Uniqueness of Weak Solutions to Systems of Conservation Laws

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Abstract. Consider a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension:

$$u_t + F(u)_x = 0. \tag{$*$}$$

Relying on the existence of the Standard Riemann Semigroup generated by $(*)$, we establish the uniqueness of entropy-admissible weak solutions to the Cauchy problem, under a mild assumption on the variation of u along space-like segments.

1 - Introduction.

We are concerned with the uniqueness of solutions to the Cauchy problem for an hyperbolic $n \times n$ system of conservation laws in one space dimension:

$$u_t + F(u)_x = 0, \tag{1.1}$$

$$u(0,x) = \bar{u}(x). \tag{1.2}$$

Let $\Omega \subseteq \mathbb{R}^n$ be an open set containing the origin, and let $F : \Omega \mapsto \mathbb{R}^n$ be a smooth map. Assume that the system (1.1) is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely nonlinear [12, 18]. For every initial data $\bar{u} \in \mathbf{L}^1$ with small total variation, a well known theorem of Glimm [10] provides the global existence of weak solutions.

In subsequent papers [7, 8, 13, 14, 15, 16, 17], several authors have shown that those solutions which satisfy various additional regularity and entropy admissibility conditions are necessarily unique. Apart from the scalar case [11, 20], however, these uniqueness results are applicable only within a class of functions which is strictly smaller than the class where an existence theorem is known.

An alternative approach to the problem of uniqueness and continuous dependence was proposed in [1]. Instead of the single Cauchy problem, one considers a whole flow of solutions of (1.1), continuously depending on the initial data. The main results in [1, 5] can be summarized as follows.

Theorem 1. *There exists a closed domain $\mathcal{D} \subset \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n)$, constants η_0, L, and a continuous semigroup $S : D \times [0,\infty[\mapsto \mathcal{D}$ with the properties:*

(i) Every function $\bar{u} \in \mathbf{L}^1$ with Tot.Var.$(\bar{u}) \le \eta_0$ lies in $\mathcal{D}$.

(ii) For all $\bar{u}, \bar{v} \in \mathcal{D}$, $t, s \ge 0$ one has $\big\| S_t\bar{u} - S_s\bar{v}\big\|_{\mathbf{L}^1} \le L\big(|t-s| + \|\bar{u}-\bar{v}\|_{\mathbf{L}^1}\big)$.

(iii) If $\bar{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small the function $u(t,\cdot) = S_t\bar{u}$ coincides with the solution of (1.1)-(1.2) obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.

(iv) Each trajectory $t \mapsto u(t,\cdot) = S_t\bar{u}$ is a weak, entropy-admissible solution of the corresponding Cauchy problem (1.1)-(1.2).

(v) Let $(u_\nu)_{\nu\ge 1}$ be a sequence of approximate solutions of (1.1)-(1.2) generated by a wave-front tracking algorithm, or by the Glimm scheme with uniformly distributed sampling. Then $\mathbf{L}^1$-$\lim_{\nu\to\infty} u_\nu(t,\cdot) = S_t\bar{u}$ for every $t \ge 0$.

The positively invariant domain $\mathcal{D}$ has the form

$$\mathcal{D} = cl\Big\{u \in \mathbf{L}^1(I\!R; I\!R^n); \quad u \text{ is piecewise constant}, \quad V(u) + C_0 \cdot Q(u) < \delta_0\Big\}, \tag{1.3}$$

for some constants $C_0, \delta_0 > 0$. Here $V(u)$ and $Q(u)$ denote the total strength of waves and the wave interaction potential of u, while cl denotes closure. On a given domain $\mathcal{D}$, the semigroup S with the above properties is unique.

Following [1], we say that a map S with the properties (i)–(iii) is a *Standard Riemann Semigroup* (SRS). The existence of such a semigroup was proved in [3] for 2×2 and recently in [5] for general $n \times n$ systems. The uniqueness of the SRS (up to the domain), together with the properties (iv)-(v), was shown in [1]. See [2] for a general survey. At this stage, given the Cauchy problem (1.1)-(1.2), in order to establish the uniqueness of a solution $u = u(t,x)$ it thus suffices to show that u coincides with the corresponding semigroup trajectory:

$$u(t,\cdot) = S_t \bar{u} \qquad \forall t \geq 0. \tag{1.4}$$

A condition which implies (1.4) was derived in [1], based on a set of local integral estimates.

In the present paper we prove that any weak, entropy-admissible solution of (1.1)-(1.2) actually satisfies (1.4), provided that the variation of u does not grow too wildly. Roughly speaking, we require that, if a space-like segment Γ is contained inside the domain of dependency of another segment $\widetilde{\Gamma}$, than the total variation of u along Γ should be controlled by the variation of u along $\widetilde{\Gamma}$. This yields a new uniqueness theorem for solutions to hyperbolic systems of conservation laws, valid within the same class of functions where global existence theorems are known.

Some basic notations are recalled below. Let $A(u) = DF(u)$ be the Jacobian matrix of F at the point u. Smooth solutions of (1.1) thus satisfy the equivalent quasilinear system

$$u_t + A(u)u_x = 0. \tag{1.5}$$

Let $\lambda_1(u) < \cdots < \lambda_n(u)$ be the eigenvalues of $A(u)$ and choose right and left eigenvectors $r_i(u)$, $l_i(u)$, $i = 1, \ldots, n$, normalized so that

$$r_i \bullet \lambda_i(u) \doteq \lim_{h\to 0} \frac{\lambda_i\big(u + hr_i(u)\big) - \lambda_i(u)}{h} \geq 0, \tag{1.6}$$

$$|r_i| \equiv 1, \qquad \big\langle l_i,\ r_j\big\rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{1.7}$$

We shall be concerned with a solution u defined on a strip $[0,T] \times I\!R$. Uniqueness will be proved under three assumptions.

(A1) **(Conservation Equations)** The function $u = u(t,x)$ is a weak solution of the Cauchy problem (1.1)-(1.2), taking values within the domain of the Standard Riemann Semigroup S. More precisely, u is a measurable function from $[0,T] \times I\!R$ into $\Omega \subset I\!R^n$, with $u(t,\cdot) \in \mathcal{D}$ for every t. The initial condition (1.2) holds, together with

$$\int\!\!\int \big(u\varphi_t + f(u)\varphi_x\big)\, dxdt = 0 \tag{1.8}$$

for every $\mathcal{C}^1$ function φ with compact support contained inside the open strip $]0,T[\, \times I\!R$.

