Shift-Differentiability of the Flow Generated by a Conservation Law

Alberto Bressan and Graziano Guerra

S.I.S.S.A., Via Beirut 4, Trieste 34014 Italy.

Abstract. The paper introduces a notion of "shift-differentials" for maps with values in the space **BV**. These differentials describe first order variations of a given function u, obtained by horizontal shifts of the points of its graph. The flow generated by a scalar conservation law is proved to be generically shift-differentiable, according to the new definition.

1 - Introduction

In the analysis of variational problems related to conservation laws [4], a major source of difficulties is the fact that the flow map: $u(0,\cdot) \mapsto u(t,\cdot)$ is usually not differentiable w.r.t. the linear structure of $\mathbf{L}^1$. As a simple example, consider Burgers' equation

$$u_t + \left[u^2/2\right]_x = 0 \tag{1.1}$$

with the family of initial conditions

$$u^\theta(0,x) = \theta x \cdot \chi_{[0,1]}(x). \tag{1.2}$$

By χ_I we denote here the characteristic function of the interval I. Assuming $\theta > 0$, the corresponding solution of (1.1)-(1.2) is

$$u^\theta(t,x) = \frac{\theta x}{1+\theta t} \cdot \chi_{[0,\ \sqrt{1+\theta t}]}(x). \tag{1.3}$$

Observe that the map $\theta \mapsto u^\theta(0,\cdot)$ describes a smooth curve in $\mathbf{L}^1$, namely a segment. However, for $t > 0$, the map $\theta \mapsto u^\theta(t,\cdot)$ is Lipschitz continuous but nowhere differentiable because the location $x^\theta(t) = \sqrt{1+(1+\theta)t}$ of the shock varies with θ.

To cope with this situation, in [3] a generalized notion of differential was studied, for maps $\theta \mapsto u^\theta$ taking values within a class of piecewise Lipschitz functions. The flow generated by a hyperbolic system of conservation laws was proved to be differentiable in this generalized sense, as long as the solutions remain piecewise Lipschitz continuous.

Aim of the present paper is to introduce a new differential structure on the space $\mathbf{BV}$ of integrable functions with bounded variation, and show that the flow generated by a scalar conservation law is generically differentiable w.r.t. this structure. We conjecture that a similar result also holds for $n \times n$ strictly hyperbolic systems.

Let a function $u \in \mathbf{BV}$ be given. In order to define a "tangent space" at u, one can follow a procedure which is now standard in differential geometry. On the family of continuous maps $\gamma : [0,\theta^*] \mapsto \mathbf{L}^1$, with $\gamma(0) = u$ and $\theta^* > 0$ (possibly depending on γ), consider the equivalence relation

$$\gamma \sim \gamma' \qquad \text{iff} \qquad \lim_{\theta\to 0} \frac{1}{\theta}\big\|\gamma(\theta) - \gamma'(\theta)\big\|_{\mathbf{L}^1} = 0. \tag{1.4}$$

Every equivalence class could be regarded as a "first-order tangent vector" at the point u. The set $\widehat{T}_u$ of all these equivalence classes, however, is extremely large, and cannot be adequately described. Therefore, one usually works with a particular subset $T_u \subset \widehat{T}_u$ of tangent vectors which admit a suitably nice representative.

The standard choice at this stage is to consider a family T_u of tangent vectors which can be put in a one-to-one correspondence with $\mathbf{L}^1(I\!R)$. More precisely, T_u is defined as the family of all equivalence classes of the maps

$$\theta \mapsto \gamma_v(\theta) \doteq u + \theta v, \qquad v \in \mathbf{L}^1(I\!R). \tag{1.5}$$

In this paper we study a different space T_u of tangent vectors, which can be put into a one-to-one correspondence with $\mathbf{L}^1(\mathrm{D}u)$. Here $\mathrm{D}u$ denotes the (signed) Radon measure corresponding to the distributional derivative of $u \in \mathbf{BV}$. When $v \in \mathcal{C}_c^1 \subset \mathbf{L}^1(\mathrm{D}u)$, to v we associate the equivalence class of the map $\theta \mapsto u^\theta$, where u^θ is implicitly defined as

$$u^\theta\big(x + \theta v(x)\big) = u(x) \tag{1.6}$$

for all $\theta \geq 0$ sufficiently small. We then show that this correspondence can be uniquely extended to the whole space $\mathbf{L}^1(\mathrm{D}u)$. Observe that in (1.5) the graph of u^θ is obtained by lifting the graph of u vertically by θv. On the other hand, in (1.6), the graph of u is shifted horizontally by θv. This motivates the term "shift-differential" used in the sequel.

