_1 - \lambda_2} \ , \tag{4.3}$$

under the condition (1.8), which reads here

$$\lambda_1 \leq A'(u) \leq \lambda_2 \ , \quad \text{for } u \in I \ . \tag{4.4}$$

for some fixed interval $I \subseteq \mathbb{R}$.

Setting $v^\varepsilon = \lambda_1 f_1^\varepsilon + \lambda_2 f_2^\varepsilon$, we rewrite system (4.2) in a relaxation form:

$$\begin{cases} \partial_t u^\varepsilon + \partial_x v^\varepsilon = 0 \ , \\ \partial_t v^\varepsilon + \partial_x((\lambda_1 + \lambda_2) v^\varepsilon - \lambda_1 \lambda_2 u^\varepsilon) = \frac{1}{\varepsilon}(A(u^\varepsilon) - v^\varepsilon) \ . \end{cases} \tag{4.5}$$

Hence the monotonicity condition (4.4) is read as the subcharacteristic condition for system (4.5), [Wh], [Li].

In particular for $\lambda_2 = -\lambda_1 = \lambda > 0$ we recover the relaxation approximation (1.9) of equation (4.1), see [JX], [Na].

Example 4.2 In the numerical approximation of equation (4.1) it could be useful to deal with more velocities, say $N > d+1$. This formally corresponds to higher order approximation schemes, see [Br2], [AN2]. Here we consider $N = 3$, $d = 1$. It is easy to show that a convergent approximation is given for $\lambda_1 < \lambda_2 < \lambda_3$, by the following **MMF**:

$$\begin{aligned} M_1(u) &= \frac{\lambda_3 u - A(u) + (\lambda_2 - \lambda_3) M_2(u)}{\lambda_3 - \lambda_1} \,, \\ M_3(u) &= \frac{\lambda_1 u - A(u) + (\lambda_2 - \lambda_1) M_2(u)}{\lambda_1 - \lambda_3} \,, \end{aligned} \tag{4.6}$$

for any Lipschitz continuous function $M_2(u)$ such that

$$0 \le M_2'(u) \le \inf \left(\frac{A'(u) - \lambda_1}{\lambda_2 - \lambda_1} \,,\ \frac{\lambda_3 - A'(u)}{\lambda_3 - \lambda_2} \right) \,, \tag{4.7}$$

for $u \in I$, for some $I \subseteq \mathbb{R}$. Observe that condition (4.7) implies the subcharacteristic condition (4.4).

For $\lambda_3 = -\lambda_1 = \lambda > 0$, and $\lambda_2 = 0$ we can choose

$$M_1' = \frac{(-A')_+}{\lambda} \,, \quad M_2' = 1 - \frac{|A'|}{\lambda} \,, \quad M_3' = \frac{(A')_+}{\lambda} \,. \tag{4.8}$$

This choice corresponds, in the relaxation limit, to the Engquist–Osher numerical scheme, see [Br2], [AN2]. The relaxation formulation is given now by

$$\begin{cases} \partial_t u^\varepsilon + \partial_x v^\varepsilon = 0 \,, \\ \partial_t v^\varepsilon + \partial_x z^\varepsilon = \frac{1}{\varepsilon}(A(u^\varepsilon) - v^\varepsilon) \,, \\ \partial_t z^\varepsilon + \lambda^2 \partial_x v^\varepsilon = \frac{1}{\varepsilon}(\lambda^2(u^\varepsilon - M_2(u^\varepsilon)) - z^\varepsilon) \,, \end{cases} \tag{4.9}$$

where $v^\varepsilon = \lambda(f_3^\varepsilon - f_1^\varepsilon)$ and $z^\varepsilon = \lambda^2(f_3^\varepsilon + f_1^\varepsilon)$.

Remark 4.3 For any choice of distinct velocities $\lambda_1 < \cdots < \lambda_N$, and for any fixed **MMF** $\mathbf{M}(u)$, we can give a correspondent relaxation formulation of the discrete Boltzmann system (1.3). In fact, set

$$v_k = \sum_{i=1}^{N} (\lambda_i)^k f_i \,, \quad \text{for } k = 0, \dots, N \,, \tag{4.10}$$

$$z_k(u) = \sum_{i=1}^{N} (\lambda_i)^k M_i(u) \ , \quad \text{for } k = 0, \dots, N-1 \ . \tag{4.11}$$

Observe that $v_0 = z_0 = u$ and $z_1 = A(u)$.

