TRAVELLING WAVE SOLUTIONS FOR A BALANCE LAW

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Abstract. We study entropy travelling wave solutions for first order hyperbolic balance laws. Results concerning existence, regularity and asymptotic stability of such solutions are proved for convex fluxes and source terms with simple isolated zeros.

1. Introduction

It is the purpose of this paper to study existence, regularity and stability of travelling wave solutions for the following balance law:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = g(u), \qquad \text{in } \mathbb{R} \times \mathbb{R}_+. \tag{1.1}$$

To prove existence we assume that f is a convex function while g has isolated simple zeros. To investigate stability, we limit ourselves for simplicity to the case where the function g has two zeros, is positive between them and negative elsewhere. The general case can be dealt with similarly.

Propagation phenomena arise in many applications (e.g. spreading of populations, combustion models, continuum mechanics). In this framework many papers have been devoted to investigate travelling waves for reaction-diffusion equations (see e.g. [1], [6]). The importance of these solutions is due to the fact that they are "permanent" (namely they exist for any time t); in typical cases permanent solutions prove to be attractors for evolution equations.

In many situations both diffusion and convection effects are present. If the effect of diffusion is small, equation (1.1) can be regarded as a sound description of the evolution phenomena under consideration. The relation between this equation and the corresponding "viscous" problem has been widely investigated in the case $g \equiv 0$ (see e.g. [15]).

Much work has been devoted to investigate equation (1.1) in recent years. Existence and uniqueness of entropy solutions to the Cauchy problem in the class of the entropy solutions was established in [7], [9]. Results concerning the asymptotic behavior of entropy solutions were proved in [4], [5], [11], [12], [13], [14]. In particular, the solution of the Cauchy problem for equation (1.1) with periodic or compactly supported initial data proves to converge in a suitable sense to travelling wave solutions of (1.1) as time diverges (see [5], [11], [13], [14]). These results suggest to study in more detail this class of solutions. This is our concern in the present paper.

We give necessary and sufficient conditions for the existence of heteroclinic, homoclinic and periodic waves. Concerning heteroclinic and homoclinic waves, these conditions depend on the stability character of the asymptotic states of the waves with respect to the ordinary differential equation

Key words and phrases. Conservation laws, entropy condition, travelling waves.

$u' = g(u)$. This is in some sense natural, since this equation describes the evolution of the solution to (1.1) along the characteristic curves.

We also prove a result concerning the regularity of travelling waves. We show that, if the speed c of the wave is different from $f'(s)$ for any s such that $g(s) = 0$ and $g'(s) > 0$, the profile of the wave has at most one discontinuity. If $c = f'(s)$ for some s such that $g(s) = 0, g'(s) > 0$, there exist waves with infinitely many jump discontinuities. For heteroclinic waves the set of discontinuity points is either bounded from below or bounded from above. For homoclinic waves with an asymptotic state different from any zero s of g with $g'(s) > 0$ the set of discontinuity points is contained in a compact subset of $\mathbb{R}$.

Concerning the stability, we restrict our attention to the case where the function g has two zeros. We show the L^∞-stability character of any wave with respect to "local" perturbations – namely, bounded perturbations with support contained in a subset of $\mathbb{R}$ which depends on the wave. It is interesting to point out that the waves whose profiles are C^1 are always asymptotically stable; moreover, the waves with more than one jump are always unstable.

In Section 2 we state existence, regularity and stability results for travelling waves. In Section 3 we prove existence and regularity theorems, while Section 4 is devoted to the proof of stability results.

2. Mathematical Framework and Results

The following assumptions on f and g will be used:

$$\text{(F)} \qquad f \in C^2(\mathbb{R}),\ f' \text{ strictly increasing; } f(0) = f'(0) = 0,\ f'(\pm\infty) = \pm\infty;$$

$$\text{(G)} \qquad \begin{cases} g \in C^1(\mathbb{R}); \\ \mathcal{Z}(g) := \{x \in \mathbb{R} : g(x) = 0\} \text{ is a countable or finite set of isolated points;} \\ g'(x) \neq 0 \quad \forall x \in \mathcal{Z}(g). \end{cases}$$

Due to assumptions (G) we have:

$$\mathcal{Z}(g) = \{\dots, s_{n-1}, s_n, s_{n+1}, \dots\}$$

with $s_i < s_j$ if $i < j$. We also set:

$$\mathcal{R}(g) := \{s_n \in \mathcal{Z}(g) : g'(s_n) > 0\},$$
$$\mathcal{A}(g) := \{s_n \in \mathcal{Z}(g) : g'(s_n) < 0\}.$$

When dealing with the Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = g(u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u = u_0 \quad \text{in } \mathbb{R} \times \{0\}, \end{cases} \tag{2.1}$$

we always assume the initial data u_0 to be in the space $BV_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. If a family u_0^ε of initial data is considered, the corresponding solution of the Cauchy problem will be denoted by $u^\varepsilon = u^\varepsilon(x, t)$.

By *travelling wave* of equation (1.1) we mean a solution of the form:

$$u(x, t) = \phi(x - ct). \tag{2.2}$$

The function ϕ is referred to as the *profile* of the wave, the constant $c \in \mathbb{R}$ as its *speed.*

Due to well-known properties of hyperbolic balance laws, we look for travelling waves with a piecewise smooth profile. The profile solves a suitable ordinary differential equation, where the wave is regular and satisfies both the Rankine-Hugoniot and the entropy condition at any point of discontinuity. More precisely, we make the following

Definition 2.1. An *entropy travelling wave* for equation (1.1) is an entropy solution $u = u(x,t)$ of equation (1.1) such that:

(i) u is of the form (2.2) with $\phi = \phi(\xi)$ piecewise C^1 – namely, there exists a countable set $J \subset \mathbb{R}$ such that $\phi \in C^1(\mathbb{R} \setminus J)$ and the set of accumulation points of J (denoted by $\mathcal{D}(J)$) has only isolated points;

(ii) if ϕ is continuously differentiable in a neighborhood of $\xi \in \mathbb{R}$, the following ordinary differential equation is satisfied:

$$(f'(\phi(\xi)) - c)\,\phi'(\xi) = g(\phi(\xi)); \tag{2.3}$$

(iii) if ϕ is not continuously differentiable in any neighborhood of $\xi \in \mathbb{R}$, there exist $\phi_\pm := \lim_{\varepsilon \to 0^+} \phi(\xi \pm \varepsilon)$ and the following conditions are satisfied:

$$c(\phi_+ - \phi_-) = f(\phi_+) - f(\phi_-) \quad \text{(Rankine-Hugoniot condition)}, \tag{RH}$$

$$\phi_+ \leqslant \phi_- \qquad \text{(entropy condition)}. \tag{E}$$

In the following we always deal with entropy travelling waves for equation (1.1).

Definition 2.2. Let ϕ be the profile of a travelling wave. Then:

(i) ϕ is a *heteroclinic wave* if

$$\lim_{\xi \to -\infty} \phi(\xi) = s_i, \tag{2.4i}$$

$$\lim_{\xi \to +\infty} \phi(\xi) = s_j, \tag{2.4ii}$$

for some s_i, s_j in $\mathcal{Z}(g)$ with $s_i \neq s_j$;

(ii) ϕ is a *homoclinic wave* if

$$\lim_{\xi \to \pm\infty} \phi(\xi) = s_i, \tag{2.5}$$

with $s_i \in \mathcal{Z}(g)$;

(iii) ϕ is a *periodic wave* if ϕ is a periodic function.

If ϕ is a heteroclinic wave, we say that ϕ "connects s_i with s_j". For shortness the corresponding travelling wave is said to be a "(s_i, s_j)-wave".

Let $h = h(s,c)$ be the function implicitly defined by the Rankine-Hugoniot condition

$$c = \frac{f(h(s,c)) - f(s)}{h(s,c) - s} \quad \text{for } c \neq f'(s),$$

or by the following equalities elsewhere:

$$h(s, f'(s)) = s, \quad h(\mp\infty, c) = \pm\infty, \quad h(s, \pm\infty) = \pm\infty.$$

The function h satisfies the folowing properties:

$h(s,\cdot)$ is strictly increasing for any $s \in \mathbb{R}$;
$h(\cdot,c)$ is strictly decreasing for any $c \in \mathbb{R}$;
$h(s, f'(s)) = s$ for any $s \in \mathbb{R}$.

The following results will be proved.

Theorem 2.3. *(Existence of heteroclinic waves with $s_i < s_j$) Let assumptions (F), (G) be satisfied. Let $s_i, s_j \in \mathcal{Z}(g)$ with $s_i < s_j$. Then heteroclinic (s_i, s_j)-waves exist if and only if one of the following conditions holds:*
(i) $j = i+2$, $s_i \in \mathcal{A}(g)$ *and* $c = f'(s_{i+1})$;
(ii) $j = i+1$ *and either* $s_i \in \mathcal{R}(g)$, $c \leqslant f'(s_i)$ *or* $s_i \in \mathcal{A}(g)$, $c \geqslant f'(s_{i+1})$.

Theorem 2.4. *(Existence of continuous heteroclinic waves with $s_i > s_j$) Let assumptions (F), (G) be satisfied. Let $s_i, s_j \in \mathcal{Z}(g)$ with $s_i > s_j$. Then continuous heteroclinic (s_i, s_j)-waves exist if and only if one of the following conditions holds:*
(i) $j = i-2$, $s_i \in \mathcal{R}(g)$ *and* $c = f'(s_{i-1})$;
(ii) $j = i-1$ *and either* $s_i \in \mathcal{A}(g)$, $c \geqslant f'(s_i)$, *or* $s_i \in \mathcal{R}(g)$, $c \leqslant f'(s_{i-1})$.

Theorem 2.5. *(Existence of discontinuous heteroclinic waves with $s_i > s_j$) Let assumptions (F), (G) be satisfied. Let $s_i, s_j \in \mathcal{Z}(g)$ with $s_i > s_j$.*
(I) Let $s_i, s_j \in \mathcal{R}(g)$. Then there are $-\infty \leqslant c_1 < c_2 \leqslant +\infty$ such that discontinuous heteroclinic (s_i, s_j)-waves with speed c exist if and only if $c \in I_1$, where $I_1 := (c_1, c_2) \cap [f'(s_j), f'(s_i)] \neq \varnothing$.
(II) Let $s_i \in \mathcal{R}(g)$ and $s_j \in \mathcal{A}(g)$. Then there are $-\infty \leqslant c_3 < c_4 \leqslant +\infty$ such that discontinuous heteroclinic (s_i, s_j)-waves with speed c exist if and only if one of the following holds:
(i) $c \in I_2$, *where* $I_2 := (c_3, c_4) \cap (f'(s_j), f'(s_i)] \neq \varnothing$;
(ii) $c = f'(s_{j-1})$ *and* $s_{i-1} < h(s_{j-2}, f'(s_{j-1}))$.
(III) Let $s_i \in \mathcal{A}(g)$ and $s_j \in \mathcal{R}(g)$. Then there are $-\infty \leqslant c_5 < c_6 \leqslant +\infty$ such that discontinuous heteroclinic (s_i, s_j)-waves with speed c exist if and only if one of the following holds:
(i) $c \in I_3$, *where* $I_3 := (c_5, c_6) \cap [f'(s_j), f'(s_i)) \neq \varnothing$;
(ii) $c = f'(s_{i+1})$ *and* $s_{j+1} < h(s_{i+2}, f'(s_{i+1}))$.
(IV) Let $s_i, s_j \in \mathcal{A}(g)$. Then discontinuous heteroclinic (s_i, s_j)-waves with speed c exist if and only if one of the following holds:
(i) $c = \dfrac{f(s_i) - f(s_j)}{s_i - s_j}$ *(shock wave);*
(ii) $c = f'(s_{j-1})$ *and* $s_i < h(s_{j-2}, f'(s_{j-1}))$.
(iii) $c = f'(s_{i+1})$ *and* $s_j > h(s_{i+2}, f'(s_{i+1}))$.