Sections 2 and 3 are concerned with the definition and the basic properties of shift differentials. In particular, we study the relation between the existence of a shift differential for a map $\theta \mapsto u^\theta$ and its differentiability in the usual sense. In Section 4 we consider the semigroup $\mathbf{S}$ generated by the scalar conservation law

$$w_t + f(w)_x = 0. \tag{1.7}$$

Denoting by $\mathbf{S}_t u = w(t, \cdot)$ the solution of (1.7) with initial condition $w(0, x) = u(x)$, we show that (generically) the flow map $u \mapsto \mathbf{S}_t u$ is shift differentiable. This result provided the initial motivation for a study of shift differentials. We believe that our approach can also be useful in the analysis of variational problems in the class of $\mathbf{BV}$ functions, in particular those related to conservation laws. A first example of "shift tangent vectors", in connection with piecewise constant approximate solutions of a 2×2 system of conservation laws, can be found in [2].

2 - Shift Tangent Vectors

Throughout the following, our basic function space is the normed space

$$X \doteq \big(\mathbf{L}^1(I\!R) \cap \mathbf{BV}(I\!R);\ \|\cdot\|_{\mathbf{L}^1(I\!R)}\big).$$

Here $\mathbf{L}^1(I\!R)$ refers to the standard Lebesgue measure. The elements of X are thus equivalence classes of functions. For sake of definiteness, we shall always deal with left continuous representatives. By $\mathbf{Lip}(I\!R)$ we denote the space of Lipschitz continuous functions. The Lipschitz constant and the $\mathbf{L}^\infty$ norm of v are denoted by $Lip(v)$ and $\|v\|_\infty$, respectively.

Given $u \in X$, for every $v \in \mathbf{L}^1(\mathrm{D}u)$ we shall construct a first-order variation $\theta \mapsto u^\theta$ of u, consistent with the definition (1.6) in the smooth case. A preliminary definition is needed.

Definition 1. Let $u \in X$ and $v \in \mathbf{L}^1(\mathrm{D}u)$. Consider a family of functions $v^\theta \in \mathbf{L}^1(\mathrm{D}u) \cap \mathbf{Lip}(I\!R)$, with $\theta \in [0, \theta^*]$. We say that v^θ *hat-converges* to v, and write $v^\theta \xrightarrow{\wedge} v$, provided that

$$\lim_{\theta\to 0} \int_{I\!R} \big|v^\theta(x) - v(x)\big|\, |\mathrm{D}u|(x) = 0, \qquad \limsup_{\theta\to 0} Lip(\theta v^\theta) < 1, \qquad \lim_{\theta\to 0} \|\theta v^\theta\|_\infty = 0. \tag{2.1}$$

Observe that the last two conditions in (2.1) essentially depend on the parametrization of the family $\{v^\theta\}$. The next lemma shows that, for every $v \in \mathbf{L}^1(\mathrm{D}u)$, it is always possible to construct a family of functions v^θ hat-converging to v.