Then we obtain the system

$$\partial_t v_k + \partial_x v_{k+1} = \frac{1}{\varepsilon}(z_k(u) - v_k) \ , \quad k = 0, \dots, N-1 \ . \tag{4.12}$$

Actually the system is closed and equivalent to system (1.3), since the linear transformations

$$\mathbf{f} \to \mathbf{v} := (v_0, \dots, v_{N-1}) \ , \ \mathbf{M} \to \mathbf{z} := (z_0, \dots, z_{N-1}) \ ,$$

are one-to-one, the corresponding matrix being the Vandermonde matrix associated to the values $\{\lambda_1, \dots, \lambda_N\}$, which is nonsingular if and only if these values are distinct. In particular there exist some constants α_i $(i = 0, \dots, N-1)$ such that

$$v_N = \sum_{i=0}^{N-1} \alpha_i v_i.$$

Example 4.4 Let us consider now the multidimensional case $d \geq 2$. First we present an example with the minimal number of velocities, i.e.: $N = d+1$. Let $\lambda > 0$ be fixed and set

$$\lambda_{ij} = -\lambda \delta_{ij} \ , \quad i = 1, \dots, d \ , \quad \lambda_{Nj} = \lambda \ ;$$

a correspondent **MMF** is given now by

$$\begin{aligned} M_N(u) &= \frac{1}{N}\left(u + \frac{1}{\lambda} \sum_{j=1}^{d} A_j(u) \right) \ , \\ M_i(u) &= M_N(u) - \frac{1}{\lambda} A_i(u) \ , \quad i = 1, \dots, d \ . \end{aligned}$$

The monotonicity is recovered just by taking

$$\lambda \geq \sup_{|u| \leq B} \sup_{j=1,\dots,d} (2d-1)|A_j'(u)|$$

for a suitable value of $B > 0$. In particular, in view of Theorem 2.1, we can choose $B = N\|u_0\|_\infty$. A relaxation formulation is now given by setting

$$u^\varepsilon = \sum_{i=1}^N f_i^\varepsilon \ , \quad v_j^\varepsilon = \lambda(f_N^\varepsilon - f_j^\varepsilon) \qquad (j = 1, \ldots, d) \ .$$

This yields, after some calculations:

$$\begin{cases} \partial_t u^\varepsilon + \sum\limits_{j=1}^d \partial_{x_j} v_j^\varepsilon = 0 \ , \\ \partial_t v_j^\varepsilon + \frac{\lambda}{N} \sum\limits_{\substack{k=1 \\ k\neq j}}^d \partial_{x_k} \left(\lambda u^\varepsilon + \sum\limits_{i=1}^d v_i^\varepsilon \right) + \frac{\lambda}{N} \partial_{x_j} \left(2 \left(\lambda u^\varepsilon + \sum\limits_{i=1}^d v_i^\varepsilon \right) - \frac{N}{\lambda} v_j^\varepsilon \right) \\ \qquad = \frac{1}{\varepsilon}(A_j(u^\varepsilon) - v_j^\varepsilon) \ , \quad j = 1, \ldots, d \ . \end{cases}$$

Remark 4.5 Let us observe that in [JX] the authors proposed the following relaxation approximation for equation (1.1):

$$\begin{cases} \partial_t u + \sum_{j=1}^d \partial_{x_j} v_j = 0 \\ \partial_t v_j + \alpha_j \partial_{x_j} u = \frac{1}{\varepsilon}(A_j(u) - v_j) \ , \quad (j = 1, \ldots, d) \ , \end{cases} \tag{4.13}$$

for a suitable choice of the constants $\alpha_j \in \mathbb{R}$. This approximation does not fit in our framework and then we are not able to prove, by using our methods, its convergence as $\varepsilon \downarrow 0$. In fact it is possible to prove, by a simple explicit computation, that it is impossible to diagonalize the left hand side of system (4.13), if $d \geq 2$.