As the above existence results show, it is always possible "to connect" s_i with s_j if $s_i > s_j$, while the same assertion does not hold if $s_i < s_j$. This is due to entropy condition (E). This condition forces the value on the right of any discontinuity of the wave to be less than the value on the left. Thus, for example, if a travelling wave has a single jump, the profile is given by two regular solution of (2.3), such that conditions (RH) and (E) are satisfied at the discontinuity point. Hence by the entropy condition the integral curve on the left of the jump must lie above the one on the right. For a fixed s_i this is possible only if $s_j \leqslant s_{i+2}$, as Theorems 2.3-2.5 shows.

The above referred limit speeds $c_1, \dots, c_6$ are related to the zeroes $s_{j-1}, s_j, s_i, s_{i+1}$ (see Remark 3.12).

Concerning the existence of homoclinic, respectively periodic waves we have the following results.

Theorem 2.6. *(Existence of homoclinic waves). Let assumptions (F),(G) be satisfied.*
(I) Let $s_i \in \mathcal{R}(g)$. Then homoclinic waves with speed c satisfying (2.5) exist if and only if $c = f'(s_i)$.
(II) Let $s_i \in \mathcal{A}(g)$. Then homoclinic waves with speed c satisfying (2.5) exist if and only if one of the following conditions holds:
(i) $c = f'(s_{i-1})$ *and* $s_i < h(s_{i-2}, f'(s_{i-1}))$,
(ii) $c = f'(s_{i+1})$ *and* $s_i > h(s_{i+2}, f'(s_{i+1}))$.

Theorem 2.7. *(Existence of periodic waves) Let assumptions (F),(G) be satisfied. Then periodic travelling waves with speed c exist if and only if $c = f'(s_i)$ for some $s_i \in \mathcal{R}(g)$.*

The above results are proved in the next section. The main idea of the proof is to study solutions of equation (2.3), thus connecting the asymptotic states of the wave along integral curves of (2.3) and jumping from one to another in such a way that the Rankine-Hugoniot and entropy conditions are satisfied.

Remark 2.8. If we drop the assumption that the zeros of g are simple, the situation becomes more intricated since many new different cases appear. More precisely, we have to split the set $\mathcal{Z}(g)$ of the zeros of g in the following way:

$$\mathcal{Z}(g) = \mathcal{A}(g) \cup \mathcal{R}(g) \cup \mathcal{P}(g) \cup \mathcal{N}(g),$$

where:

$$\begin{aligned}
\mathcal{A}(g) &:= \{s \in \mathcal{Z}(g) : g(\sigma)(\sigma - s) \leqslant 0 \text{ in a neighborhood of } s\},\\
\mathcal{R}(g) &:= \{s \in \mathcal{Z}(g) : g(\sigma)(\sigma - s) \geqslant 0 \text{ in a neighborhood of } s\},\\
\mathcal{P}(g) &:= \{s \in \mathcal{Z}(g) : g(\sigma) \geqslant 0 \text{ in a neighborhood of } s\},\\
\mathcal{N}(g) &:= \{s \in \mathcal{Z}(g) : g(\sigma) \leqslant 0 \text{ in a neighborhood of } s\}.
\end{aligned}$$

In this case the role of $\mathcal{R}(g)$ is played by the subset of $\mathcal{R}(g)$ defined as follows:

$$\mathcal{R}_1(g) := \{s \in \mathcal{R}(g) : \frac{f'(\sigma) - f'(s)}{g(\sigma)} \in L^1(s - \varepsilon, s + \varepsilon) \text{ for some } \varepsilon > 0\}.$$

Let ϕ be a given a travelling wave ϕ. In order to state the regularity results, we set:

$$D := \{\xi \in \mathbb{R} : \phi(\xi-) \neq \phi(\xi+)\}; \tag{2.6}$$

$$C := \{\xi \in \mathbb{R} : \phi \text{ is continuous at } \xi \text{ and } \phi \text{ is not } C^1 \text{ in any neighborhood of } \xi\}. \tag{2.7}$$

Then the following holds:

Theorem 2.9. *Let ϕ be a travelling wave with $\phi \notin C^1(\mathbb{R})$.*
(I) Let $c \neq f'(s)$ for any $s \in \mathcal{Z}(g)$. Then $D = \{\xi_0\}$ and $\phi \in C^1(\mathbb{R} \setminus \{\xi_0\})$ for some $\xi_0 \in \mathbb{R}$.
(II) Let $c = f'(s)$ for some $s \in \mathcal{A}(g)$.
(i) If $\phi \in C(\mathbb{R})$ then card$(C) \leq 2$ and $\phi \in C^1(\mathbb{R} \setminus C)$;
(ii) if $\phi \notin C(\mathbb{R})$ then $D = \{\xi_0\}$ and $\phi \in C^1(\mathbb{R} \setminus \{\xi_0\})$ for some $\xi_0 \in \mathbb{R}$.
(III) Let $c = f'(s)$ for some $s \in \mathcal{R}(g)$.
(i) If ϕ is heteroclinic then there is a unique $\xi_0 \in D$ such that whether $\phi \in C^1(-\infty, \xi_0)$ or $\phi \in C^1(\xi_0, +\infty)$;
(ii) if ϕ is homoclinic with $s_i \in \mathcal{A}(g)$ (s_i as in (2.5)), then there exists $\xi_1, \xi_2 \in D$ such that $\phi \in C^1(\mathbb{R} \setminus [\xi_1, \xi_2])$.

Now we turn to the stability of travelling waves. As already mentioned in the Introduction, for the sake of simplicity we make the following assumptions:

$$\text{(G}'\text{)} \qquad \begin{cases} g \in C^1(\mathbb{R}), \quad |g(v)| \leqslant A + B|v| \text{ for some } A, B \geqslant 0;\\ g(0) = g(1) = 0,\ g(s) \neq 0 \quad \forall s \notin \{0, 1\},\ g'(0) > 0 \text{ and } g'(1) < 0. \end{cases}$$

If assumption (G′) holds, the results stated in Theorems 2.3-2.7 can be summarized as follows:

c	<0	$=0$	>0
(0,1)	$\exists!$	$\exists\infty$	$\nexists$
(1,0)	$\nexists$	$\exists\infty$	$\exists!$
homoclinic periodic	$\nexists$	$\exists\infty$	$\nexists$

The symbols ! and ∞ refer to the number of possible profiles for the travelling waves.
Let us discuss the stability character of these waves.
Given a travelling profile ϕ_c of equation (1.1) with speed c we set:

$$I(\phi_c,\sigma,t) := \mathbb{R}\setminus\bigcup_{x\in D}(x-ct-\sigma, x-ct+\sigma)\quad(\sigma>0, t>0),$$

where D is given by (2.6).
Finally, given a bounded piecewise smooth function ψ, a compact set $K\subset\mathbb{R}$ and $\varepsilon>0$ we define:

$$\Lambda(\psi,K,\varepsilon) := \{v\in BV_{\mathrm{loc}}(\mathbb{R})\cap L^\infty(\mathbb{R}) : \mathrm{supp}(v-\psi)\subset K, \|v-\psi\|_{L^\infty(\mathbb{R})}<\varepsilon\}.$$

Theorem 2.10. *(Stability character of (0,1)-waves). Let assumptions (F), (G′) be satisfied. Let ϕ_c be a (0,1)-wave of equation (1.1) with speed c.*
(i) Let $c<0$ and the following hold:
(a) $u_0\geqslant 0$ in $\mathbb{R}$;
(b) $\liminf_{x\to+\infty} u_0(x) = l > 0$,
(c) $\lim_{x\to-\infty}\dfrac{u_0(x)}{\phi_c(x)} = 1$.
Then for the corresponding solution of problem (2.1) there holds:

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}|u(x,t)-\phi_c(x-ct)| = 0.$$

(ii) Let $c=0$. Then there is a compact set $K\subset\mathbb{R}$ with the following property: for any $\varepsilon>0$ there exist $u_0^\varepsilon\in\Lambda(\phi,K,\varepsilon)$ and $\sigma>0$ such that

$$\liminf_{t\to+\infty}\sup_{x\in I(\phi_0,\sigma,t)}|u^\varepsilon(x,t)-\phi_0(x)| > l > 0,$$

where l is a constant independent of ε.

Remark 2.11. In Theorem 2.10(i) if hypothesis (c) is replaced by the following one:
(c′) $u_0(x) = \phi_c(x)$ for $x<k$ for some $k\in\mathbb{R}$,
the solution u proves to converge exponentially fast to the wave ϕ_c.

Remark 2.12. In Theorem 2.10(ii) the instability of (0,1)-waves with zero speed is stated. Moreover, for *continuous* travelling waves ϕ_0, (these are unique up to translations), a result similar to that stated under (i) can be proved. More precisely, let:

(a) $u_0 \geqslant 0$ in $\mathbb{R}$;

(b) $\liminf_{x\to+\infty} u_0(x) = l > 0$;

(c) $u_0(x) = 0$ for $x < k$ for some $k \in \mathbb{R}$.

Then the entropy solution of problem (2.1) proves to converges uniformly to $\phi_0(x+\delta)$, for some $\delta \in \mathbb{R}$.

Let us state the following results.

Theorem 2.13. *(Stability character of (1,0)-waves). Let assumptions (F), (G′) be satisfied. Let ϕ_c be a (1,0)-wave of (1.1) with speed c.*

(i) Let $c \in [0, f(1))$. Then there is a compact set $K \subset \mathbb{R}$ with the following property: for any $\varepsilon > 0$ there exist $u_0^\varepsilon \in \Lambda(\phi_0, K, \varepsilon)$ and $\sigma > 0$ such that the corresponding solution of problem (2.1) satisfies:

$$\liminf_{t\to+\infty} \sup_{x\in I(\phi_c,\sigma,t)} |u^\varepsilon(x,t) - \phi_c(x-ct)| > l > 0,$$

where l is a constant independent of ε.

(ii) Let $c = f(1)$. Then for any compact $K \subset \mathbb{R}$ there is $\varepsilon > 0$ with the following property: if

(a) $u_0 > -\varepsilon$ in $\mathbb{R}$

and

(b) $u_0 = \phi_{f(1)}$ in $\mathbb{R} \setminus K$,

then there exist $T > 0$, $\delta \in \mathbb{R}$ such that for the solution of problem (2.1) there holds:

$$u(x,t) = \phi_{f(1)}(x - f(1)t + \delta) \quad \text{for any } (x,t) \in \mathbb{R} \times [T, +\infty).$$

(iii) Let $c > f(1)$. Assume that:

(a) $u_0 \geqslant 0$ in $\mathbb{R}$;

(b) $\liminf_{x\to-\infty} u_0(x) = l > 0$,

(c) $u_0 = \phi_c$ in $(k, +\infty)$ $(k \in \mathbb{R})$.

Then for the solution of problem (2.1) there holds:

$$\lim_{t\to+\infty} \sup_{x\in I(\phi_c,\sigma,t)} |u(x,t) - \phi_c(x-ct)| = 0 \quad \text{for any } \sigma > 0.$$

Remark 2.14. In Theorem 2.13(iii), if $c > f'(1)$ the rate of convergence to the wave proves to be exponential.