Lemma 1. *Let $r = r(\eta)$ be continuous and strictly increasing, with $r(0) = 0$. Then, for all $v \in \mathbf{L}^1(\mathrm{D}u)$ there exist functions $v^\theta \in \mathbf{C}_c^\infty(I\!R)$, $\theta \in [0, \theta^*]$ satisfying*

$$Lip(v^\theta) \le \frac{1}{r(\theta)}, \qquad \|v^\theta\|_\infty \le \frac{1}{r(\theta)}, \qquad \lim_{\theta\to 0} v^\theta = v \quad in \quad \mathbf{L}^1(\mathrm{D}u). \tag{2.2}$$

Proof. Take any sequence of functions $v_k \in \mathbf{C}_c^\infty(I\!R)$ converging to v in $\mathbf{L}^1(\mathrm{D}u)$ and define $\alpha_k = \max_{i\le k}\{Lip(v_i), \|v_i\|_\infty\} + k$. Choose the values θ_k so that $r(\theta_k) = (1/\alpha_k)$. The interpolated functions

$$v^\theta(x) = \begin{cases} \frac{\theta-\theta_{k+1}}{\theta_k-\theta_{k+1}} v_{k-1}(x) + \frac{\theta_k-\theta}{\theta_k-\theta_{k+1}} v_k(x) & \text{if} \quad \theta \in (\theta_{k+1}, \theta_k] \\ v(x) & \text{if} \quad \theta = 0 \end{cases}$$

then satisfy all required properties. □

Applying the above lemma with $r(\eta) = \sqrt{\eta}$, from (2.2) it follows $Lip(\theta v^\theta) \le \sqrt{\theta}$ and $\|\theta v^\theta\|_\infty \le \sqrt{\theta}$. Therefore, $v^\theta \xrightarrow{\wedge} v$.

We now introduce a convenient notation for certain functions, obtained by shifting horizontally the graph of u.

Definition 2. If $u \in X$ and $Lip(\theta v^\theta) \le \alpha < 1$, we denote by $v^\theta \star u$ the function implicitly defined by

$$(v^\theta \star u)\big(x + \theta v^\theta(x)\big) = u(x). \tag{2.3}$$

Remark 1. Setting $y_\theta(x) = x + \theta v^\theta(x)$, one easily checks that y_θ is a Lipschitz function with $Lip(y_\theta) \le 1 + \alpha$. Its inverse $x_\theta(y)$ is also Lipschitz continuous, with constant $Lip(x_\theta) \le 1/(1-\alpha)$. Hence, the function $v^\theta \star u$ in (2.3) is well defined. In particular, if $v^\theta \xrightarrow{\wedge} v$, then for all $\theta > 0$ sufficiently small the definition (2.3) is meaningful.

With the above notation, we can now introduce the main definition in this paper.

Definition 3. Fix $u \in X$ and consider a path $\theta \mapsto u^\theta \in \mathbf{L}^1(I\!R)$, defined on some interval $\theta \in [0, \theta^*]$, with $u^0 = u$. We say that the path u^θ *generates the shift tangent vector* $v \in \mathbf{L}^1(\mathrm{D}u)$ if, for some functions $v^\theta \xrightarrow{\wedge} v$, one has

$$\lim_{\theta \to 0} \frac{1}{\theta} \big\| u^\theta - v^\theta \star u \big\|_{\mathbf{L}^1(I\!R)} = 0. \tag{2.4}$$

Roughly speaking, this means that in first approximation the functions u^θ are obtained by shifting the graph of u horizontally by the amount θv.

Example 1. Let $u(x) = x \cdot \chi_{[0,1]}(x)$ and define

$$u^\theta(x) = (1+\theta)^2 x \cdot \chi_{[0,1+\theta]}(x).$$

Then the curve $\theta \mapsto u^\theta$ generates the shift tangent vector $v \in \mathbf{L}^1(\mathrm{D}u)$, with

$$v(x) = \begin{cases} -2x & \text{if} \quad x \in [0,1)\,, \\ 1 & \text{if} \quad x = 1. \end{cases}$$

Indeed, one can check that (2.4) holds, choosing for example $v^\theta(x) = \max\big\{-2x,\ 1 + (x-1)/2\theta\big\}$.

We will show that the above definition does not depend on the choice of the approximating functions v^θ. Moreover, if it exists, the vector $v \in \mathbf{L}^1(\mathrm{D}u)$ is uniquely determined by the curve $\theta \mapsto u^\theta$. A few preliminary lemmas are needed.