(A2) **(Entropy Condition)** Let u have an approximate jump discontinuity at some point $(\tau,\xi) \in\,]0,T[\times I\!R$. More precisely, assume that there exists states $u^-, u^+ \in \Omega$ and a speed $\lambda \in I\!R$ such that, calling

$$U(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \xi + \lambda(t-\tau), \\ u^+ & \text{if} \quad x > \xi + \lambda(t-\tau), \end{cases} \tag{1.9}$$

there holds

$$\lim_{\rho\to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \Big| u(t,x) - U(t,x) \Big|\, dxdt = 0. \tag{1.10}$$

Then, for some $i \in \{1,\ldots,n\}$, one has the entropy inequality [12]:

$$\lambda_i(u^-) \geq \lambda \geq \lambda_i(u^+). \tag{1.11}$$

A precise statement of our regularity assumption requires some preliminary definitions. Fix a constant $\hat\lambda$, strictly greater than all wave speeds. By a *space-like segment* we mean a set of the form

$$\Gamma \doteq \Big\{(t,x); \quad x \in [a,b], \;\; t = \phi(x) = \alpha x + \beta\Big\},$$

for some α, β with $|\alpha| < 1/\hat\lambda$. In particular, a *horizontal segment* is a set of the form

$$\widetilde\Gamma \doteq \big\{(\tau,x); \quad x \in [c,d]\big\},$$

for some fixed τ. We say that the above segment Γ lies inside the domain of dependency of $\widetilde\Gamma$ if

$$[a,b] \subseteq [c,d], \qquad 0 \leq \phi(a) - \tau \leq \frac{a-c}{\hat\lambda}, \qquad 0 \leq \phi(b) - \tau \leq \frac{d-b}{\hat\lambda}.$$

The total variation of the function u along the segment Γ is written Tot.Var.$(u;\Gamma)$. The following assumption implies that the variation of u does not grow too wildly, as time increases.

(A3) **(Tame Variation Assumption)** There exists a constant K such that

$$\text{Tot.Var.}(u;\Gamma) \leq K \cdot \text{Tot.Var.}(u;\widetilde\Gamma) \tag{1.12}$$

for every horizontal segment $\widetilde{\Gamma}$ and every space-like segment Γ in the domain of dependency of $\widetilde{\Gamma}$.

By standard interaction estimates, it is well known that every solution of (1.1) constructed by wave-front tracking or by the Glimm scheme satisfies the above assumption. More generally, we expect that any approximation scheme allowing control of the total variation will produce solutions which satisfy (A3). It is conceivable that (A3) may be deduced from (A1)-(A2), for arbitrary solutions with bounded variation. Based on the theory of generalized characteristics [6], the detailed analysis in [19] shows that the property (A3) actually holds for every *countably regular* BV solution of a strictly hyperbolic, genuinely nonlinear 2×2 system of conservation laws.

The main result proved in this paper is the following.

Theorem 2. *Let the system (1.1) generate the Standard Riemann Semigroup* $S:\mathcal{D}\times[0,\infty[\,\mapsto\mathcal{D}$. *Then, for every* $\bar u\in\mathcal{D}$, $T>0$, *the Cauchy problem (1.1)-(1.2) has a unique weak solution* $u:[0,T]\mapsto\mathcal{D}$ *satisfying the assumptions* **(A1)–(A3)**.

A proof of Theorem 2 will be given in Section 3.

2 - Preliminary results.

Lemma 1. *Let* $u:[0,T]\times I\!\!R\mapsto I\!\!R^n$ *be any function which satisfies the Tame Variation Assumption (A3), with Tot.Var.*$\big(u(0,\cdot)\big)<\infty$. *Then the map* $t\mapsto u(t,\cdot)$ *is Lipschitz continuous, i.e.*

$$\big\|u(t,\cdot)-u(s,\cdot)\big\|_{\mathbf{L}^1}\leq L'|t-s| \tag{2.1}$$

for some constant L' *and all* $t,s\in[0,T]$.

Proof. The condition (1.12) implies

$$\text{Tot.Var.}\big(u(\tau,\cdot)\big)\leq K\cdot\text{Tot.Var.}\big(u(0,\cdot)\big)\qquad\forall\tau\in[0,T]. \tag{2.2}$$

Consider any interval $[\tau,\ \tau+h]\subseteq[0,T]$. For each $x\in I\!\!R$, we can apply (1.12) choosing $\widetilde{\Gamma}$ to be the segment with endpoints $(\tau,x-h\hat\lambda)$, $(\tau,x+h\hat\lambda)$, and Γ the segment with endpoints $(\tau+h,x)$, $(\tau,x+h\hat\lambda)$. This yields

$$\begin{aligned}\big|u(\tau+h,x)-u(\tau,x)\big|&\leq\big|u(\tau+h,x)-u(\tau,x+h\hat\lambda)\big|+\big|u(\tau,x+h\hat\lambda)-u(\tau,x)\big|\\&\leq(K+1)\cdot\text{Tot.Var.}\big(u(\tau);\ [x-h\hat\lambda,\ x+h\hat\lambda]\big).\end{aligned} \tag{2.3}$$

Introduce the function

$$\Phi(x)\doteq\text{Tot.Var.}\big(u(\tau);\ ]-\infty,x]\big).$$

Integrating (2.3) and recalling (2.2) one obtains

$$\begin{aligned}\int_{-\infty}^{\infty} \big|u(\tau+h,x)-u(\tau,x)\big|\,dx &\le (K+1)\int_{-\infty}^{\infty}\big[\Phi(x+h\hat\lambda)-\Phi(x-h\hat\lambda)\big]\,dx\\ &= 2h\hat\lambda(K+1)\cdot \text{Tot.Var.}\big(u(\tau)\big)\\ &\le 2h\hat\lambda K(K+1)\cdot \text{Tot.Var.}\big(u(0)\big).\end{aligned}$$

This yields (2.1) with Lipschitz constant $L' = 2\hat\lambda K(K+1)\cdot \text{Tot.Var.}\big(u(0)\big)$.