Remark 2.15. According to [10], compactly supported perturbations of the solution of the Riemann problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0, \\ u(x,0) = \begin{cases} u_l & x < 0, \\ u_r & x \geqslant 0, \end{cases} \end{cases}$$

disappear in finite time. Theorem 2.13(iii) gives a similar result for equation (1.1).

Concerning homoclinic and periodic waves we have the following results.

Theorem 2.16. *(Stability character of homoclinic and periodic waves). Let assumptions (F), (G') be satisfied. Let ϕ be either a homoclinic, or a periodic wave of equation (1.1). Then there is a compact set $K \subset \mathbb{R}$ with the following property: for any $\varepsilon > 0$ there exist $u_0^\varepsilon \in \Lambda(\phi, K, \varepsilon)$ and $\sigma > 0$ such that*

$$\liminf_{t \to +\infty} \sup_{x \in I(\phi,\sigma,t)} |u^\varepsilon(x,t) - \phi(x - ct)| > l > 0,$$

where l is a constant independent of ε.

Observe that the perturbations we consider are always bounded and local – namely, their support lies in some subset of $\mathbb{R}$, possibly depending on the structure of the wave.

Theorems 2.10, 2.13 and 2.16 are proved in Section 4. The main tools of the proof are comparison of entropy solutions of equation (1.1) (see [7]) and the technique of generalized characteristics (see [2], [3]).

3. Proof of Existence and Regularity Results

In this Section we prove Theorems 2.3-2.7 and 2.9.

Let $A \subset \mathbb{R}$. We shall denote by $\mathcal{D}(A)$ the set of accumulation points of A. We also denote by $\mathrm{Cl}(A)$ the closure of A and by $\mathrm{Int}(A)$ the set of interior point of A.

Let us first recall a useful property of entropy solutions to (1.1).

Proposition 3.1 [14]**.** *Let u be an entropy solution to (1.1). Suppose that $u(x_1+, t) < u(x_2-, t)$ for some $x_1 < x_2$, $t > 0$. Then for any $\overline{u} \in (u(x_1+, t), u(x_2-, t))$ there exists $\overline{x} \in (x_1, x_2)$ such that $u(\overline{x}-, t) = \overline{u}$.*

Now we state two lemmas which explain the structure of the discontinuities of a travelling wave. Lemma 3.2 shows that a profile ϕ can have a jump discontinuity only if the value at the left and at the right of the jump is, respectively, greater and less than a given constant, which depends on the speed. Lemma 3.3 describes the set D of the discontinuities of ϕ.

Lemma 3.2. *Let ϕ be a travelling wave with speed c and let ξ_0 be such that $\phi(\xi_0-) \neq \phi(\xi_0+)$. Then*

$$\phi(\xi_0-) > (f')^{-1}(c) > \phi(\xi_0+).$$

Proof. Since f' is increasing, there exists a unique ϕ^* such that $f'(\phi^*) = c$. By the Rankine-Hugoniot condition we have

$$f'(\phi^*) = c = \frac{f(\phi(\xi_0-)) - f(\phi(\xi_0+))}{\phi(\xi_0-) - \phi(\xi_0+)}.$$

Since f is convex, either

$$\phi(\xi_0-) < \phi^* < \phi(\xi_0+)$$

or

$$\phi(\xi_0+) < \phi^* < \phi(\xi_0-).$$

The conclusion follows by condition (E). $\square$

Lemma 3.3. *Let ϕ be a travelling wave and let $D := \{\xi \in \mathbb{R} : \phi(\xi_0-) \neq \phi(\xi_0+)\}$. Then*
(i) D is a set of isolated points;
(ii) $\mathcal{D}(D) \subset \{\xi \in \mathbb{R} : \phi(\xi) = (f')^{-1}(c)\}$;
(iii) if card$(D) > 1$ then $c = f'(s_k)$ for some $s_k \in \mathcal{R}(g)$.

Proof. (i) Let $\xi_0 \in D$. Suppose that there is $\{\xi_n\}_{n\in\mathbb{N}} \subset D \setminus \{\xi_0\}$ such that $\xi_n \to \xi_0$ as $n \to \infty$. We can suppose without restriction that $\{\xi_n\}$ is strictly decreasing. Then by Lemma 3.2

$$\phi(\xi_n-) > (f')^{-1}(c) \quad \text{and} \quad \phi(\xi_0+) < (f')^{-1}(c).$$

This implies

$$(f')^{-1}(c) \leqslant \lim_{n\to\infty} \phi(\xi_n-) = \phi(\xi_0+) < (f')^{-1}(c),$$

which is clearly impossible.

(ii) Let $\{\xi_n\} \subset D$, $\xi_n \to \xi_0$ as $n \to \infty$. Then, by (i), ϕ is continuous at ξ_0 and:

$$\phi(\xi_0) = \lim_{n\to\infty} \phi(\xi_n-) \geqslant (f')^{-1}(c) \geqslant \lim_{n\to\infty} \phi(\xi_n+) = \phi(\xi_0).$$

Hence the conclusion follows.

(iii) Let card$(D) > 1$. Then there exist $\xi_1 < \xi_2$ such that $\xi_1, \xi_2 \in D$. Then by Lemma 3.2

$$\phi(\xi_1+) < (f')^{-1}(c) < \phi(\xi_2-).$$

By Proposition 3.1 there exists $\overline{\xi} \in (\xi_1, \xi_2)$ such that $\phi(\overline{\xi}) = (f')^{-1}(c)$. Let

$$A := \{\xi \in (\xi_1, \xi_2) : \phi(\xi) = (f')^{-1}(c)\},$$
$$\overline{\xi}_1 := \inf A.$$

Then $\phi(\overline{\xi}_1+) = (f')^{-1}(c)$. Hence by Lemma 3.2 ϕ is continuous in $\overline{\xi}_1$ and $\phi(\overline{\xi}_1) = (f')^{-1}(c)$, which implies $\overline{\xi}_1 = \min A$. Moreover, we have $\overline{\xi}_1 > \xi_1$ since $\xi_1 \notin A$. Also the following holds:

$$\phi(\xi) < (f')^{-1}(c) \quad \forall \xi \in (\xi_1, \overline{\xi}_1). \tag{3.1}$$

In fact:
(a) $\phi(\xi) \neq (f')^{-1}(c)$ for any $\xi \in (\xi_1, \overline{\xi}_1)$ by definition of ξ_1;
(b) if there is $\xi^* \in (\xi_1, \overline{\xi}_1)$, by Proposition 3.1 there exists $\overline{\xi} \in (\xi_1, \xi^*)$ such that $\phi(\overline{\xi}) = (f')^{-1}(c)$ contradicting the definition of $\overline{\xi}_1$.

By Lemma 3.2 ϕ cannot be discontinuous in $(\xi_1, \overline{\xi}_1)$. Since ϕ is piecewise C^1 we have

$$\phi'(\phi(\xi)) = \frac{g(\phi(\xi))}{f'(\phi(\xi)) - c} \quad \text{a.e. in } (\xi_1, \overline{\xi}_1).$$

Since g has only isolated zeros and ϕ is continuous in $(\xi_1, \overline{\xi}_1)$, we can find $\xi \in (\xi_1, \overline{\xi}_1)$ such that $g(\phi(\xi))$ is either strictly positive or strictly negative in $(\xi, \overline{\xi}_1)$. This implies that $\phi'(\xi)$ is either strictly positive or strictly negative in $(\xi, \overline{\xi}_1)$. Since

$$(f')^{-1}(c) = \phi(\overline{\xi}_1) = \phi(\xi) + \int_{\xi}^{\overline{\xi}_1} \phi'(s)ds,$$

and $\phi(\xi) < (f')^{-1}(c)$, we obtain

$$\phi'(\xi) > 0 \quad \forall \xi \in (\xi, \overline{\xi}_1).$$

Hence

$$\phi'(\phi) > 0 \quad \forall \phi \in (\phi_0, (f')^{-1}(c)),$$

where $\phi_0 := \phi(\xi)$, thus

$$g(\phi) < 0 \quad \forall \phi \in (\phi_0, (f')^{-1}(c)). \tag{3.2}$$

Defining $\overline{\xi}_2 := \sup A$ and arguing in a similar way, we also obtain

$$g(\phi) > 0 \quad \forall \phi \in ((f')^{-1}(c), \phi_1), \tag{3.3}$$

for some $\phi_1 > (f')^{-1}(c)$. From (3.2), (3.3) and (G) the conclusion follows. □

Now we can prove Theorem 2.3.

Proof of Theorem 2.3. Let ϕ be a (s_i, s_j)-wave with $s_i < s_j$. Then

$$\phi'(\phi) = \frac{g(\phi)}{f'(\phi) - c} \geqslant 0 \quad \forall \phi \in (s_i, s_j) \setminus \{(f')^{-1}(c)\}. \tag{3.4}$$

In fact suppose that there exists $\overline{\phi} \in (s_i, s_j) \setminus \{(f')^{-1}(c)\}$ with $\phi'(\overline{\phi}) < 0$. Since conditions (2.4) hold, there exists $\xi_1 < \xi_2$ such that:

$$\phi(\xi_1+) < \overline{\phi} < \phi(\xi_2-). \tag{3.5}$$

Set $\overline{\xi} := \inf\{\xi \in (\xi_1, \xi_2) : \phi(\xi-) = \overline{\phi}\}$ (this set is not empty by Proposition 3.1). Then there exists $\{\xi^n\}$ converging to $\overline{\xi}$ such that $\phi(\xi^n) = \overline{\phi}$ for any n. It follows that $\phi(\overline{\xi}+) = \overline{\phi}$ and $\xi_1 < \overline{\xi}$ by (3.5). Since $\overline{\phi} \neq (f')^{-1}(c)$, there exists a neighborhood I of $\overline{\xi}$ such that $\phi \in C(I)$: otherwise there would exists $\{\xi^n\}$ converging to $\overline{\xi}$ with $\{\xi^n\} \in D$, and $\overline{\phi} = (f')^{-1}(c)$ by Lemma 3.2(ii). Since $\phi'(\overline{\phi}) < 0$, there exists $\xi_2 < \overline{\xi}$ such that $\phi(\xi_2) > \overline{\phi}$. Therefore there is $\xi_3 \in (\xi_1, \xi_2)$ with $\phi(\xi_3) = \overline{\phi}$, contradicting the definition of $\overline{\xi}$. Hence (3.4) holds.

If there is $s_0 \in \mathcal{Z}(g) \cap (s_i, s_j)$, it follows by (3.4) that $f'(\phi) - c$ changes sign at s_0. By monotonicity of f', $s_0 = f'(c)$ and $g'(s_0) > 0$.

Therefore, given a ϕ (s_i, s_j)-wave with speed c, there can be at most one zero between s_i and s_j. This proves (3.4).

Thus one of the following holds:

(i) $j = i + 2$, $g'(s_{i+1}) > 0$ and $c = f'(s_{i+1})$;

(ii) $j = i + 1$ and either $s_i \in \mathcal{R}(g)$, $c \leqslant f'(s_i)$ or $s_i \in \mathcal{A}(g)$, $c \geqslant f'(s_{i+1})$.

If (i) holds, there exists a monotonically increasing solution implicitly defined by:

$$\int_{\phi_0}^{\phi(\xi)} \frac{f'(s) - c}{g(s)} d\,s = \xi - \xi_0. \tag{3.6}$$

with $\phi_0 = s_{i+1}$.