Lemma 2. *For all $u \in X$ and $\psi \in \mathbf{Lip}(I\!R) \cap \mathbf{L}^\infty(I\!R)$ one has*

$$\int_{I\!R} u(x)\psi'(x)\mathrm{d}x = -\int_{I\!R} \psi(x)\mathrm{D}u(x). \tag{2.5}$$

Indeed, when $\psi \in \mathbf{C}^1_c(I\!R)$, (2.5) follows simply by the definition of the measure $\mathrm{D}u$. In the general case, (2.5) is proved by an approximation argument, using the fact that u is not only in $\mathbf{BV}(I\!R)$ but also in $\mathbf{L}^1(I\!R)$.

Lemma 3. *Let $u \in X$ and, for a fixed $\theta > 0$, assume $Lip(\theta v^\theta) \leq \alpha < 1$, $Lip(\theta \tilde{v}^\theta) \leq \alpha < 1$. Then, for all $\varphi \in \mathbf{C}^0_c(I\!R)$, one has*

$$\frac{1}{\theta}\int_{I\!R} \big[(v^\theta \star u)(y) - (\tilde{v}^\theta \star u)(y)\big]\varphi(y)\mathrm{d}y = -\int_{I\!R} \big[v^\theta(x) - \tilde{v}^\theta(x)\big]\varphi_\theta(x)\mathrm{D}u(x), \tag{2.6}$$

where

$$\varphi_\theta(x) = \int_0^1 \varphi\Big(x + \theta\big[s v^\theta(x) + (1-s)\tilde{v}^\theta(x)\big]\Big)\mathrm{d}s. \tag{2.7}$$

Proof. Define $y_\theta(x) = x + \theta v^\theta(x)$, $\tilde{y}_\theta(x) = x + \theta\tilde{v}^\theta(x)$ and denote by $x_\theta(y)$, $\tilde{x}_\theta(y)$ the corresponding inverse functions. We thus have $(v^\theta \star u)(y) = u(x_\theta(y))$ and $(\tilde{v}^\theta \star u)(y) = u(\tilde{x}_\theta(y))$. We now fix $\varphi \in \mathbf{C}^0_c(I\!\!R)$ and compute

$$\int_{I\!\!R} (v^\theta \star u)(y)\varphi(y)\mathrm{d}y = \int_{I\!\!R} u\big(x_\theta(y)\big)\varphi(y)\mathrm{d}y = \int_{I\!\!R} u\big(x_\theta(y_\theta(x))\big)\varphi\big(y_\theta(x)\big)\Big|\frac{\mathrm{d}}{\mathrm{d}x}y_\theta(x)\Big|\mathrm{d}x,$$

performing the change of variable $y = y_\theta(x)$. By setting $\psi(y) = \int_{-\infty}^y \varphi(\xi)\mathrm{d}\xi$ and observing that $x_\theta(y_\theta(x)) = x$ and $\frac{\mathrm{d}}{\mathrm{d}x}y_\theta(x) = 1 + \theta\frac{\mathrm{d}}{\mathrm{d}x}v^\theta(x) \geq \frac{1}{1-\alpha} > 0$, we obtain

$$\int_{I\!\!R} (v^\theta \star u)(y)\varphi(y)\mathrm{d}y = \int_{I\!\!R} u(x)\frac{\mathrm{d}}{\mathrm{d}x}\psi(y_\theta(x))\mathrm{d}x.$$

Since the composition $\psi \circ y_\theta$ lies in $\mathbf{Lip}(I\!\!R) \cap \mathbf{L}^\infty(I\!\!R)$, we can apply Lemma 2 and deduce

$$\int_{I\!\!R} (v^\theta \star u)(y)\varphi(y)\mathrm{d}y = -\int_{I\!\!R} \psi(y_\theta(x))\mathrm{D}u(x).$$