Lemma 2. *Let $w :]a,b[\mapsto I\!R^n$ be a function such that*

$$\sup_{a<\xi_1<\xi_2<b}\left|\int_{\xi_1}^{\xi_2} w(x)\,dx\right| \le C_2(b-a)\cdot V^2, \tag{2.4}$$

$$\textit{Tot. Var.}\big(w;\ ]a,b[\big) < 2KV \tag{2.5}$$

for some constants C_2, K, V, with $V < 1/4$. Then

$$\int_a^b \big|w(x)\big|\,dx \le \big(C_2+4K\big)(b-a)V^{3/2}. \tag{2.6}$$

Proof. It suffices to prove (2.6) assuming that w is a smooth function. By a density argument, the same inequality must then hold for every BV function. Let x, h be given, with

$$0 < h < (b-a)/2, \qquad\qquad 0 < x < (a+b)/2.$$

Integrating the identity

$$w(x) = w(y) - \int_x^y w_x(z)\,dz$$

for $y \in [x,\ x+h]$, we obtain

$$h\,|w(x)| \le \left|\int_x^{x+h} w(y)\,dy\right| + \int_x^{x+h}\int_x^y \big|w_x(z)\big|\,dzdy.$$

Dividing by h and integrating once more we arrive at

$$\begin{aligned}\int_a^{(a+b)/2}\big|w(x)\big|\,dx &\le \frac{1}{h}\int_a^{(a+b)/2}\left|\int_x^{x+h} w(y)\,dy\right|dx + \frac{1}{h}\int_a^{(a+b)/2}\int_x^{x+h}\int_x^y \big|w_x(z)\big|\,dzdydx\\ &\doteq E_1+E_2.\end{aligned} \tag{2.7}$$

The first term on the right hand side of (2.7) is bounded by

$$E_1 \le \frac{b-a}{2h}\cdot \sup_{a<\xi_1<\xi_2<b}\left|\int_{\xi_1}^{\xi_2} w(x)\,dx\right| \le C_2\frac{(b-a)^2}{2h}\cdot V^2. \tag{2.8}$$

Introducing the (nondecreasing) function

$$\Phi(x) \doteq \int_a^x \big|w_x(z)\big| dz = \text{Tot.Var.}\big(w;\ [a,x]\big),$$

the second term can be estimated by

$$\begin{aligned}
E_2 &= \frac{1}{h}\int_a^{(a+b)/2}\int_x^{x+h}\big[\Phi(y)-\Phi(x)\big]\ dydx \\
&\leq \int_a^{(a+b)/2}\frac{1}{h}\int_x^{x+h}\big[\Phi(x+h)-\Phi(x)\big]\ dydx \\
&= \left(\int_{(a+b)/2}^{(a+b)/2+h} - \int_a^{a+h}\right)\Phi(x)\ dx \qquad (2.9)\\
&\leq h\cdot\Phi\Big(\frac{a+b}{2}+h\Big) \\
&\leq h\cdot\text{Tot.Var.}\big(w;\ ]a,b[\big) \\
&\leq h\cdot 2KV\,.
\end{aligned}$$

Choosing

$$h \doteq (b-a)V^{1/2}\ <\ \frac{b-a}{2}, \tag{2.10}$$

from the estimates (2.7)-(2.9) we deduce

$$\int_a^{(a+b)/2}\big|w(x)\big|\ dx \leq \Big(\frac{C_2}{2}+2K\Big)(b-a)V^{3/2}.$$

An entirely similar argument shows that this estimate is also valid for the subinterval $\big[(a+b)/2,\ b\big]$, completing the proof.

Lemma 3. *Assume that the function u satisfies the Tame Variation Assumption (A3). Let (τ,ξ) be a point of discontinuity for the function $x\mapsto u(\tau,x)$ (with τ fixed) and denote by $u(\tau,\xi\pm)$ its left and right limits at $x=\xi$. Then we have*

$$\big|u(t,x)-u(\tau,\xi+)\big| \leq (K+1)\cdot \mathit{Tot.Var.}\big(u;\ ]\xi,2x-\xi]\big) \qquad \mathit{if}\quad x>\xi+\hat\lambda(t-\tau),\ t\geq\tau, \tag{2.11}$$

$$\big|u(t,x)-u(\tau,\xi-)\big| \leq (K+1)\cdot \mathit{Tot.Var.}\big(u;\ [2x-\xi,\xi[\big) \qquad \mathit{if}\quad x>\xi-\hat\lambda(t-\tau),\ t\geq\tau. \tag{2.12}$$

In particular if u satisfies the conditions (1.9)-(1.10) in (A2) for some u^+, u^- and λ, then $u^\pm = u(\tau,\xi\pm)$.

Proof. Consider for instance a point (t,x) with $x\geq\xi+\hat\lambda(t-\tau)$. The estimate (2.11) is then obtained from (1.14), choosing $\widetilde\Gamma$ to be the segment with endpoints $\big(\tau,x-\hat\lambda(t-\tau)\big)$, $(\tau,2x-\xi)$ and Γ the segment with endpoints (t,x), $(\tau,2x-\xi)$.

Observing that the total variation in the right hand side of (2.11) converges to zero uniformly as $(x,t)\to(\tau,\xi)$, we deduce

$$\lim_{\rho\to 0+}\frac{1}{\rho^2}\int_\tau^{\tau+\rho/\hat\lambda}\int_{\xi+\hat\lambda(t-\tau)}^{\xi+\rho}\big|u(t,x)-u(\tau,\xi+)\big|\,dxdt\le\lim_{\rho\to 0+}\frac{K+1}{\hat\lambda}\cdot\mathrm{Tot.Var.}\big(u;\,]\xi,\xi+2\rho[\big)=0. \tag{2.13}$$

Comparing (2.13) with (1.10), we conclude that $u^+=u(\tau,\xi+)$. Similarly, $u^-=u(\tau,\xi-)$.

Lemma 4. *Let u be a weak solution of (1.1). Let the conditions (1.9)-(1.10) hold at some point (τ,ξ). Then the states u^-,u^+ and the speed λ satisfy the Rankine-Hugoniot equations*

$$\lambda(u^+-u^-)=f(u^+)-f(u^-). \tag{2.14}$$

Together with the entropy condition in (A2), Lemma 4 shows that the states u^-,u^+ are connected either by an admissible shock or by a contact discontinuity, propagating with the speed λ.