If (ii) holds, let $s_i \in \mathcal{R}(g)$ (the other case can be treated similarly). If $c < f'(s_i)$, there is a regular (s_i, s_{i+1})- wave given by (3.6) with $\phi_0 \in (s_i, s_{i+1})$. If $c = f'(s_i)$ there is a continuous wave defined by (3.6) with $\phi_0 = s_i$ for $\xi > \xi_0$ and by $\phi(\xi) \equiv s_i$ for any $\xi \leqslant \xi_0$. This completes the proof. □

In the proof of Theorem 2.3, we prove the existence of continuous travelling waves. It is worth mentioning that when $c = f'(s_k)$ for some $s_k \in \mathcal{R}(g)$, there are also infinitely many other travelling waves with discontinuous profile. The following example explain more clearly this phenomenon.

Example 3.4. Let $f(s) = \frac{1}{2}s^2$, $g(s) = s(1-s^2)$ for $s \in [-1,1]$. Then a continuous profile $\phi_0 = \phi_0(\xi)$ of a (-1,1)-wave with speed $c = 0$ is given by

$$\phi_0(\xi) = \frac{e^{2\xi}-1}{e^{2\xi}+1} \quad (\xi \in \mathbb{R}).$$

Since for any $\delta \in \mathbb{R}$ $\phi_0(\xi+\delta)$ is another profile, we can construct other travelling waves piecing together different continuous profiles, in such a way that conditions (RH) and (E) are satisfied. For example, we can define for any $a > 0$

$$\phi_a(\xi) := \begin{cases} \dfrac{e^{2(\xi+a)}-1}{e^{2(\xi+a)}+1}, & \xi < 0, \\ \dfrac{e^{2(\xi-a)}-1}{e^{2(\xi-a)}+1}, & \xi > 0. \end{cases}$$

It is easily checked that ϕ_a is a (-1,1)-wave for (1.1). Similar waves can be constructed when $c = f'(s_k)$ for some $s_k \in \mathcal{R}(g)$. In what follows we refer to these phenomena "discontinuous oscillations around the value s_k" or simply "oscillations around s_k".

Now we prove Theorem 2.4, concerning existence of continuous heteroclinic waves with $s_i > s_j$.

Proof of Theorem 2.4. For a fixed speed c of the wave, a solution of (2.3) can cross at most one zero of g. Thus only two cases arise:

(i) $j = i-2$, $g'(s_{i-1}) < 0$ and $c = f'(s_{i-1})$;

(ii) $j = i-1$.

In case (i) a monotone decreasing solution of (2.3) is easily seen to exist, which satisfies (2.4).

Now suppose that (ii) holds; moreover, let $g'(s_i) < 0$. If $c \neq f'(s_i)$ and $c \neq f'(s_j)$, the values s_i and s_j are not assumed by any solution of (2.3). Therefore a possible profile must remain in the strip $\mathbb{R} \times (s_j, s_i)$. A solution of (2.3) with values in this interval, connecting s_i with s_j exists if and only if $c > f'(s_i)$.

Let $c = f'(s_j)$. Since the only solution of (2.3) satisfying (2.4i) (respectively (2.4ii)) is $\phi \equiv s_i$ ($\phi \equiv s_j$ respectively), there is no wave with such a speed. If $c = f'(s_i)$, there is a decreasing solution crossing s_i, which tends to s_j at $+\infty$. Hence a continuous travelling exists.

The case $g'(s_i) > 0$ can be treated by the same arguments. $\square$

To prove Theorem 2.5, concerning discontinuous heteroclinic waves with $s_i > s_j$, more work is needed. Roughly speaking, this is due to the fact that we are going from a higher value s_i to a lower one s_j. Hence discontinuities satisfying (E), i.e. decreasing discontinuities can be used to join the asymptotic states. This leads to a more complicated situation, since many different ways of connecting s_i with s_j arise.

Lemma 3.5. *Let assumption of Theorem 2.5 be satisfied. Let ϕ be a travelling wave satisfying (2.4) with more than one discontinuity. Then:*

$$(f')^{-1}(c) \in \{s_{j-1}, s_j, s_i, s_{i+1}\} \cap \mathcal{R}(g). \tag{3.7}$$

Proof. Since ϕ has more than one discontinuity, by Lemma 3.3(iii) $c = f'(s_n)$ for some $s_n \in \mathcal{R}(g)$.

If $s_n > s_{i+1}$, starting at $-\infty$ from s_i no jump is possible by Lemma 3.2 unless $\phi(\xi)$ is less than s_n. Therefore $\phi(\xi)$ is given by a solution of (2.3) and is less than s_{i+1} for any $\xi \in \mathbb{R}$. Thus ϕ cannot be discontinuous.

If $s_n = s_{i-1} \neq s_j$ and $s_{i-1} \in \mathcal{R}(g)$, condition (2.4) is satisfied only if $\phi(\xi) = s_i$ for $\xi < \xi_1 := \inf D$. By Lemma 3.3(ii), ξ_1 is a discontinuity point. Thus

$$f'(s_{i-1}) = \frac{f(s_i) - f(\phi(\xi_1+))}{s_i - \phi(\xi_1+)}.$$

Since ϕ has a second discontinuity, $\phi(\xi_1+) \in (s_{i-2}, s_{i-1})$. Let ξ_2 be a second point of jump. Then

$$f'(s_{i-1}) = \frac{f(\phi(\xi_2-)) - f(\phi(\xi_2+))}{\phi(\xi_2-) - \phi(\xi_2+)}.$$

Since $\phi(\xi_2-) < s_i$ and f is strictly convex, we have $\phi(\xi_2+) > \phi(\xi_1+) > s_{i-1}$. Therefore ϕ cannot tend to s_j as $\xi \to +\infty$.

Now let $s_n < s_{i-1}$. Then $\xi_1 \in D$, $\phi(\xi_1-) \in (s_{i-1}, s_{i+1})$ and $\phi(\xi_1+) \in (s_{n-1}, s_{n+1})$. Therefore $\phi(\xi) \in (s_{n-1}, s_{n+1})$ for any $\xi > \xi_1$. Since ϕ satisfies (2.4ii), we have either $s_n = s_j$ or $s_n = s_{j-1}$. This completes the proof. $\square$

Lemma 3.6. *Let the assumptions of Theorem 2.5 be satisfied. Let ϕ be a heteroclinic travelling wave with more than one discontinuity; moreover, let $c = f'(s_n)$ satisfy (3.7). Then there exists a unique $\xi_0 \in \mathbb{R}$ such that:*
(i) if $s_n = s_{i+1}$ or $s_n = s_i$:

$$\phi(\xi) \in (s_{n-1}, s_{n+1}) \quad \forall \xi < \xi_0,$$
$$\phi(\xi) < s_{n-1} \quad \forall \xi > \xi_0;$$

(ii) if $s_n = s_{j-1}$ or $s_n = s_j$:

$$\phi(\xi) > s_{n+1} \quad \forall \xi < \xi_0,$$
$$\phi(\xi) \in (s_{n-1}, s_{n+1}) \quad \forall \xi > \xi_0.$$

Proof. We only consider the case (i), since (ii) can be treated similarly.

If $s_n = s_{i+1} \in \mathcal{R}(g)$, ϕ is given by (3.6) near for any $\xi < \xi_0$ for some $\xi_0 \in \mathbb{R}$, $\phi_0 \in (s_i, s_{i+1})$. Set:

$$\xi_1 := \sup\{\xi \in \mathbb{R} : \phi(\xi) \in (s_i, s_{i+2})\}.$$

Then $\xi_1 \in D$ and $\phi(\xi_1+) < s_i$. Since $\phi < s_{i+1} = (f')^{-1}(c)$, by Lemma 3.2 ϕ is continuous until $\phi(\xi) < s_i$ and thus it is a regular solution of (2.3). Since no nonconstant solution of (2.3) assumes the value s_i, we conclude that $\phi(\xi) < s_i$ for any $\xi > \xi_1$.

If $s_n = s_i \in \mathcal{R}(g)$, then $\phi(\xi) \in (s_{i-1}, s_{i+1})$ for every $\xi > \xi_2$ for some ξ_2. Let

$$\xi_1 := \sup\{\xi \in \mathbb{R} : \phi(\xi) \in (s_{i-1}, s_{i+1})\}.$$

Then the same argument as above gives to the conclusion. $\square$

From Lemmas 3.5 and 3.6 we can deduce that the discontinuous heteroclinic waves are always made by two parts, one of which is C^1.

Definition 3.7. Let ϕ be a discontinuos heteroclinic wave with $s_i > s_j$. If ϕ has a single discontinuity, we denote by ξ_0 the unique jump point. Otherwise we define ξ_0 as in Lemma 3.6.

The next two lemmas give sufficient conditions to ensure that (2.4) is satisfied.

Lemma 3.8. *Let ϕ be a travelling wave satisfying (2.4i) for some $s_i \in \mathcal{Z}(g)$; let ξ_0 be chosen as in Definition 3.7. Then one of the following properties holds:*
(α) $c \neq f'(s_i)$ and either $c < f'(s_i)$, $s_i \in \mathcal{R}(g)$, or $c > f'(s_i)$, $s_i \in \mathcal{A}(g)$;
(β) $\phi(\xi) = s_i$ for $\xi < \xi_0$;
(γ) $c = f'(s_i)$ and $s_i \in \mathcal{R}(g)$;
(δ) $c = f'(s_{i+1})$ and $s_{i+1} \in \mathcal{R}(g)$.

Proof. There are four possible profiles on the left of ξ_0, such that (2.4i) holds:
(a) ϕ is a regular nonconstant solution of (2.3);
(b) ϕ is the constant solution s_i for $\xi < \xi_0$;
(c) $\phi \to s_i$ as $\xi \to -\infty$ oscillating around s_i;
(d) $\phi \to s_i$ as $\xi \to -\infty$ oscillating around s_{i+1} and then increasing to s_i.

It is easily seen that conditions (a), (b), (c) and (d) are satisfied if and only if (α), (β), (γ) and (δ) hold respectively. Hence the conclusion follows. □

Lemma 3.9. *Let ϕ be a travelling wave which satisfies (2.4ii) for some $s_j \in \mathcal{Z}(g)$ and let ξ_0 be choosen in Definition 3.7. Then one of the following holds:*
(α) $c \neq f'(s_j)$ and either $c < f'(s_j)$, $s_j \in \mathcal{A}(g)$, or $c > f'(s_j)$, $s_j \in \mathcal{R}(g)$;
(β) $\phi(\xi) = s_j$ for $\xi > \xi_0$;
(γ) $c = f'(s_j)$ and $s_j \in \mathcal{R}(g)$;
(δ) $c = f'(s_{j-1})$ and $s_{j-1} \in \mathcal{R}(g)$.

The proof is similar to that of Lemma 3.8, hence we omit it.

From Lemmas 3.8 and 3.9 one can deduce the speed of the wave ϕ, as well as the value $\phi(\xi_0-)$ such that conditions (2.4) are satisfied. This is the content of the next two lemmas.

Lemma 3.10. *Let ϕ be a travelling wave such that (2.4i) holds for some $s_i \in \mathcal{Z}(g)$.*
(I) Let $s_i \in \mathcal{R}(g)$. Then $c \leqslant f'(s_i)$ and $\phi(\xi_0-) \in (\max\{(f')^{-1}(c), s_{i-1}\}, s_{i+1})$. Moreover, for a fixed $c \leqslant f'(s_i)$, for any $\overline{\phi} \in (\max\{(f')^{-1}(c), s_{i-1}\}, s_{i+1})$ there is a wave ϕ such that $\phi(\xi_0-) = \overline{\phi}$.
(II) Let $s_i \in \mathcal{A}(g)$. Then one of the following holds:
(i) $c < f'(s_i)$ and $\phi(\xi_0-) = s_i$;
(ii) $c = f'(s_{i+1})$ and $\phi(\xi_0-) \in (s_{i+1}, s_{i+2})$.
Moreover if $c = f'(s_{i+1})$ and $\overline{\phi} \in (s_{i+1}, s_{i+2})$ there is a wave ϕ such that $\phi(\xi_0-) = \overline{\phi}$.