A similar identity holds for $\tilde{v}^\theta \star u$. Subtracting these two equalities and dividing by θ we now obtain

$$\frac{1}{\theta}\int_{I\!\!R} \big[(v^\theta \star u)(y) - (\tilde{v}^\theta \star u)(y)\big]\varphi(y)\mathrm{d}y = -\frac{1}{\theta}\int_{I\!\!R} \big[\psi(y_\theta(x)) - \psi(\tilde{y}_\theta(x))\big]\mathrm{D}u(x). \tag{2.8}$$

This completes the proof, observing that

$$\begin{aligned}
\psi\big(y_\theta(x)\big) - \psi\big(\tilde{y}_\theta(x)\big) &= \int_0^1 \varphi\Big(x + \theta\big[sv^\theta(x) + (1-s)\tilde{v}^\theta(x)\big]\Big)\cdot\theta\big[v^\theta(x) - \tilde{v}^\theta(x)\big]\mathrm{d}s \\
&= \varphi_\theta(x)\cdot\theta\big[v^\theta(x) - \tilde{v}^\theta(x)\big].
\end{aligned}$$

□

Theorem 1. *Let $u \in X$, $v, \tilde{v} \in \mathbf{L}^1(\mathrm{D}u)$ and assume $v^\theta \xrightarrow{\wedge} v$, $\tilde{v}^\theta \xrightarrow{\wedge} \tilde{v}$. Then one has*

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_{I\!\!R} \big|(v^\theta \star u)(y) - (\tilde{v}^\theta \star u)(y)\big|\,\mathrm{d}y = \int_{I\!\!R} \big|v(x) - \tilde{v}(x)\big||\mathrm{D}u|(x). \tag{2.9}$$

Proof. We first establish the inequality

$$\limsup_{\theta\to 0}\frac{1}{\theta}\int_{I\!\!R} \big|(v^\theta \star u)(y) - (\tilde{v}^\theta \star u)(y)\big|\mathrm{d}y \leq \int_{I\!\!R} \big|v(x) - \tilde{v}(x)\big||\mathrm{D}u|(x). \tag{2.10}$$

Consider any $\varphi \in \mathbf{C}^0_c(I\!\!R)$, with $\|\varphi\|_\infty \leq 1$. By (2.7) this implies $|\varphi_\theta(x)| \leq 1$ for all x. From (2.6) we deduce

$$\frac{1}{\theta}\int_{I\!\!R} \big[(v^\theta \star u)(y) - (\tilde{v}^\theta \star u)(y)\big]\varphi(y)\mathrm{d}y \leq \int_{I\!\!R} \big|v^\theta(x) - \tilde{v}^\theta(x)\big||\mathrm{D}u|(x).$$

Taking the supremum over all φ with $\|\varphi\|_\infty \le 1$ we obtain

$$\frac{1}{\theta}\int_{I\!R} \big|(v^\theta \star u)(y) - (\tilde v^\theta \star u)(y)\big| \mathrm{d}y \le \int_{I\!R} \big|v^\theta(x) - \tilde v^\theta(x)\big| |\mathrm{D}u|(x).$$

Since $v^\theta \to v$ and $\tilde v^\theta \to \tilde v$ in $\mathbf{L}^1(\mathrm{D}u)$, the inequality (2.10) is proved by taking the upper limit as $\theta \to 0$.

Next, consider the converse inequality:

$$\int_{I\!R} \big|v(x) - \tilde v(x)\big| |\mathrm{D}u|(x) \le \liminf_{\theta\to 0} \frac{1}{\theta}\int_{I\!R} \big|(v^\theta \star u)(y) - (\tilde v^\theta \star u)(y)\big| \mathrm{d}y. \tag{2.11}$$

To prove (2.11), we first observe that, for all $\varphi \in \mathbf{C}^0_c(I\!R)$, one has

$$\lim_{\theta\to 0}\int_{I\!R} \big[v^\theta(x) - \tilde v^\theta(x)\big] \varphi_\theta(x)\mathrm{D}u(x) = \int_{I\!R} \big[v(x) - \tilde v(x)\big] \varphi(x)\mathrm{D}u(x). \tag{2.12}$$

Indeed, the assumptions of hat-convergence imply $v^\theta \to v$, $\tilde v^\theta \to \tilde v$ in $\mathbf{L}^1(\mathrm{D}u)$, while $\varphi_\theta \to \varphi$ uniformly on $I\!R$, because of (2.7) and the fact that $\|\theta v^\theta\|_\infty$, $\|\theta \tilde v^\theta\|_\infty \to 0$.