Proof of Lemma 4. To derive (2.14), we integrate the conservation laws (1.1) on a family of parallelograms with a fixed center (τ,ξ):

$$D^s\doteq\Big\{(x,t):\quad |t-\tau|\le s,\quad \big|x-\xi-\lambda(t-\tau)\big|\le s\Big\}\qquad (s>0).$$

From the basic equations (1.1) we deduce

$$\begin{aligned}\int_{\xi-s+\lambda s}^{\xi+s+\lambda s}u(\tau+s,x)\,dx-\int_{\xi-s-\lambda s}^{\xi+s-\lambda s}u(\tau-s,x)\,dx+\int_{\xi-s}^{\xi+s}\big(f(u)-\lambda u\big)(t,\xi+s+\lambda(t-\tau))\,dt\\ -\int_{\xi-s}^{\xi+s}\big(f(u)-\lambda u\big)(t,\xi-s+\lambda(t-\tau))\,dt=0.\end{aligned}$$

Integrating once more for $s\in]0,\rho]$ and dividing by ρ^2 we obtain

$$\begin{aligned}&\frac{1}{\rho^2}\int_0^\rho\int_{\xi-s+\lambda s}^{\xi+s+\lambda s}u(\tau+s,x)\,dxds-\frac{1}{\rho^2}\int_0^\rho\int_{\xi-s-\lambda s}^{\xi+s-\lambda s}u(\tau-s,x)\,dxds\\ &\quad+\frac{1}{\rho^2}\int_0^\rho\int_{\xi-s}^{\xi+s}\big(f(u)-\lambda u\big)(t,\xi+s+\lambda(t-\tau))\,dtds\\ &\quad-\frac{1}{\rho^2}\int_0^\rho\int_{\xi-s}^{\xi+s}\big(f(u)-\lambda u\big)(t,\xi-s+\lambda(t-\tau))\,dtds\\ &\quad=0.\end{aligned} \tag{2.15}$$

By (1.9)-(1.10), each of the four terms on the left hand side of (2.15) has a limit when $\rho\to 0$. For instance

$$\frac{1}{\rho^2}\int_0^\rho\int_{\xi-s+\lambda s}^{\xi+s+\lambda s}u(\tau+s,x)\,dxds\to\frac{u^-+u^+}{2}$$

$$\frac{1}{\rho^2}\int_0^\rho\int_{\xi-s}^{\xi+s}\big(f(u)-\lambda u\big)(t,\xi+s+\lambda(t-\tau))dtds \quad\to\quad f(u^+)-\lambda u^+.$$

We thus obtain

$$\frac{u^-+u^+}{2}-\frac{u^-+u^+}{2}+\big(f(u^+)-\lambda u^+\big)-\big(f(u^-)-\lambda u^-\big)=0,$$

proving (2.14).

We conclude this section by recalling two local integral estimates that characterize the trajectories of a Standard Riemann Semigroup.

Two types of local approximate solutions for (1.1) will be considered. One is derived from the self-similar solution of a Riemann problem, the other is obtained by "freezing" the coefficients of the corresponding quasilinear hyperbolic system in a neighborhood of a given point.

Let $w : I\!R \mapsto I\!R^n$ be any BV function and fix any point $\xi \in I\!R$ where w has a jump. Call $\omega = \omega(t,x)$ the unique self-similar entropy solution of the Riemann problem

$$u_t + F(u)_x = 0, \qquad\qquad u(0,x)=\begin{cases} w(\xi-) & \text{if} \quad x<0,\\ w(\xi+) & \text{if} \quad x>0.\end{cases} \tag{2.16}$$

As usual, let $\hat\lambda$ be an upper bound for all wave speeds. For $t\geq\tau$, define

$$U^\sharp(t,x)\doteq\begin{cases}\omega(t,\ x-\xi) & \text{if} \quad |x-\xi|\leq\hat\lambda t,\\ w(x) & \text{if} \quad |x-\xi|>\hat\lambda t.\end{cases} \tag{2.17}$$

Observe that the function $t\mapsto U^\sharp(t,\cdot)$ is Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance, and approaches w as $t\to 0+$.

Next, call $\widetilde A\doteq DF\big(w(\xi)\big)$ the Jacobian matrix of F computed at the point $w(\xi)$. For $t\geq 0$, define $U^\flat(t,x)$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$U^\flat_t+\widetilde A U^\flat_x=0, \qquad\qquad U^\flat(0)=w. \tag{2.18}$$

Lemma 5. *Let S be a Standard Riemann Semigroup generated by the system (1.1), with domain $\mathcal{D}$ as in (1.3). Then, there exists a constant C_1 such that the following holds. For every function $w\in\mathcal{D}$, every $\xi\in I\!R$ and $h,\rho>0$, with the above definitions one has*

$$\frac{1}{h}\int_{\xi-\rho+h\hat\lambda}^{\xi+\rho-h\hat\lambda}\Big|(S_hw)(x)-U^\sharp(h,x)\Big|\,dx\ \leq\ C_1\cdot \text{Tot.Var.}\big(w;\ ]\xi-\rho,\ \xi[\,\cup\,]\xi,\ \xi+\rho[\big), \tag{2.19}$$

$$\frac{1}{h}\int_{\xi-\rho+h\hat\lambda}^{\xi+\rho-h\hat\lambda}\Big|(S_hw)(x)-U^\flat(h,x)\Big|\,dx\leq C_1\cdot\Big(\text{Tot.Var.}\big(w;\ ]\xi-\rho,\ \xi+\rho[\big)\Big)^2. \tag{2.20}$$

For a proof, see [1, p.217]. Observe that any solution $U^\flat$ of the linear system (2.18) will certainly satisfy the Tame Variation Assumption (A3), possibly with a larger constant K, depending only on the first order derivatives of F. In particular, for every $\xi \in I\!R$, $\rho > 0$ we can assume

$$\text{Tot.Var.}\big(U^\flat(\rho/\hat\lambda);\ ]\xi-\rho,\ \xi+\rho[\big) \leq K \cdot \text{Tot.Var.}\big(U^\flat(0);\ ]\xi-2\rho,\ \xi+2\rho[\,\big). \tag{2.21}$$

3 - Proof of Theorem 2.

It suffices to show that $u(T,\cdot) = S_T\bar u$. Indeed, the same assumptions (A1)–(A3) hold on any interval $[0,T'] \subset [0,T]$.

By (A3) and Lemma 1, u is a BV function on the strip $]0,T[\,\times I\!R$, in the sense that the distributional derivatives $D_t u, D_x u$ are Radon measures. By a well known structure theorem [9, 20], there exists a set $\widetilde{\mathcal{N}} \subset]0,T[\,\times I\!R$ of 1-dimensional Hausdorff measure zero such that, at every point $(\tau,\xi) \notin \widetilde{\mathcal{N}}$, u either is continuous or has an approximate jump discontinuity. We claim that, at every such point of discontinuity, the jump cannot have a horizontal tangent. In other words, one can never have (1.10) in the case

$$U(t,x) = \begin{cases} u^- & \text{if} \quad t < \tau, \\ u^+ & \text{if} \quad t > \tau, \end{cases} \qquad u^+ \neq u^-. \tag{3.1}$$

Indeed, for every positive constant c (arbitrarily large), from (3.1), (1.9) and (2.1) we deduce

$$c|u^+ - u^-| = \lim_{\rho\to 0+} \frac{1}{\rho^2}\int_\tau^{\tau+\rho}\int_\xi^{\xi+c\rho} |u^+ - u^-|\,dxdt = \lim_{\rho\to 0+} \frac{1}{\rho^2}\int_\tau^{\tau+\rho}\int_\xi^{\xi+c\rho} \big|u(t,x) - u(-t,x)\big|\,dxdt$$

$$\leq \lim_{\rho\to 0+} \frac{1}{\rho^2}\int_\tau^{\tau+\rho} L'\cdot 2t\ dt = L'.$$

Hence $u^+ = u^-$, against the second assumption in (3.1).