Proof. (I) Note that $s_i \in \mathcal{R}(g)$ implies $s_{i+1} \in \mathcal{A}(g)$; thus case ($\delta$) of Lemma 3.8 cannot occur. If $c < f'(s_i)$, there are solutions of (2.3) connecting s_i with each value contained in the interval $(\max\{(f')^{-1}(c), s_{i-1}\}, s_{i+1})$; hence in this case the conclusion follows. If $c = f'(s_i)$, oscillations around s_i can occur. The value ξ_0 is the one that takes the profile outside the strip (s_{i-1}, s_{i+1}); thus we have $\xi_0 \in (s_i, s_{i+1})$.

If $c > f'(s_i)$, case (β) of Lemma 3.8 must hold. Since $s_i < (f')^{-1}(c)$ there are no discontinuities.

(II) Case (γ) of Lemma 3.8 cannot occur. If $c < f'(s_i)$, the wave satisfies case (β) of Lemma 3.8, thus conclusion follows. If $c < f'(s_i)$, the profile cannot jump from s_i. If $c > f'(s_i)$ and $c \neq f'(s_{i+1})$, the wave remains under the value s_{i+1}, thus no jump is possible. If $c = f'(s_{i+1})$, one of (α), (β) and (δ) holds, so that property (ii) is satisfied. Hence the conclusion. □

Lemma 3.11. *Let ϕ be a travelling wave such that (2.4ii) for some $s_j \in \mathcal{Z}(g)$.*

(I) Let $s_j \in \mathcal{R}(g)$. Then $c \geqslant f'(s_j)$ and $\phi(\xi_0-) \in (\max\{(f')^{-1}(c), h(s_{j+1}, c)\}, h(s_{j-1}, c))$. Moreover, for a fixed $c \geqslant f'(s_j)$, for any $\overline{\phi} \in (\max\{(f')^{-1}(c), h(s_{j+1}, c)\}, h(s_{j-1}, c))$ there is a wave ϕ such that $\phi(\xi_0-) = \overline{\phi}$.

(II) Let $s_j \in \mathcal{A}(g)$. Then one of the following holds:
(i) $c = f'(s_{j-1})$ and $\phi(\xi_0-) \in (s_{j-1}, h(s_{j-2}, f'(s_{j-1})))$;
(ii) $c > f'(s_j)$ and $\phi(\xi_0-) = h(s_j, c)$. Moreover, if $c = f'(s_{j-1})$ and $\overline{\phi} \in (s_{j-1}, h(s_{j-2}, f'(s_{j-1})))$ there is a wave ϕ such that $\phi(\xi_0-) = \overline{\phi}$.

Proof. The proof is similar to that of the previous Lemma. Note that in the statement of Lemma 3.11, the value $\phi(\xi_0+)$ of the profile on the right hand side goes into $\phi(\xi_0-)$ by use of the Rankine-Hugoniot condition. $\square$

Proof of Theorem 2.5. The conclusions of Theorem 2.5 follow by Lemmas 3.10 and 3.11. Indeed, travelling waves of (1.1) exist if and only if there is some value $\phi(\xi_0-)$ which can be simultaneously reached both from the right and from the left. Hence the speed c is determined requiring that the intersection between the sets of possible values for $\phi(\xi_0-)$ given in Lemmas 3.10 and 3.11 is not empty.

(I) Let $s_i, s_j \in \mathcal{R}(g)$. A necessary and sufficient condition of existence of waves is:

$$\begin{cases} f'(s_j) \leqslant c \leqslant f'(s_i), \\ (\max\{(f')^{-1}(c), s_{i-1}\}, s_{i+1}) \cap (\max\{(f')^{-1}(c), h(s_{j+1}, c)\}, h(s_{j-1}, c)) \neq \varnothing. \end{cases}$$

This can be reformulated as follows:

$$\begin{cases} f'(s_j) \leqslant c \leqslant f'(s_i), \\ \max\{(f')^{-1}(c), s_{i-1}\} < h(s_{j-1}, c), \\ \max\{(f')^{-1}(c), h(s_{j+1}, c)\} < s_{i+1}. \end{cases} \tag{3.8}$$

Since $h(\cdot, c)$ is decreasing, it follows that

$$(f')^{-1}(c) = h((f')^{-1}(c), c) \leqslant h(s_j, c) < h(s_{j-1}, c),$$
$$(f')^{-1}(c) \leqslant s_i < s_{i+1}.$$

Then conditions (3.8) are satisfied if and only if

$$\begin{cases} f'(s_j) \leqslant c \leqslant f'(s_i), \\ s_{i-1} < h(s_{j-1}, c), \\ h(s_{j+1}, c) < s_{i+1}. \end{cases}$$

Let c_1 and c_2 be defined by as follows

$$h(s_{j-1}, c_1) = s_{i-1},$$
$$h(s_{j+1}, c_2) = s_{i+1}.$$

For c_1 and c_2 the following holds:
(i) $h(s_{j-1}, c) > s_{i-1}$ if and only if $c > c_1$;
(ii) $h(s_{j+1}, c) < s_{i+1}$ if and only if $c < c_2$;

(iii) $c_1 < c_2$;
(iv) $c_1 \in (f'(s_{j-1}), f'(s_{i-1}))$;
(v) $c_2 \in (f'(s_{j+1}), f'(s_{i+1}))$.
By (i), (ii) and (iii) we obtain that (s_i, s_j)-waves exist if and only if

$$c \in I_1 := (c_1, c_2) \cap [f'(s_j), f'(s_i)];$$

(iv) and (v) imply that $I_1 \neq \varnothing$. This proves the result in case (I).
The other cases can be dealt with similarly. Hence the conclusion. $\square$

Remark 3.12. The limits speed $c_1, \ldots, c_6$ are uniquely determined by the following equalities:

$$\begin{aligned} h(s_{j-1}, c_1) &= s_{i-1}, \quad & h(s_{j+1}, c_2) &= s_{i+1}, \\ h(s_j, c_3) &= s_{i-1}, \quad & h(s_j, c_4) &= s_{i+1}, \\ h(s_{j-1}, c_5) &= s_i, \quad & h(s_{j+1}, c_6) &= s_i. \end{aligned}$$

Using Lemmas 3.10 and 3.11 we can prove also Theorem 2.6, concerning the existence of homoclinic waves.

Proof of Theorem 2.6. (I) Let $s_i \in \mathcal{R}(g)$. Applying Lemma 3.10(I) and Lemma 3.11(I) with $s_j = s_i$ and arguing as in the proof of Theorem 2.5 proves the result.

(II) Let $s_i \in \mathcal{A}(g)$. Then by Lemma 3.10(II) and Lemma 3.11(II) we obtain that to have existence of homoclinic waves one of the following conditions must hold:
(i) $c = f'(s_{i-1})$ and $s_i \in (s_{i-1}, h(s_{i-2}, c))$;
(ii) $c = f'(s_{i+1})$ and $h(s_i, c) \in (s_{i+1}, s_{i+2})$.
Thus the conclusion follows. $\square$

Now we prove the existence of periodic travelling waves.

Proof of the Theorem 2.7. Let ϕ be a periodic wave with speed c and period $T \in \mathbb{R}$. There is no periodic classical solution of (2.3), thus ϕ has to be discontinuous somewhere. Since ϕ is periodic, ϕ has infinitely many discontinuities. Moreover, by Lemma 3.3 $c = f'(s_i)$ for some $s_i \in \mathcal{R}(g)$. In this case infinitely many periodic waves exist (oscillating around s_i). Hence the conclusion. $\square$

We conclude this section proving the regularity results stated in Theorem 2.9.
Let ϕ be a fixed wave. Set

$$E := \{\xi \in \mathbb{R} : \phi(\xi) = (f')^{-1}(c)\}. \tag{3.9}$$

Lemma 3.13. *Let D, C and E be defined by (2.6), (2.7) and (3.9) respectively. Then the following holds:*

$$\mathcal{D}(C) \subset \mathcal{D}(D) \subset C \subset \partial E.$$

Proof. The proof requires several steps.

(i) $C \subset E$. Let $\xi \in C \setminus \mathcal{D}(C)$. Then there is $B = B(\xi, \varepsilon)$ such that $\phi \in C(B) \cap C^1(B \setminus \{\xi\})$. In $B \setminus \{\xi\}$ there is an integral curve of (2.3), thus $\phi(\xi) = (f')^{-1}(c)$. This gives $C \setminus \mathcal{D}(C) \subset E$. Since E is closed, $\mathrm{Cl}(C \setminus \mathcal{D}(C)) \subset E$. Moreover,

$$C = (C \setminus \mathcal{D}(C)) \cup (\mathcal{D}(C) \setminus \mathcal{D}(\mathcal{D}(C))) \subset (C \setminus \mathcal{D}(C)) \cup \mathcal{D}(C \setminus \mathcal{D}(C)) = \mathrm{Cl}(C \setminus \mathcal{D}(C)).$$

Hence claim (i) follows.

(ii) $C \subset \partial E$. In fact, $\phi \in C^\infty(\mathrm{Int}(E))$.

(iii) $\mathcal{D}(D) \subset C$. Let $\xi \in \mathcal{D}(D)$. Then there exists $\{\xi_n\} \subset D$ such that $\xi_n \to \xi$ as $n \to \infty$. Then $\phi(\xi) = (f')^{-1}(c)$, ϕ is continuous in ξ and ϕ is not C^1 in any neighborhood of ξ. Thus $\xi \in C$.

(iv) $\mathcal{D}(C) \subset \mathcal{D}(D)$. Let $\xi_0 \in \mathcal{D}(C)$. Let there exist $B = B(\xi_0, \varepsilon)$ such that $\phi \in C(B)$. Then there exists $\xi_1 \in B$ such that $\phi(\xi_1) = (f')^{-1}(c)$ (otherwise $\xi_0 \in \mathrm{Int}(E)$ and $\xi_0 \notin C$). Thus ϕ is C^1 in a neighborhood $U \subset B$ of ξ_1. Let $\xi_1 > \xi_0$. Then ϕ is an integral curve of (2.3) in U. Since $\phi(\xi_0) = (f')^{-1}(c)$, it can be easily seen that there is no $\xi \in (\xi_0, \xi_0 + \varepsilon)$ such that $\phi(\xi) = (f')^{-1}(c)$. Supposing $\xi_1 < \xi_0$ and proceeding as before, we conclude that there is no $\xi \in (\xi_0 - \varepsilon, \xi_0)$ such that $\phi(\xi) = (f')^{-1}(c)$. Hence ξ is an isolated point of C. This completes the proof. □

Lemma 3.14. *Let $c \neq f'(s)$ for any $s \in \mathcal{Z}(g)$. Then $C = \varnothing$.*

Proof. By Lemma 3.3(iii), $\mathrm{card}(D) \leqslant 1$. If $\xi \in C$ there exists $B = B(\xi, \varepsilon)$ such that $\phi \in C(B) \cap C^1(B \setminus \{\xi\})$. Thus ϕ is given by an integral curve of (2.3) in $B \setminus \{\xi\}$ continuous in ξ. Since no such curve exists, the conclusion follows. □

Lemma 3.15. *Let $c = f'(s)$ for some $s \in A(g)$. Then:*

(i) if $\mathrm{card}(D) = 1$ then $C = \varnothing$;

(ii) $\mathrm{card}(C) \leqslant 2$.