By Lemma 3, from (2.12) we deduce

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_{I\!R} \big[(v^\theta \star u)(y) - (\tilde v^\theta \star u)(y)\big] \varphi(y)\mathrm{d}y = -\int_{I\!R} \big[v(x) - \tilde v(x)\big] \varphi(x)\mathrm{D}u(x). \tag{2.13}$$

From (2.13) it now follows that, for all $\varphi \in \mathbf{C}^0_c(I\!R)$ with $\|\varphi\|_\infty \le 1$,

$$-\int_{I\!R} \big[v(x) - \tilde v(x)\big] \varphi(x)\mathrm{D}u(x) \le \liminf_{\theta\to 0} \frac{1}{\theta}\int_{I\!R} \big|(v^\theta \star u)(y) - (\tilde v^\theta \star u)(y)\big| \mathrm{d}y. \tag{2.14}$$

Since the right hand side of (2.14) does not depend on φ, we obtain the inequality (2.11) by taking the supremum over all $\varphi \in \mathbf{C}^0_c(I\!R)$ with $\|\varphi\|_\infty \le 1$. This completes the proof of (2.9). □

Remark 2. In the same setting of Definition 3, let (2.4) hold for some family of functions $v^\theta \stackrel{\wedge}{\longrightarrow} v$. Consider any other family $\tilde v^\theta \stackrel{\wedge}{\longrightarrow} v$. Then by Theorem 1 it follows

$$\begin{aligned}\limsup_{\theta\to 0}\frac{1}{\theta}\big\|u^\theta - \tilde v^\theta \star u\big\|_{\mathbf{L}^1(I\!R)} &\le \limsup_{\theta\to 0}\frac{1}{\theta}\big\|u^\theta - v^\theta \star u\big\|_{\mathbf{L}^1(I\!R)} + \limsup_{\theta\to 0}\frac{1}{\theta}\big\|v^\theta \star u - \tilde v^\theta \star u\big\|_{\mathbf{L}^1(I\!R)} \\ &\le 0.\end{aligned} \tag{2.15}$$

This shows that the definition of shift tangent vector does not depend on the choice of v^θ in (2.4).

Remark 3. Still in the setting of Definition 3, assume that the same curve $\theta \mapsto u^\theta$ generates the shift tangent vectors $v, \tilde v \in \mathbf{L}^1(\mathrm{D}u)$. Choosing functions $v^\theta \stackrel{\wedge}{\longrightarrow} v$, $\tilde v^\theta \stackrel{\wedge}{\longrightarrow} \tilde v$, from Theorem 1 it now follows

$$\begin{aligned}\|v - \tilde v\|_{\mathbf{L}^1(\mathrm{D}u)} &= \lim_{\theta\to 0}\frac{1}{\theta}\big\|v^\theta \star u - \tilde v^\theta \star u\big\|_{\mathbf{L}^1(I\!R)} \\ &\le \lim_{\theta\to 0}\frac{1}{\theta}\big\|v^\theta \star u - u^\theta\big\|_{\mathbf{L}^1(I\!R)} + \lim_{\theta\to 0}\frac{1}{\theta}\big\|u^\theta - \tilde v^\theta \star u\big\|_{\mathbf{L}^1(I\!R)} \\ &= 0.\end{aligned}$$

Therefore, v and $\tilde{v}$ must coincide almost everywhere w.r.t. the measure $\mathrm{D}u$. This proves that the shift tangent vector generated by a curve $\theta \mapsto u^\theta$, if it exists, is uniquely determined in the space $\mathbf{L}^1(\mathrm{D}u)$.