Taking the projection of $\widetilde{\mathcal{N}}$ on the t-axis, we conclude that there exists a set $\mathcal{N} \subset [0,T]$ of measure zero (containing the endpoints 0 and T) such that, at every point $(\tau,\xi) \in [0,T] \times I\!R$ with $\tau \notin \mathcal{N}$, the following property holds.

(P) Either u is continuous at (τ,ξ), or it has an approximate jump discontinuity, in the sense that (1.9)-(1.10) hold for some states u^-, u^+ and some $\lambda \in I\!R$. In this second case, by Lemma 3, one has $u^- = u(\tau,\xi-)$, $u^+ = u(\tau,\xi+)$. Moreover, by Lemma 4 the Rankine-Hugoniot equations (2.14) hold. Hence, by (A2) the entropy condition (1.11) holds as well.

In the next section of the proof we establish the basic relation

$$\liminf_{h\to 0+} \frac{1}{h}\big\|u(\tau+h) - S_h u(\tau)\big\|_{\mathbf{L}^1} = 0 \qquad \text{for all} \quad \tau \notin \mathcal{N}. \tag{3.2}$$

Let any $\tau \in [0, T[$ be given. Fix any $\xi \in I\!R$ and $\rho > 0$. Define $U^\flat$ to be the solution of the linear hyperbolic problem with constant coefficients

$$U_t^\flat + \widetilde{A} U_x^\flat = 0, \qquad U^\flat(\tau) = u(\tau), \tag{3.3}$$

where $\widetilde{A} \doteq DF\big(u(\tau,\xi)\big)$. Call $\tilde\lambda_1, \ldots \tilde\lambda_n$ the eigenvalues of $\widetilde{A}$ and let $\tilde l_i$, $\tilde r_i$ be the corresponding left and right eigenvectors, normalized as in (1.7). Consider the time step $h = \rho/\hat\lambda$, where $\hat\lambda$ is the given upper bound for all wave speeds. By the assumption (A3), for every (t, x) in the interior of the trapezoidal region

$$D_\rho \doteq \Big\{(t,x); \quad t \in [\tau,\ \tau+h], \quad |x - \xi| \le 2\rho - \hat\lambda(t-\tau)\Big\},$$

the same argument used in (2.3) yields the estimate

$$\big|u(t,x) - u(\tau,\xi)\big| \le (K+1)\cdot \text{Tot.Var.}\big(u(\tau);\ ]\xi - 2\rho,\ \xi + 2\rho[\,\big). \tag{3.4}$$

For every $i = 1, \ldots, n$ and every choice of ξ_1, ξ_2, with $\xi - \rho < \xi_1 < \xi_2 < \xi + \rho$, we now estimate the quantity

$$E_i \doteq \int_{\xi_1}^{\xi_2} \big\langle \tilde l_i,\ u(\tau+h, x) - U^\flat(\tau+h, x)\big\rangle\, dx. \tag{3.5}$$

Observing that (3.3) yields

$$\big\langle \tilde l_i,\ U^\flat(\tau+h, x)\big\rangle = \big\langle \tilde l_i,\ U^\flat(\tau, x - h\tilde\lambda_i)\big\rangle = \big\langle \tilde l_i,\ u(\tau, x - h\tilde\lambda_i)\big\rangle,$$

and integrating (1.1) over the domain

$$D_i \doteq \Big\{(t,x); \ \ t \in [\tau,\ \tau+h], \ \ \xi_1 + (t-\tau-h)\tilde\lambda_i \le x \le \xi_2 + (t-\tau-h)\tilde\lambda_i\Big\},$$

we obtain

$$\begin{aligned} E_i = &\int_\tau^{\tau+h} \Big\langle \tilde l_i,\ \big(F(u) - \tilde\lambda_i u\big)(t,\ \xi_1 + (t-\tau-h)\tilde\lambda_i)\Big\rangle\, dt \\ &- \int_\tau^{\tau+h} \Big\langle \tilde l_i,\ \big(F(u) - \tilde\lambda_i u\big)(t,\ \xi_2 + (t-\tau-h)\tilde\lambda_i)\Big\rangle\, dt. \end{aligned} \tag{3.6}$$

To estimate the quantity in (3.6), consider the states

$$u' \doteq u(t,\ \xi_1 + (t-\tau-h)\tilde\lambda_i), \qquad u'' \doteq u(t,\ \xi_2 + (t-\tau-h)\tilde\lambda_i), \qquad \tilde u \doteq u(\tau, \xi).$$

We then have

$$\Big\langle \tilde{l}_i,\ F(u'') - F(u') - \tilde{\lambda}_i(u''-u')\Big\rangle = \Big\langle \tilde{l}_i,\ DF(\tilde{u})\cdot(u''-u') - \tilde{\lambda}_i(u''-u')\Big\rangle + \Big\langle \tilde{l}_i,\ A^*(u''-u')\Big\rangle,$$

where the matrix A^* is defined by

$$A^* \doteq \int_0^1 \Big[DF\big(su''+(1-s)u'\big) - DF(\tilde{u})\Big]\, ds.$$

Therefore,

$$\left|\Big\langle \tilde{l}_i,\ F(u'') - F(u') - \tilde{\lambda}_i(u''-u')\Big\rangle\right| \ \le\ C\cdot\Big(\max\{|u'-\tilde{u}|,\ |u''-\tilde{u}|\}\Big)^2, \tag{3.7}$$

for some constant C depending only on the second order derivatives of the function F. Applying (3.7) to the integrand in (3.6) and recalling (3.4) we thus obtain

$$\begin{aligned}|E_i| &\le Ch\cdot\Big(\sup_{(t,x)\in\mathcal{D}_i}\big|u(t,x)-u(\tau,\xi)\big|\Big)^2\\ &\le Ch(K+1)^2\cdot\Big(\text{Tot.Var.}\big(u(\tau);\ ]\xi-2\rho,\ \xi+2\rho[\big)\Big)^2.\end{aligned}$$

Since the above estimate holds for all $i=1,\ldots,n$, we conclude

$$\sup_{\xi-\rho<\xi_1<\xi_2<\xi+\rho}\left|\int_{\xi_1}^{\xi_2}\Big(u(\tau+h,x)-U^\flat(\tau+h,x)\Big)\,dx\right| \le 2C_2\rho\cdot\Big(\text{Tot.Var.}\big(u(\tau);\ ]\xi-2\rho,\ \xi+2\rho[\big)\Big)^2, \tag{3.8}$$

for some constant C_2 and all $\xi\in\mathbb{R}$ and $\rho>0$, with $h=\rho/\hat{\lambda}$.