Proof. (i) Let $\xi_0 \in D$. Suppose by contradiction that $C \cap (-\infty, \xi_0) \neq \varnothing$. Let $\xi_1 := \max\{\xi \in C : \xi < \xi_0\} < \xi_0$. Then $\phi \in C^1(\xi_1, \xi_0)$ and is an integral curve of (2.3). Since $\phi(\xi_0-) > (f')^{-1}(c) = \phi(\xi_1)$, ϕ is increasing in (ξ_1, ξ_0). Thus $s \in \mathcal{R}(g)$.

(ii) If $\mathrm{card}(D) = 1$ the conclusion is a consequence of (ii). Let $D = \varnothing$. Then $\phi \in C(\mathbb{R})$ and ϕ is monotone in $\mathbb{R}$. Hence either E is empty or $E = (\xi_1, \xi_2)$ with $-\infty \leqslant \xi_1 \leqslant \xi_2 \leqslant +\infty$. Since by Lemma 3.13 $C \subset \partial E$, $\mathrm{card}(C) \leqslant 2$. □

Lemma 3.16. *Let $c = f'(s)$ with $s \in \mathcal{R}(g)$.*

(i) If $\lim\limits_{\xi \to -\infty} \phi(\xi) = s_i \neq s$, there exists ξ_- such that $\phi \in C^1(-\infty, \xi_-)$.

(ii) If $\lim\limits_{\xi \to +\infty} \phi(\xi) = s_j \neq s$, there exists ξ_+ such that $\phi \in C^1(\xi_+, +\infty)$.

Proof. (i) For any ε there exists ξ_ε such that $|\phi(\xi) - s_i| < \varepsilon$ for any $\xi < \xi_\varepsilon$. Let ε be sufficently small. Either $\phi(\xi) < s = (f')^{-1}(c)$ or $\phi(\xi) > s = (f')^{-1}(c)$ in $(-\infty, \xi_\varepsilon)$. Thus $(-\infty, \xi_\varepsilon) \cap (D \cup C) = \varnothing$.

The proof of (ii) is similar. □

The proof of Theorem 2.9 is an easy consequence of Lemmas 3.14, 3.15 and 3.16.

4. Proof of Stability Results

In this Section we prove Theorems 2.10, 2.13 and 2.16; wide use is made of the theory of generalized characteristic (see [2], [3]).

Let us introduce some notations. In the following $U(\cdot; s)$ is the solution of the Cauchy problem

$$\begin{cases} \dfrac{dU}{dt} = g(U), \\ U(0) = s. \end{cases}$$

Moreover, we denote by $X(\cdot; x_0, s)$ the solution to the Cauchy problem

$$\begin{cases} \dfrac{dX}{dt} = f'(U(t;s)), \\ X(0) = x_0. \end{cases}$$

Let us first recall a useful lemma concerning entropy solutions of (1.1).

Lemma 4.1 [14]**.** *Let $u = u(x,t)$ be an entropy solution of (1.1) defined for $(x,t) \in \mathbb{R} \times [0, T^*)$. Let $(x,t) \in \mathbb{R} \times [0,T^*)$; denote by $\eta(x,t)$ the point on the x-axis intercepted by the minimal backward characteristic through (x,t).*

Let $\{(x_n, t_n)\}_{n\in\mathbb{N}} \subset \mathbb{R} \times (0, T^*)$ be a sequence such that:
(a) $t_n \to T^*$ as $n \to +\infty$;
(b) $A_1 \leqslant \eta(x_n, t_n) \leqslant A_2$ for some A_1, A_2;
(c) $U(-t_n; u(x_n,t_n)) \to s \in \mathcal{Z}(g)$.

Then there exists a global characteristic of the form $\zeta(t) = x^* + f'(s)t$ with $x^* \in [A_1, A_2]$.

Proof of Theorem 2.10. (i) Let us First prove that:

$$\lim_{t\to+\infty} \sup_{x>k} |1 - u(x,t)| = 0 \quad \forall k \in \mathbb{R}. \tag{4.1}$$

Since ϕ_c satisfies

$$\phi_c' = \frac{g(\phi_c)}{f'(\phi_c) - c} \quad \text{and} \quad \lim_{\xi\to-\infty} \phi_c(\xi) = 0,$$

we have

$$\lim_{\xi\to-\infty} \phi_c(\xi) e^{-\alpha\xi} = \begin{cases} 0 & \forall \alpha < -\dfrac{g'(0)}{c}, \\ +\infty & \forall \alpha > -\dfrac{g'(0)}{c}. \end{cases}$$

Let $c < \gamma < 0$; let ϕ_γ be a (0,1)-wave with speed γ. Then for any fixed $\alpha \in (-\frac{g'(0)}{c}, -\frac{g'(0)}{\gamma})$ we have

$$\lim_{\xi\to-\infty} \frac{\phi_\gamma(\xi)}{\phi_c(\xi)} = \lim_{\xi\to-\infty} \frac{\phi_\gamma(\xi)e^{-\alpha\xi}}{\phi_c(\xi)e^{-\alpha\xi}} = 0. \tag{4.2}$$

By (4.2) and hypothesis (c) we obtain:

$$\lim_{x\to-\infty} \frac{u_0(x)}{\phi_\gamma(x)} = +\infty.$$

Thus there exists $L_1 > 0$ such that

$$u_0(x) > \phi_\gamma(x) \quad \forall x < -L_1.$$

Let $\sigma \in (0, \min\{l, 1\})$ be fixed. By assumpton (b) there exists $L_2 > 0$ such that

$$u_0(x) > \sigma \quad \forall x > L_2.$$

Set $L := \max\{L_1, L_2\}$. Then we have

$$u_0(x) \geqslant v_0(x),$$

where:

$$v_0(x) := \begin{cases} \phi_\gamma(x) & x < -L, \\ 0 & -L \leqslant x \leqslant L, \\ \psi(x) & L < x \leqslant L + \delta, \\ \sigma & x > L + \delta. \end{cases}$$

Here $\psi = \psi(x)$ is implicitly defined by the equality:

$$x - L = \int_0^{\psi(x)} \frac{f'(s)}{g(s)} ds$$

and δ is such that $\psi(L + \delta) = \sigma$. Define:

$$v_0^1 = \begin{cases} \phi_\gamma(x) & x < -L, \\ 0 & x \geqslant -L. \end{cases}$$

Since $v_0 \geqslant v_0^1$, we have $v(x,t) \geqslant v^1(x,t)$, where v and v^1 are the entropy solutions of the problem (2.1) with data v_0 and v_0^1 respectively.

The solution $v^1(x,t)$ has a shock $\zeta = \zeta(t)$ starting from $(-L, 0)$. On the left of ζ the solution is given by $\phi_\gamma(x - \gamma t)$. Since

$$\zeta'(t) = \frac{f(u(\zeta(t)-,t)) - f(u(\zeta(t)+,t))}{u(\zeta(t)-) - u(\zeta(t)+)} \geqslant \frac{f(\phi_\gamma(\zeta(t) - \gamma t))}{\phi_\gamma(t) - \gamma t} > 0,$$

ζ is increasing, and thus $\phi_\gamma(\zeta(t) - \gamma t)$ is increasing. This implies that ζ' is increasing too, hence ζ is convex. Thus we obtain

$$\zeta(t) \geqslant \xi(t) := \zeta'(0)t - L = \frac{f(\phi_\gamma(-L))}{\phi_\gamma(-L)} t - L.$$

For any given any $k \in \mathbb{R}$ we have

$$(4.3) \quad 0 \leqslant \lim_{t \to +\infty} \sup_{k < x < \xi(t)} (1 - v(x,t)) = \lim_{t \to +\infty} \sup_{k < x < \xi(t)} (1 - v^1(x,t)) = \\ \lim_{t \to +\infty} \sup_{k < x < \xi(t)} (1 - \phi_\gamma(x - \gamma t)) = \lim_{t \to +\infty} (1 - \phi_\gamma(\xi(t) - \gamma t)) = 0.$$

Moreover, since $v_0(x) \geqslant \min\{\max\{\psi(x), 0\}, \sigma\}$, comparison results entail:

$$v(x,t) \geqslant \min\{\max\{\psi(x), 0\}, U(\sigma; t)\}.$$

Therefore

$$(4.4) \qquad 0 \leqslant \lim_{t \to +\infty} \sup_{x > \xi(t)} (1 - v(x,t)) \leqslant \lim_{t \to +\infty} (1 - U(\sigma; t)) + \lim_{t \to +\infty} (1 - \psi(\xi(t))) = 0.$$

By (4.3) and (4.4) it follows that

$$\lim_{t\to+\infty}\sup_{x>k}|1-v(x,t)|=0. \tag{4.5}$$

Let $M:=\sup_{\mathbb{R}}|u_0|$. Since $u(x,t)\leqslant U(M;t)$ we obtain

$$1-U(M;t)\leqslant 1-u(x,t)\leqslant 1-v(x,t),$$

which implies

$$\lim_{t\to+\infty}\sup_{x>k}|1-u(x,t)|\leqslant\lim_{t\to+\infty}\sup_{x>k}|1-U(M;t)|+\lim_{t\to+\infty}\sup_{x>k}|1-v(x,t)|=0,\quad\forall k\in\mathbb{R}.$$

This proves (4.1).

Let us investigate the asymptotic behavior of u for $x<k$ for some $k\in\mathbb{R}$. Let $(x,t)\in\mathbb{R}\times\mathbb{R}_+$; let $\eta(x,t)$ be the point of x-axis intercepted by the minimal backward characteristic $\zeta(t)$ through (x,t).

Let $k\in\mathbb{R}$ be fixed. We have

$$\lim_{t\to+\infty}\sup_{x<k}\eta(x,t)=-\infty. \tag{4.6}$$

In fact, note that $\eta(x,t)\leqslant\eta(k,t)$ for any $x\leqslant k$. Suppose by contradiction that

$$\limsup_{t\to+\infty}\eta(k,t)>-\infty.$$

Then there would exist an increasing sequence $t_n\to+\infty$ such that $\eta(k,t_n)$ decreases to $l>-\infty$. Let $X_n(t)$ be the minimal backward characteristics taken from (k,t_n). We can choose the sequence $\{t_n\}$ such that:

$$\lim_{n\to+\infty}U(-t_n;u(k;t_n))=0.$$

In fact suppose that there exists $\varepsilon>0$ such that $U(-t_n;u(k;t_n))\geqslant\varepsilon$ for any $n>n_\varepsilon$. Define:

$$\zeta(t):=l+\int_0^t f'(U(\tau;\varepsilon))d\tau.$$

We obtain

$$\zeta(t)\leqslant X_n(t)\quad\forall t\in[0,t_n].$$

For $t=t_n$ we have $\zeta(t_n)\leqslant X_n(t)=k$. However $\zeta(t)\to+\infty$ as $t\to+\infty$.

Applying Lemma 4.1 to the sequence $\{(k,t_n)\}_{n\in\mathbb{N}}$ we conclude that the straight line $x=l$ is a global characteristic. This implies that $u(x,t)=0$ for any $x\leqslant l, t\geqslant 0$, contradicting hypothesis (c). Thus (4.6) holds.

Now the following holds:

$$x=\eta(x,t)+\int_0^t f'(U(\tau-t;u(x,t))d\tau=\eta(x,t)+ct+\int\limits_{U(-t;u(x,t))}^{u(x,t)}H(s)ds,$$

where $H(s) = \dfrac{f'(s) - cs}{g(s)}$.