Corollary 1. *Assume that the curve $\theta \mapsto u^\theta$ (with $u^0 = u$) generates the shift tangent vector $v \in \mathbf{L}^1(\mathrm{D}u)$. Then for all $\varphi \in \mathbf{C}_c^0(I\!R)$ one has*

$$\lim_{\theta\to 0} \frac{1}{\theta} \int_{I\!R} \big[u^\theta(y) - u(y)\big]\varphi(y)\mathrm{d}y = -\int_{I\!R} v(x)\varphi(x)\mathrm{D}u(x). \tag{2.16}$$

Proof. Take v^θ hat-converging to v and set $\tilde{v}^\theta \equiv 0$ so that $\tilde{v}^\theta \star u = u$ for all θ. Using (2.13) we obtain

$$\begin{aligned}\lim_{\theta\to 0} \frac{1}{\theta} \int_{I\!R} \big[u^\theta(y) - u(y)\big]\varphi(y)\mathrm{d}y &= \lim_{\theta\to 0} \frac{1}{\theta} \int_{I\!R} \big[u^\theta(y) - (v^\theta \star u)(y)\big]\varphi(y)\mathrm{d}y \\ &\quad + \lim_{\theta\to 0} \frac{1}{\theta} \int_{I\!R} \big[(v^\theta \star u)(y) - (\tilde{v}^\theta \star u)(y)\big]\varphi(y)\mathrm{d}y \\ &= 0 - \int_{I\!R} [v(x) - 0]\varphi(x)\mathrm{D}u(x).\end{aligned}$$

□

Denote by μ_θ the absolutely continuous measure having density $(u^\theta - u)/\theta$ w.r.t. Lebesgue measure. The above corollary states that, if u^θ generates the shift tangent vector v, then the measures μ_θ converge weakly to the measure having density $-v$ w.r.t. the measure $\mathrm{D}u$.

On the other hand, according to the standard definition, a function $w \in \mathbf{L}^1(I\!R)$ is the differential of the curve $\theta \mapsto u^\theta$ at $\theta = 0$ if

$$\lim_{\theta\to 0} \left\| \frac{u^\theta - u}{\theta} - w \right\|_{\mathbf{L}^1(I\!R)} = 0. \tag{2.17}$$

The following theorem describes the relations between the shift differentiability of a curve and its differentiability in the usual sense. By $w\mathrm{d}x$ we denote here the measure having density w w.r.t. Lebesgue measure and by $-v\mathrm{D}u$ the measure having density $-v$ w.r.t. the measure $\mathrm{D}u$.

Theorem 2. *Consider a map $\theta \mapsto u^\theta \in X$, with $u^0 = u$, generating the shift tangent vector $v \in \mathbf{L}^1(\mathrm{D}u)$. Then this map is differentiable at $\theta = 0$ in the usual sense if and only if $v\mathrm{D}u$ is absolutely continuous w.r.t. Lebesgue measure. In this case we have $v\mathrm{D}u = -w\mathrm{d}x$ with w defined by* (2.17)*.*

Viceversa, let (2.17) *hold. Then the curve $\theta \mapsto u^\theta$ generates a shift tangent vector $v \in \mathbf{L}^1(\mathrm{D}u)$ if and only if the measure $w\mathrm{d}x$ is absolutely continuous w.r.t. the measure $\mathrm{D}u$, and the equality $w\mathrm{d}x = -v\mathrm{D}u$ holds.*

Proof. **1.** Assume that the curve $\theta \mapsto u^\theta$ generates the shift tangent vector v and that (2.17) holds. Corollary 1 then implies

$$\int_{I\!R} w(x)\varphi(x)\mathrm{d}x = -\int_{I\!R} v(x)\varphi(x)\mathrm{D}u(x) \qquad \forall \varphi \in \mathcal{C}_c^0.$$

Hence we have the identity of measures $w\mathrm{d}x = -v\mathrm{D}u$. This proves the "only if" parts of the theorem.