We now derive an additional estimate which is appropriate at points of discontinuity. Fix any $\tau\notin\mathcal{N}$. Let $\xi_1'<\cdots<\xi_N'$ be any finite collection of points where $u(\tau,\cdot)$ is discontinuous. Since $\tau\notin\mathcal{N}$, for each $\alpha=1,\ldots,N$ by the property (P) the two states $u_\alpha^\pm \doteq u(\tau,\xi_\alpha'\pm)$ are connected by an entropy-admissible shock, travelling with some speed λ^α. Moreover, introducing the functions

$$U_\alpha(t,x) = \begin{cases} u_\alpha^- & \text{if} \quad x<\xi_\alpha'+\lambda^\alpha(t-\tau),\\ u_\alpha^+ & \text{if} \quad x>\xi_\alpha'+\lambda^\alpha(t-\tau),\end{cases} \tag{3.9}$$

one has

$$\lim_{\rho\to 0+}\frac{1}{\rho}\int_{\tau-\rho}^{\tau+\rho}\int_{\xi_\alpha-\rho}^{\xi_\alpha+\rho}\big|u(t,x)-U_\alpha(t,x)\big|\,dxdt = 0 \qquad (\alpha=1,\ldots,N). \tag{3.10}$$

We claim that there exists a sequence of positive numbers h_ν decreasing to zero such that

$$\lim_{\nu\to\infty}\sum_{\alpha=1}^N\frac{1}{h_\nu}\int_{\xi_\alpha'-\hat{\lambda}h_\nu}^{\xi_\alpha'+\hat{\lambda}h_\nu}\big|u(\tau+h_\nu,x)-U_\alpha(\tau+h_\nu,x)\big|\,dx = 0. \tag{3.11}$$

Indeed, in the opposite case, we would have

$$\sum_{\alpha=1}^{N} \frac{1}{h} \int_{\xi'_\alpha - \hat\lambda h}^{\xi'_\alpha + \hat\lambda h} |u(\tau+h,x) - U_\alpha(\tau+h,x)|\, dx \geq \delta,$$

for some $\delta > 0$ and all h small enough. Therefore,

$$\liminf_{h\to 0+} \sum_{\alpha=1}^{N} \frac{1}{h^2} \int_0^h \int_{\xi'_\alpha - \hat\lambda s}^{\xi'_\alpha + \hat\lambda s} \big|u(\tau+s,x) - U_\alpha(\tau+s,x)\big|\, dxds \geq \delta > 0,$$

in contradiction with (3.10).

We are now in the position to prove the basic estimate (3.2). Fix any $\varepsilon \in]0,\ 1/4[$. Let $\xi'_1 < \ldots < \xi'_N$ be the points where the total variation measure of $u(\tau)$ exceeds the value ε, i.e.

$$\big|u(\tau,\xi'_\alpha+) - u(\tau,\xi'_\alpha-)\big| \geq \varepsilon.$$

Define the corresponding functions U_α as in (3.9). By the previous arguments, there exists a sequence $h_\nu \to 0+$ such that

$$\sum_{\alpha=1}^{N} \frac{1}{h_\nu} \int_{\xi'_\alpha - \hat\lambda h_\nu}^{\xi'_\alpha + \hat\lambda h_\nu} \Big|u(\tau+h_\nu,x) - U_\alpha(\tau+h_\nu,x)\Big|\, dx \leq \varepsilon \tag{3.12}$$

for every ν. ¿From (2.19) in Lemma 5 it follows that the semigroup trajectory satisfies an entirely similar estimate:

$$\begin{aligned}
&\sum_{\alpha=1}^{N} \frac{1}{h_\nu} \int_{\xi'_\alpha - \hat\lambda h_\nu}^{\xi'_\alpha + \hat\lambda h_\nu} \Big|\big(S_{h_\nu} u(\tau)\big)(x) - U_\alpha(\tau+h_\nu,x)\Big|\, dx \\
&\quad \leq C_1 \cdot \sum_{\alpha=1}^{N} \text{Tot.Var.}\Big(u(\tau);\ ]\xi'_\alpha - 2\hat\lambda h_\nu,\ \xi'_\alpha[\cup]\xi'_\alpha,\ \xi'_\alpha + 2\hat\lambda h_\nu[\Big) \\
&\quad \leq \varepsilon,
\end{aligned} \tag{3.13}$$

provided that h_ν is sufficiently small. Combining (3.12) and (3.13) we deduce

$$\sum_{\alpha=1}^{N} \frac{1}{h_\nu} \int_{\xi'_\alpha - \hat\lambda h_\nu}^{\xi'_\alpha + \hat\lambda h_\nu} \Big|u(t,x) - \big(S_{h_\nu} u(\tau)\big)(x)\Big|\, dx \ \leq\ 2\varepsilon. \tag{3.14}$$

Next, we consider all the remaining points, where $u(\tau)$ either is continuous or has a jump of size $< \varepsilon$. For each ν, define $\rho_\nu \doteq \hat\lambda h_\nu$. Choose $\bar\nu$ large enough so that, for all $\nu > \bar\nu$, the estimate (3.14) holds together with the properties

$$4\rho_\nu < \min_{\alpha=2,\cdots,N} |\xi'_\alpha - \xi'_{\alpha-1}|, \tag{3.15}$$

$$\text{Tot.Var.}\big(u(\tau);\,]a,b[\big) \leq \varepsilon < \frac{1}{4} \qquad \text{whenever} \quad b-a \leq 4\,\rho_\nu, \quad]a,b[\,\cap\,\{\xi'_1,\ldots\xi'_N\} = \emptyset. \tag{3.16}$$

Such a $\bar\nu$ certainly exists, because there are only finitely many points ξ'_α and the total variation of $u(\tau)$ is bounded.