For a travelling waves $\phi = \phi_c$ we have

$$x - ct - x_0 = \int_{\phi_c(x_0)}^{\phi_c(x-ct)} H(s)ds.$$

From the above equalities we obtain

$$0 = \eta(x,t) - x_0 + \int_{U(-t;u(x,t))}^{u(x,t)} H(s)ds - \int_{\phi_c(x_0)}^{\phi_c(x-ct)} H(s)ds.$$

Choosing $x_0 := \eta(x,t)$ gives

$$0 = \int_{U(-t;u(x,t))}^{u(x,t)} H(s)ds - \int_{\phi_c(\eta(x,t))}^{\phi_c(x-ct)} H(s)ds = \int_{U(-t;u(x,t))}^{\phi_c(\eta(x,t))} H(s)ds - \int_{u(x,t)}^{\phi_c(x-ct)} H(s)ds,$$

hence

$$\int_{u(x,t)}^{\phi_c(x-ct)} H(s)ds = \int_{U(-t;u(x,t))}^{\phi_c(\eta(x,t))} H(s)ds.$$

Since $\eta(x,t) \to -\infty$ as $t \to +\infty$ uniformly in $(-\infty,k)$, $\phi_c(\eta(x,t)) \to 0$ as $t \to +\infty$ uniformly in $(-\infty,k)$. Moreover, since hypothesis (c) holds, $U(-t;u(x,t)) \to 0$ as $t \to +\infty$ uniformly in $(-\infty,k)$. Thus we obtain

$$\lim_{t\to+\infty} \int_{u(x,t)}^{\phi_c(x-ct)} H(s)ds = 0 \quad \text{uniformly in } (-\infty,k).$$

Since $\phi_c(\xi) \in (0,1)$ and $u(x,t) \in (0,1)$ for $x < k$ (for some $k \in \mathbb{R}$), we have

$$0 \leqslant (\min_{s\in(0,1)} H(s))|\phi_c(x-ct) - u(x,t)| \leqslant \int_{u(x,t)}^{\phi_c(x-ct)} H(s)ds.$$

Since the right hand side vanishes as $t \to +\infty$ uniformly in $(-\infty,k)$, we have

$$\lim_{t+\infty} \sup_{x<k} |\phi_c(x-ct) - u(x,t)| = 0. \tag{4.7}$$

The conclusion follows by (4.1) and (4.7).

(ii) Let c=0. Let ϕ_0 be the unique continuous (0,1)-wave such that

$$\phi_0(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ \psi(x) & \text{for } x > 0, \end{cases}$$

where ψ is implicitly defined by

$$x = \int_0^{\psi(x)} \frac{f'(s)}{g(s)} d\,s. \tag{4.8}$$

Let $\delta > 0$ be fixed. Consider problem (2.1) with initial data

$$u_0^\delta(x) := \begin{cases} 0 & \text{for } x < -\delta \\ \psi(x) & \text{for } x \geqslant -\delta, \end{cases}$$

where ψ is defined by (4.8).

The corresponding entropy solution u^δ is:

$$u^\delta(x,t) = \begin{cases} 0 & \text{for } x < \zeta(t), \\ \psi(x) & \text{for } \zeta(t) \leqslant x, \end{cases}$$

where $\zeta(t)$ is a shock curve, whose derivative satisfies Rankine-Hugoniot condition:

$$\zeta'(t) = \frac{f(\psi(\zeta(t))}{\psi(\zeta(t))} < 0.$$

Since $\zeta(t)$ is decreasing, $\psi(\zeta(t))$ and $\zeta'(t)$ are also decreasing. Hence the shock curve ζ is concave, thus u^δ converges to $\psi(x)$ on compact sets of $\mathbb{R}$. The conclusion follows noting that for any $\varepsilon > 0$ and compact set $K \subset \mathbb{R}$, $u_0^\delta \in \Gamma(\phi_0, K, \varepsilon)$ for δ sufficently small.

Next let $\phi_0(x)$ be a discontinuous (0,1)-wave with zero speed. We can suppose that

$$\inf\{x \in \mathbb{R} : \phi(y) > 0 \quad \forall y > x\} = 0.$$

For any $\eta > 0$, let x_η satisfy

$$|\phi_0(x)| < \eta \quad \forall x < x_\eta.$$

Fix η such that $[-2\eta, 0] \subset \phi_0(\mathbb{R})$. There is a finite number of $x \in [x_\eta, 0]$ such that $\inf_{y\in\mathbb{R}} \phi_0(y) = \phi_0(x+)$. Let x_0 be the minimum of such values. Given $\varepsilon \in (0, \phi_0(x_0-))$ we consider problem (2.1) with initial data:

$$u_0^\varepsilon(x) := \begin{cases} \phi_0(x) & \text{for } x \in \mathbb{R} \setminus [x_o - \delta, x_0], \\ \phi_0(x_0-) - \varepsilon & \text{for } x \in [x_0 - \delta, x_0]. \end{cases}$$

Here δ is the smallest positive constant such that $\phi_0(x_0 - \delta) = \phi_0(x_0-) - \varepsilon$. Set $\alpha := \phi_0(x_0-) - \varepsilon$. The corresponding entropy solution of (2.1):

$$u(x,t) = \begin{cases} \phi_0(x) & \text{for } x_n \leqslant x \text{ or } x < \min\{X(t; x_n - \delta, \alpha), \zeta(t)\}, \\ \psi(x) & \text{for } \zeta(t) \leqslant x < x_n, \\ U(t;\alpha) & \text{for } X(t; x_n - \delta, \alpha) \leqslant x < \zeta(t), \end{cases}$$

where $\psi(x)$ is defined by the equality

$$x - x_0 = \int_{\phi(x_0+)}^{\psi(x)} \frac{f'(s)}{g(s)},$$

and $\zeta(t)$ is a shock curve with $\zeta(0)=x_n$.

Let us prove that there exists $T>0$ such that

$$\zeta(T)=X(T;x_n-\delta,\alpha). \tag{4.9}$$

The shock curve $\zeta(t)$ has negative velocity for small $t>0$. In fact,

$$\zeta'(t)=\frac{f(\psi(\zeta(t)))-f(U(t;\alpha))}{\psi(\zeta(t))-U(t;\alpha)}<0.$$

This implies that $\psi(\zeta(t)+)$ is a decreasing function for small t. Moreover, there exists $t_1>0$ such that $U(t_1;\alpha)=\phi_0(x_0-)$ and

$$X(t_1;x_0-\delta,\alpha)=x_0-\delta+\int_{\phi_0(x_n)-\varepsilon}^{\phi_0(x_n)}\frac{f'(s)}{g(s)}ds=x_0-\delta+\int_{x_0-\delta}^{x_0}d\xi=x_0.$$

Suppose by contradiction that there exists a time t_2 such that

$$\begin{gathered}X(t_2;x_0-\delta,\alpha)<\zeta(t_2),\\ \zeta'(t_2)=0,\qquad \zeta'(t)<0\quad\forall t<t_2.\end{gathered} \tag{4.10}$$

Then we have

$$0=\frac{f(\psi(\zeta(t_2)+))-f(U(t_2;\alpha))}{\psi(\zeta(t_2)+)-U(t_2;\alpha)}<\frac{f(\phi_0(x_0+))-f(U(t_2;\alpha))}{\phi_0(x_0+)-U(t_2;\alpha)}.$$

This implies $U(t_2;\alpha)>\phi_0(x_0-)$. Hence $t_2>t_1$ and by the monotonicity of X

$$x_0=X(t_1;x_0-\delta,\alpha)<X(t_2;x_0-\delta,\alpha)<\zeta(t_2),$$

contradicting (4.10). Then (4.9) holds.

For $t=T$ the derivative of ζ satisfies

$$\zeta'(T)=\frac{f(\psi(\zeta(T)+))-f(\phi_0(\zeta(T)-))}{\psi(\zeta(T)+)-\phi_0(\zeta(T)-)}<0.$$

Then ζ is decreasing for $T<t<T+\tau$, for some $\tau>0$. This implies that $\zeta'(t)\leqslant 0$ for any t. In fact we have

$$\psi(\zeta(t)+)<\psi(\zeta(T)+)<\phi_0(x_n+),\qquad \phi_0(x_n-)<\phi_0(\zeta(t)-).$$

Choosing a compact set K such that $x_0\in$Int(K) gives the result. $\square$

Proof of Theorem 2.13.

(i) Given $c\in(0,f(1))$, let $\phi_c(\xi)$ be a (0,1)-wave with speed c. Then there exists $\xi_0\in\mathbb{R}$ such that $\phi_c(\xi)=1$ for any $\xi<\xi_0$ and ϕ_c is strictly increasing for $\xi>\xi_0$ (thus $\phi_c(\xi)<0$ for $\xi>\xi_0$). We can suppose $\xi_0=0$.

Given $\varepsilon > 0$, let $\alpha := \phi_c(0+) + \varepsilon$. Consider the Cauchy problem (2.1) with initial data:

$$u_0(x) = \begin{cases} \phi_c(x) & \text{for } x \leqslant 0 \text{ or } x > \delta, \\ \alpha & \text{for } 0 < x \leqslant \delta, \end{cases}$$

where δ is the smallest positive constant such that $\phi_c(\delta) = \alpha$. The corresponding entropy solution of (2.1) has the following structure:

$$u(x,t) = \begin{cases} 1 & \text{for } x \leqslant \zeta(t), \\ U(t;\alpha) & \text{for } \zeta(t) < x \leqslant X(t;\delta,\alpha), \\ \phi_c(x) & \text{for } x > \max\{\zeta(t), X(t;\delta,\alpha)\}, \end{cases}$$

where ζ is a shock curve.

As in the proof of Theorem 2.10(ii), it can be shown that the curve $\zeta(t)$ is increasing and convex for $t > 0$. Hence there exists $\sigma > 0$ such that

$$\sup_{x \in I(\phi_c,\sigma,t)} |\phi_c(x) - u(x,t)| \geqslant 1 \quad \forall t > t_1,$$

for some t_1.

(ii) Let $c = f(1)$. Then the unique profile connecting 1 to 0 is given by two constant states joined by a single shock at some $\xi_0 \in \mathbb{R}$. Let $\xi_0 = 0$.

(α) Let $L > 0$, $\varepsilon = \varepsilon(L) > 0$ to be determined. Consider problem (2.1) with initial data:

$$u_0^1(x) := \begin{cases} 1 & \text{for } x < 0, \\ -\varepsilon & \text{for } 0 \leqslant x < L, \\ \psi(x) & \text{for } L \leqslant x < L + \delta, \\ 0 & \text{for } x \geqslant L + \delta, \end{cases}$$

where $\psi(x)$ is implicitly defined by

$$x - L = \int_{-\varepsilon}^{\psi(x)} \frac{f'(s)}{g(s)} ds,$$

and δ is chosen so that $\psi(L + \delta) = 0$. Then

$$u_0^1(x) \geqslant v_0^1(x) := \begin{cases} 1 & \text{for } x < 0, \\ -\varepsilon & \text{for } x > 0. \end{cases}$$

The entropy solution to (2.1) with data v_0^1 is given by two constant states joined by a single shock ζ with derivative

$$\zeta'(t) = \frac{f(1) - f(U(-\varepsilon;t))}{1 - U(-\varepsilon;t))}.$$

By convexity of f there exists a unique $\overline{s} < 0$ such that $f(\overline{s}) = 0$. Let $0 < \varepsilon < -\overline{s}$ fixed. Then there exists a unique $t_\varepsilon > 0$ such that $\zeta'(t_\varepsilon) = 0$. Hence

$$\zeta(t_\varepsilon) = \int_0^{t_\varepsilon} \zeta'(\tau) d\tau = \int_0^{t_\varepsilon} \frac{f(1) - f(U(-\varepsilon;\tau))}{1 - U(-\varepsilon;\tau)} d\tau = \int_{-\varepsilon}^{\overline{s}} f(1) - f(s)(1-s)g(s) ds.$$

For any ε sufficiently small $\zeta(t_\varepsilon) \geqslant L+\delta$.