More precisely, take a function $\mathbf{f} = (f_1, \ldots, f_N)$ and some constants $\lambda_{ij} \in \mathbb{R}$ $(i = 1, \ldots, N; \ j = 1, \ldots, d; \ N \geq d+1)$ such that

$$\partial_t f^i + \sum_{j=1}^d \lambda_{ij} f^i = 0 \ . \tag{4.14}$$

Let $b_{ij} \in \mathbb{R}$ $(i = 1, \ldots, N; \ j = 0, \ldots, d)$ be some real coefficients such that the functions

$$\begin{aligned} u &= \sum_{i=1}^N b_{i0} f^i \ , \\ v_j &= \sum_{i=1}^N b_{ij} f^i \ , \qquad j = 1, \ldots, d \end{aligned} \tag{4.15}$$

are solutions of the homogeneous counterpart of system (4.13), i.e.:

$$\begin{cases} \partial_t u + \sum_{j=1}^d \partial_{x_j} v_j = 0 \ , \\ \partial_t v_j + \alpha_j \partial_{x_j} u = 0 \ , \qquad (j = 1, \ldots, d) \ . \end{cases} \tag{4.16}$$

Then it is easy to check that there exists a unique $k \in \{1, \ldots, d\}$, such that $b_{ik} \neq 0$ for some $i \in \{1, \ldots, N\}$, which gives $v_j = 0$ if $j \neq k$, In fact, from (4.15)–(4.16) we have

$$\partial_t u + \sum_{j=1}^d \partial_{x_j} v_j = \sum_{i=1}^N \left(b_{i0} \partial_t f^i + \sum_{j=1}^d b_{ij} \partial_{x_j} f^i \right) = 0 \ ,$$

$$\partial_t v_j + \alpha_j \partial_{x_j} u = \sum_{i=1}^N \left(b_{ij} \partial_t f^i + \alpha_j b_{i0} \partial_{x_j} f^i \right) = 0 \ ,$$

for $j = 1, \ldots, N$. So, by using (4.14) and the independence of $\partial_{x_j} f^i$, we obtain the following conditions on the coefficients, for any $i = 1, \ldots, d$, $j = 1, \ldots, N$:

$$b_{ij} \lambda_{ij} = a_j b_{i0} \ , \quad b_{ij} = b_{i0} \lambda_{ij}, \quad b_{ij} \lambda_{il} = 0 \quad l \neq j \ . \tag{4.17}$$

From (4.17) it is easy to conclude that there exists no more than a unique value $k \in \{1, \ldots, d\}$ such that $\alpha_l = \lambda_{il} = b_{il} = 0$ if $l \neq k$, for any $i \in \{1, \ldots, N\}$, and α_k, λ_{ik}, $b_{ik} \neq 0$ for some $i \in \{1, \ldots, N\}$.

Example 4.6 Let us conclude our presentation by an extension of Example 4.2 to the multidimensional case. Take $N = 2d + 1$ and set, for $j = 1, \ldots, d$,

$$\lambda_{ij} = \begin{cases} \lambda_i \delta_{ij} \ , & i = 1, \ldots, d \ , \\ 0 \ , & i = d+1 \ , \\ -\lambda_{i-(d+1)} \delta_{ij} \ , & i = d+2, \ldots, 2d+1 \ . \end{cases} \tag{4.18}$$

Here the constant values λ_i ($i = 1, \ldots, d$) will be chosen later. For the maxwellian function $\mathbf{M}$ we choose

$$\begin{aligned} M_i'(u) &= \frac{1}{\lambda_i} (A_i'(u))_+ \ , \quad i = 1, \ldots, d \ , \\ M_{d+1}'(u) &= 1 - \sum_{j=1}^d \frac{|A_j'(u)|}{\lambda_j} \ , \end{aligned} \tag{4.19}$$

$$M_i'(u) = \frac{1}{\lambda_{i-(d+1)}} \left(-A'_{i-(d+1)}(u) \right)_+ \ , \quad i = d+2, \ldots, 2d+1 \ .$$

According to this choice, the function $\mathbf{M}$ is a **MMF** on a suitable interval $I \subseteq \mathbb{R}$, if

$$1 \geq \sum_{j=1}^{d} \frac{|A_j'(u)|}{\lambda_j}$$

for any $u \in I$.

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