For any given $\nu > \bar\nu$ we can now select a countable (but locally finite) set of points $\{\xi_k;\ k \geq 1\}$ such that the intervals

$$I_k \doteq [\xi_k - \rho_\nu,\ \xi_k + \rho_\nu], \qquad J_k \doteq [\xi_k - 2\rho_\nu,\ \xi_k + 2\rho_\nu]$$

satisfy the properties:

(i) $\bigcup_k I_k \ \supseteq\ I\!\!R \setminus \bigcup_{\alpha=1}^N [\xi'_\alpha - \rho_\nu,\ \xi'_\alpha + \rho_\nu]$.

(ii) $\xi'_\alpha \notin J_k$ for every α, k.

(iii) Every point $x \in I\!\!R$ is contained in at most three distinct intervals J_k.

For each k, call $U^\flat_k$ the solution of (3.3) with $\widetilde A = DF\big(u(\tau,\xi_k)\big)$, and define

$$V_k \doteq \text{Tot.Var.}\big(u(\tau);\ \xi_k - 2\rho_\nu,\ \xi_k + 2\rho_\nu[\,\big).$$

By (3.16), $V_k < 1/4$. ¿From the estimates (3.8), (1.12) and (2.21) it follows

$$\sup_{\xi_k-\rho_\nu<\xi_1<\xi_2<\xi_k+\rho_\nu} \left| \int_{\xi_1}^{\xi_2} \Big(u(\tau+h_\nu,x) - U^\flat(\tau+h_\nu,x)\Big)\,dx \right| \leq 2C_2\rho_\nu \cdot V_k^2, \tag{3.17}$$

$$\text{Tot.Var.}\big(u(\tau+h_\nu) - U^\flat(\tau+h_\nu);\ ]\xi_k-\rho_\nu,\ \xi_k+\rho_\nu[\,\big) \leq 2KV_k. \tag{3.18}$$

Applying Lemma 2 to the function $w = u(\tau+h_\nu) - U^\flat_k(\tau+h_\nu)$, we obtain

$$\int_{\xi_k-\rho_\nu}^{\xi_k+\rho_\nu} \Big|u(\tau+h_\nu,x) - U^\flat_k(\tau+h_\nu,x)\Big|\,dx \leq C_3 \cdot 2\rho_\nu V_k^{3/2}, \tag{3.19}$$

with $C_3 = C_2 + 4K$. On the other hand, applying (2.20) with $w = u(\tau)$ we deduce

$$\frac{1}{h_\nu}\int_{\xi_k-\rho_\nu}^{\xi_k+\rho_\nu} \Big|\big(S_{h_\nu}u(\tau)\big)(x) - U^\flat_k(\tau+h_\nu,x)\Big|\,dx \leq C_2\Big(\text{Tot.Var.}\big(u(\tau);\ ]\xi_k-\rho_\nu,\xi_k+\rho_\nu[\,\big)\Big)^2 \tag{3.20}$$

The two inequalities (3.19) and (3.20) together yield

$$\frac{1}{h_\nu}\int_{\xi_k-\rho_\nu}^{\xi_k+\rho_\nu} \Big|u(\tau+h_\nu,x) - \big(S_{h_\nu}u(\tau)\big)(x)\Big|\,dx \leq \big(2\hat\lambda C_3 + C_2\big)\,\text{Tot.Var.}\big(u(\tau);\ ]\xi_k-\rho_\nu,\ \xi_k+\rho_\nu[\,\big)\,\varepsilon^{1/2}. \tag{3.21}$$

Recalling that each point x can belong to at most three of the intervals J_k, summing over k we obtain

$$\sum_k \frac{1}{h_\nu}\int_{\xi_k-\rho_\nu/2}^{\xi_k+\rho_\nu/2}\Big|u(\tau+h_\nu,x)-\big(S_{h_\nu}u(\tau)\big)(x)\Big|\,dx \le 3\,\big(2\hat\lambda C_3+C_2\big)\,\varepsilon^{1/2}\cdot \text{Tot.Var.}\big(u(\tau)\big). \quad (3.22)$$

¿From (3.14) and (3.22) we finally obtain

$$\frac{1}{h_\nu}\int_{-\infty}^{\infty}\Big|u(\tau+h_\nu,x)-\big(S_{h_\nu}u(\tau)\big)(x)\Big|\,dx \le 2\varepsilon+3\,\big(2\hat\lambda C_1+C_2\big)\cdot \text{Tot.Var.}\big(u(\tau)\big)\;\varepsilon^{1/2}, \quad (3.23)$$

for every h_ν with $\nu>\bar\nu$. Therefore,

$$\liminf_{h\to 0+}\ \frac{1}{h}\big\|u(\tau+h)-S_h u(\tau)\big\|_{\mathbf{L}^1}\le 2\varepsilon+3(2\hat\lambda C_3+C_2)\cdot \text{Tot.Var.}\big(u(\tau)\big)\varepsilon^{1/2}.$$

Since $\varepsilon>0$ was arbitrary, this establishes (3.2).

We can now complete the proof of Theorem 2.

Let L, L' be the Lipschitz constants of the semigroup S, and of the map $t\mapsto u(t,\cdot)$, so that

$$\big\|S_t\bar u-S_s\bar v\big\|\le L\big(\|\bar u-\bar v\|+|t-s|\big), \quad (3.24)$$

$$\big\|u(t)-u(s)\big\|\le L'|t-s|. \quad (3.25)$$

Fix any $\varepsilon>0$. Choose an open set $J\subset I\!R$ such that

$$J\supseteq \mathcal{N}\cup\{0,T\},\qquad \text{meas}(J)<\varepsilon. \quad (3.26)$$

Consider the time

$$\tau\ \doteq\ \sup\Big\{t\in[0,T];\ \ \big\|S_{T-t}u(t)-S_T\bar u\big\|\le\varepsilon Lt+L(L'+L)\cdot\text{meas}\big([0,t]\cap J\big)\Big\}. \quad (3.27)$$

We claim that $\tau=T$. To show this, assume that, on the contrary, $\tau<T$. By continuity, we then have

$$\big\|S_{T-\tau}u(\tau)-S_T\bar u\big\|\le\varepsilon L\tau+L(L'+L)\cdot\text{meas}\big([0,\tau]\cap J\big)\Big\}. \quad (3.28)$$

Two cases may occur.