Since:

$$u_0^1(x) \geqslant v_0^2(x) = \begin{cases} \psi(x) & \text{for } x < L+\delta, \\ 0 & \text{for } x > L+\delta, \end{cases}$$

and v_0^2 is a stationary solution of (1.1) we have

$$u^1(x,t) \geqslant \max\{v^1(x,t), v_0^2(x)\}.$$

For $t = t_\varepsilon$ this gives

$$u^1(x,t_\varepsilon) \geqslant 0.$$

Then if $\varepsilon > 0$ is sufficently small, $u^1(x,t) \geqslant 0$ for $t = t_\varepsilon$.

(β) Let $M > 1$, $L > 0$. Consider the Cauchy problem (2.1) with initial data:

$$u_0^2(x) := \begin{cases} 1 & \text{for } x < 0, \\ M & \text{for } 0 \leqslant x < L, \\ 0 & \text{for } x \geqslant L. \end{cases} \tag{4.11}$$

There is a shock curve ζ starting from the point $(L,0)$, whose speed is:

$$\zeta'(t) = \frac{f(u(\zeta(t)-,t)}{u(\zeta(t)-,t)} \leqslant \frac{f(U(M;t))}{U(M;t)}. \tag{4.12}$$

Let ξ be the classical characteristic of equation $\xi(t) = f'(1)t$. From (4.12) we deduce that $\zeta'(t) - \xi'(t) < l < 0$ for some l, independent of t, for large t. Thus there exists $t_1 > 0$ such that $\zeta(t_1) = \xi(t_1)$. For $t \geqslant t_1$, the solution is given by:

$$u^2(x,t) = \begin{cases} 1 & \text{for } x < f(1)(t-t_1) + \zeta(t_1), \\ 0 & \text{for } x > f(1)(t-t_1) + \zeta(t_1). \end{cases}$$

(γ) Let $u_0 \in BV_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfy (a) and (b). Combining (α)-(β) with comparison results, we conclude that there is a time $t_0 > 0$ such that:

$$0 \leqslant u(x,t_0) \leqslant 1.$$

Therefore we can suppose $0 \leqslant u_0(x) \leqslant 1$ and $K = [0,L]$, for some $L > 0$. Let ζ_- be the minimal forward characteristic from $(0,0)$ and ζ_+ the maximal one from $(L,0)$.

Suppose by contradiction that

$$\zeta_-(t) < \zeta_+(t) \quad \forall t \geqslant 0.$$

If

$$\liminf_{t\to+\infty} u(\zeta_-(t)+,t) = l > 0,$$

there exists a time t_1 so that $u(\zeta_-(t)+,t) > \dfrac{l}{2}$ for every $t > t_1$. Then

$$\zeta_-'(t) > \frac{f(1) - f(\frac{l}{2})}{1 - \frac{l}{2}} > f(1) \geqslant \zeta_+'(t),$$

whence it follows that ζ_- and ζ_+ intersect.

This implies

$$\liminf_{t\to+\infty} u(\zeta_-(t)+,t) = 0.$$

Given $\nu \in (0, f(1))$, let z be the straight line $z(t) = \nu t + L$. Since $\zeta_-'(t) \geqslant f(1)$, z and ζ_- intersect at $t = t_2 > 0$ and $\zeta_-(t) > z(t)$ for every $t > t_2$.

Let $t_3 > t_2$ be such that $u(\zeta_-(t_3)+, t_3) < (f')^{-1}(\nu)$. The backward maximal characteristic $y = y(t)$ from $(\zeta_-(t_3), t_3)$ is classical for $t < t_3$ and convex. Moreover, it lies on the right hand side of the straight line $x = \zeta_-(t_3) + \nu(t - t_3)$ for $t < t_3$. Thus:

$$y(0) \geqslant \zeta_-(t_3) - \nu t_3 = [\zeta_-(t_3) - (\nu t_3 + L)] + L = [\zeta(t_3) - z(t_3)] + L > L,$$

which is a contradiction. Hence there exist a time $T > 0$ such that:

$$\zeta_-(T) = \zeta_+(T).$$

Fot $t > T$, the entropy solution is a single shock starting from $(\zeta_-(T), T)$.

(iii) Let $c \in (f(1), f'(1))$ (the other case can be dealth with similarly). Then there exists $\xi_0 \in \mathbb{R}$ such that $\phi_c(\xi) = 1$ for any $\xi < \xi_0$, ϕ_c is decreasing for $\xi > \xi_0$ and $\phi_0 := \lim_{\xi\to\xi_0^+} \phi(\xi) = h(1, c)$. Let $\xi_0 = 0$.

(α') Let $u^1(x,t)$ be the entropy solution of problem (2.1) with data:

$$u_0^1(x) := \min\{\varepsilon, \phi_c(x)\}, \tag{4.13}$$

with $\varepsilon \in (0, \phi_0)$. Then $u^1(x,t)$ is:

$$u^1(x,t) = \begin{cases} U(t;\varepsilon) & \text{for } x \leqslant \zeta(t), \\ \psi(x - ct) & \text{for } x > \zeta(t), \end{cases}$$

where ψ is defined by:

$$\xi = \int_{\phi_0}^{\psi(\xi)} \frac{f'(s) - c}{g(s)}\, ds,$$

and $\zeta(t)$ is the forward generalized characteristic starting from $(k, 0)$. Let x_0 be such that $\psi(x_0) = (f')^{-1}(c)$. The curve $\zeta(t)$ is a classical characteristic in $(0, t_0)$, where t_0 satisfies $U(t_0;\varepsilon) = (f')^{-1}(c)$.

It can be proved that $\zeta(t)$ is a shock for $t > t_0$ and that $\lim_{t\to+\infty} ct - \zeta(t) = 0$. Hence we obtain

$$\lim_{t\to+\infty} \sup_{x\in I(\phi_c,\sigma,t)} |u^1(x,t) - \phi_c(x - ct)| = 0, \tag{4.14}$$

for any $\sigma > 0$.

(β') Consider the Cauchy problem (2.1) with data:

$$u_0^2(x) := \begin{cases} \sigma & \text{for } x \leqslant -L, \\ 0 & \text{for } -L < x \leqslant L, \\ \psi(x) & \text{for } L < x \leqslant L + \delta, \\ \phi_c(x) & \text{for } x > L + \delta, \end{cases}$$

with $L > 0$, $\sigma \in (0,1)$ and ψ defined by

$$x - L = \int_0^{\psi(x)} \frac{f'(s)}{g(s)} ds.$$

Here δ satisfies $\psi(L+\delta) = \phi_c(L+\delta)$ (note that ψ is increasing, ϕ_c is decreasing and $\phi_c(L) > 0 = \psi(L)$).

Let u^2 be the corresponding entropy solution. Then there exist $T > 0$ and $\varepsilon > 0$ such that $u^2(x,T) \geqslant u_0^1(x)$ with u_0^1 given by (4.13). It follows that

$$u^2(x, t-T) \geqslant u^1(x,t) \quad \forall t > 0, \tag{4.15}$$

with u^1 satisfying (4.14).

(γ') Let $M > 0$ be such that $u_0(x) \leqslant M$. Consider the Cauchy problem (2.1) with initial data $u_0^3(x)$ given by:

$$u_0^3(x) := \begin{cases} L & \text{for } x < k, \\ \phi_c(x) & \text{for } x \geqslant k, \end{cases}$$

with $L := \max\{M, 1\}$. The entropy solution is

$$u^3(x,t) = \begin{cases} U(t;L) & \text{for } x < \zeta(t), \\ \phi_c(x-ct) & \text{for } x \geqslant \zeta(t), \end{cases}$$

where $\zeta(t)$ is a shock curve with $\zeta(0) = k$.

By the same argument used in (α'), we obtain

$$\lim_{t\to+\infty} \sup_{x \in I(\phi_c,\sigma,t)} |u^2(x,t) - \phi_c(x-ct)| = 0, \tag{4.16}$$

for every $\sigma > 0$.

(δ') Let u_0 satisfy hypotheses (a), (b) and (c). Then there exist u_0^2 and u_0^3 as in (β')-(γ') such that $u_0^2 \leqslant u_0 \leqslant u_0^3$. It can be seen that there exists $t_1 > 0$ such that

$$u_0^1(x) \leqslant u(x,t_1) \leqslant u_0^2(x).$$

Then the conclusion follows by (4.14), (4.15) and (4.16), due to comparison results. □

The proof of Theorem 2.16 is very similar to that given for stationary waves connecting 0 with 1 (see Theorem 2.10(ii)). We omit the details.

References

1. D. G. Aronson and H. F. Weinberger, *Nonlinear Diffusion in Population Genetics, Combustion and Nerve Pulse Propagation*, Partial Differential Equations and Related Topics, Lecture Notes in Mathematics 446, Springer-Verlag, 1975, pp. 5–49.
2. C. M. Dafermos, *Characteristics in Hyperbolic Conservation Laws*, Nonlinear Analysis and Mechanics Vol.1 (R. J. Knops, eds.), Research Notes in Mathematics 17, Pitman, 1977, pp. 1–58.
3. C. M. Dafermos, *Generalized Characteristics and the Structure of Solutions of Hyperbolic Conservation Laws*, Indiana Univ. Math. **26** (1977), 1097–1119.
4. C. M. Dafermos, *Large Time Behavior of Solutions of Hyperbolic Balance Laws*, Bull. Greek Math. **25** (1984), 15–29.
5. H. Fan and J. K. Hale, *Large-time Behavior in Inhomogeneus Conservation Laws*, Arch. Rat. Mech. Anal. **125** (1993), 201–216.

6. P. C. Fife, *Mathematical Aspects in Reacting and Diffusing Systems*, Lecture Notes in Biomathematics 28, Springer-Verlag, 1979.
7. S. N. Kružkov, *First Order Quasilinear Equations in Several Independent Variables*, Mat. Sb. **51** (1960), 99-128 (Russian); English transl. in Amer. Math. Soc. Transl. (2) **42** (1964), 199-231.
8. S. N. Kružkov and N. S. Petrosjan, *Asymptotic Behaviour of the Solutions of the Cauchy Problem for Nonlinear First Order Equations*, Uspekhi Mat. Nauk. **42** (1987), 3–40 (Russian); English transl. in Russian Math. Surveys **42** (1987), 1–47.
9. P. D. Lax, *Hyperbolic Systems of Conservation Laws II*, Comm. Pure Appl. Math. **10** (1957), 537–566.
10. T. P. Liu, *Invariants and Asymptotic Behavior of Solutions of a Conservation Law*, Proc. Amer. Math. Soc. **71** (1978), 227–231.
11. A. N. Lyberopoulos, *A Poincaré-Bendixson Theorem for Scalar Balance Laws*, Proc. R. Soc. Ed. **124A** (1994), 589–607.
12. R. Natalini and A. Tesei, *On a Class of Perturbed Conservation Laws*, Adv. Appl. Math. **13** (1992), 429–453.
13. C. Sinestrari, *Asymptotic Profile of Solutions of Conservation Laws with Source*, Differential and Integral Equations **9** (1996), 499–525.
14. C. Sinestrari, *Large Time Behaviour of Solutions of Balance Laws with Periodic Initial Data*, NoDEA **2** (1995), 111–131.
15. Z. Xin, *Asymptotic Stability of Rarefaction Waves for 2*2 Viscous Hyperbolic Conservation Laws*, J. Diff. Eq. **73** (1988), 45–77.

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