CASE 1: $\tau\in J$. Since J is an open set, there exists $h\in]0,\ T-\tau]$ such that $[\tau,\ \tau+h]\subset J$. By (3.24)-(3.25) and (3.28) it follows

$$\begin{aligned}\big\|S_{T-\tau-h}u(\tau+h)-S_T\bar u\big\| &\le \big\|S_{T-\tau-h}u(\tau+h)-S_{T-\tau-h}u(\tau)\big\|+\big\|S_{T-\tau-h}u(\tau)-S_{T-\tau-h}S_h u(\tau)\big\|\\ &\qquad+\big\|S_{T-\tau}u(\tau)-S_T\bar u\big\|\\ &\le L(L'h)+L(Lh)+\varepsilon L\tau+L(L'+L)\cdot\text{meas}\big([0,\tau]\cap J\big)\\ &< \varepsilon L(\tau+h)+L(L'+L)\cdot\text{meas}\big([0,\ \tau+h]\cap J\big).\end{aligned}$$

This contradicts the maximality of τ.

CASE 2: $\tau \notin J$. Since $\mathcal{N} \subset J$, we thus have $\tau \notin \mathcal{N}$. By (3.2), there exists $h \in]0,\ T-\tau]$ such that

$$\big\|u(\tau+h) - S_h u(\tau)\big\| < \varepsilon h.$$

In turn, this implies

$$\begin{aligned}\big\|S_{T-\tau-h} u(\tau+h) - S_T \bar{u}\big\| &\leq \big\|S_{T-\tau-h} u(\tau+h) - S_{T-\tau-h} S_h u(\tau)\big\| + \big\|S_{T-\tau} u(\tau) - S_T \bar{u}\big\| \\ &< \varepsilon L h + \varepsilon L \tau + L(L'+L) \cdot \mathrm{meas}\big([0,\tau] \cap I\big) \\ &\leq \varepsilon L(\tau+h) + L(L'+L) \cdot \mathrm{meas}\big([0,\ \tau+h] \cap J\big),\end{aligned}$$

again contradicting the maximality of τ.

Using (3.28) with $\tau = T$ and then (3.27), we obtain

$$\big\|u(T) - S_T \bar{u}\big\| \leq \varepsilon L T + L(L'+L) \cdot \mathrm{meas}(J) \leq \varepsilon\big(LT + L(L'+L)\big).$$

Since $\varepsilon > 0$ was arbitrary, the theorem is proved.

Remarks. The assumption (A2) can be relaxed, requiring (1.11) to hold only for τ outside a set $\mathcal{N} \subset [0,T]$ of measure zero.

Another basic assumption in Theorem 2 is the existence of a Standard Riemann Semigroup $S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$ such that $u(t,\cdot) \in \mathcal{D}$ for all t. Due to the finite propagation speed, this can be replaced by the following local condition.

For every (τ,ξ), there exists $\rho > 0$ and a semigroup S such that, for all $t \in [\tau,\ \tau+\rho/\hat{\lambda}]$, the domain $\mathcal{D}$ of S contains some function w such that

$$w(x) = u(t,x) \qquad\qquad \text{for all} \quad x \in J(t),$$

where $J(t) \doteq \big[\xi - \rho + \hat{\lambda}(t-\tau),\ \xi + \rho - \hat{\lambda}(t-\tau)\big]$ and $\hat{\lambda}$ is an upper bound for all wave speeds.

The proof is obtained in the same way as before, replacing (3.2) with

$$\liminf_{h\to 0+} \frac{1}{h} \int_{J(\tau+h)} \Big| u(\tau+h,x) - \big(S_h w\big)(x)\Big|\, dx = 0,$$

where $w \in \mathcal{D}$ is any function coinciding with $u(\tau)$ on $J(\tau)$. Relying on the construction in [4], the present uniqueness result can thus be extended to some 2×2 conservation laws with large data.

References

[1] A. Bressan, The unique limit of the Glimm scheme, *Arch. Rational Mech. Anal.* **130** (1995), 205-230.

[2] A. Bressan, The semigroup approach to systems of conservation laws, *Mathematica Contemporanea*, to appear.

[3] A. Bressan and R.M. Colombo, The semigroup generated by 2×2 conservation laws, *Arch. Rational Mech. Anal.* **133** (1995), 1-75.

[4] A. Bressan and R.M. Colombo, Unique solutions of 2×2 conservation laws with large data, *Indiana Univ. Math. J.* **44** (1995), 677-725.

[5] A. Bressan, G. Crasta, and B. Piccoli, Well posedness of the Cauchy problem for $n \times n$ systems of conservation laws, preprint S.I.S.S.A., Trieste 1996.

[6] C. Dafermos, Generalized characteristics in hyperbolic systems hyperbolic systems of conservation laws, *Arch. Rat. Mech. Anal.* **107** (1989), 127-155.

[7] C. Dafermos and X. Geng, Generalized characteristics, uniqueness and regularity of solutions in a hyperbolic system of conservation laws, *Ann. Inst. Henri Poincaré - Nonlinear Analysis* **8** (1991), 231-269.

[8] R. J. DiPerna, Uniqueness of solutions to hyperbolic conservation laws, *Indiana Univ. Math. J.* **28** (1979), 137–188.

[9] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, C.R.C. Press, 1992.

[10] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 697-715.

[11] S. Kruzkov, First-order quasilinear equations with several space variables, *Mat. Sb.* **123** (1970), 228–255. English transl. in *Math. USSR Sb.* **10** (1970), 217–273.

[12] P.D. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* **10** (1957), 537-566.

[13] P.G. LeFloch and Z.P. Xin, Uniqueness via the adjoint problems for systems of conservation laws, *Comm. Pure Appl. Math.* **46** (1993), 1499–1533.

[14] T. P. Liu, Uniqueness of weak solutions of the Cauchy problem for general 2×2 conservation laws, *J. Differential Equations* **20** (1976), 369–388.

[15] O. Oleinik, Discontinuous solutions of nonlinear differential equations *Usp. Mat. Nauk.* **12** (1957), 3-73; English transl. in *Amer. Math. Soc. Transl. Ser.2,* **26**, 95-172.

[16] O. Oleinik, On the uniqueness of the generalized solution of the Cauchy problem for a nonlinear system of equations occurring in mechanics, *Usp. Mat. Nauk (N.S.)* **12** (1957), 169–176 (in Russian).

[17] M. Schatzman, Continuous Glimm functionals and uniqueness of solutions of the Riemann problem, *Indiana Univ. Math. J.* **34** (1985), 533-589.

[18] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983.

[19] K. Trivisa, A priori estimates in hyperbolic systems of conservation laws via generalized characteristics, preprint, Brown University, Providence 1996.

[20] A. I. Volpert, The space BV and quasilinear equations, *Math. USSR Sb.* **2** (1967), 257–267.