2. Next, consider any function $v \in \mathbf{L}^1(\mathrm{D}u)$. Choose $v^\theta \xrightarrow{\wedge} v$ with the additional property that $\lim_{\theta\to 0} Lip(\theta v^\theta) = 0$. This choice is possible because of Lemma 1. Then define

$$y_{\theta,s}(x) = x + s\theta v^\theta(x) \qquad s \in [0,1]. \tag{2.18}$$

By Remark 1, for θ sufficiently small the map $x \mapsto y_{\theta,s}(x)$ is invertible for every $s \in [0,1]$. Denoting by $x_{\theta,s}(y)$ its inverse, there holds

$$\left|\frac{\mathrm{d}}{\mathrm{d}x} y_{\theta,s}(x)\right| \le \frac{3}{2}, \qquad \left|\frac{\mathrm{d}}{\mathrm{d}y} x_{\theta,s}(y)\right| \le 2 \qquad \text{a. e.} \tag{2.19}$$

By (2.19), the composed function $w\big(y_{\theta,s}(x)\big)$ satisfies

$$\big\| w \circ y_{\theta,s}\big\|_{\mathbf{L}^1(I\!R)} \ \le\ 2\|w\|_{\mathbf{L}^1(I\!R)}. \tag{2.20}$$

Observe that $y_{\theta,s}(x) \to x$ as $\theta \to 0$, uniformly w.r.t. x and $s \in [0,1]$. Together with (2.19), this yields

$$\lim_{\theta\to 0} \big\| w \circ y_{\theta,s} - w\big\|_{\mathbf{L}^1(I\!R)} = 0. \tag{2.21}$$

Indeed, (2.21) is clear if w is continuous with compact support. The general case where $w \in \mathbf{L}^1(I\!R)$ is proved by a standard approximation argument.

3. Now assume that the functions $w \in \mathbf{L}^1(I\!R)$ and $v \in \mathbf{L}^1(\mathrm{D}u)$ satisfy $w\mathrm{d}x = -v\mathrm{D}u$. If $v^\theta \xrightarrow{\wedge} v$ is as in step **2**, we claim that

$$\lim_{\theta\to 0} \frac{1}{\theta}\big\| u + \theta w - v^\theta \star u\big\|_{\mathbf{L}^1(I\!R)} = 0. \tag{2.22}$$

To show this, take any $\varphi \in \mathbf{C}_c^0(I\!R)$ with $\|\varphi\|_\infty \le 1$. Introducing the function $\psi(x) = \int_{-\infty}^x \varphi(\xi)\mathrm{d}\xi$, we compute

$$\begin{aligned}\frac{1}{\theta}\int_{I\!R}\big[u(y) + \theta w(y) - (v^\theta \star u)(y)\big]\varphi(y)\mathrm{d}y &= \frac{1}{\theta}\int_{I\!R}\Big[u(y)\varphi(y) + \theta w(y)\varphi(y) - u\big(x_{\theta,1}(y)\big)\varphi(y)\Big]\mathrm{d}y\\ &= \int_{I\!R} w(y)\varphi(y)\mathrm{d}y + \frac{1}{\theta}\int_{I\!R} u(x)\frac{\mathrm{d}}{\mathrm{d}x}\Big[\psi(x) - \psi\big(y_{\theta,1}(x)\big)\Big]\mathrm{d}x.\end{aligned} \tag{2.23}$$

Here the second equality is obtained performing the change of variable $y = y_{\theta,1}(x)$ in the last term of the integrand. Since $\psi - \psi \circ y_{\theta,1} \in \mathbf{Lip}(I\!R) \cap \mathbf{L}^\infty(I\!R)$ we can apply Lemma 2 and deduce

$$\begin{aligned}
\frac{1}{\theta}\int_{I\!R} u(x)\frac{\mathrm{d}}{\mathrm{d}x}\Big[\psi(x) - \psi\big(y_{\theta,1}(x)\big)\Big]\mathrm{d}x &= \int_{I\!R}\Big[\int_0^1 \varphi(x + s\theta v^\theta(x))\mathrm{d}s\Big] v^\theta(x)\mathrm{D}u(x) \\
&\le \|v - v^\theta\|_{\mathbf{L}^1(\mathrm{D}u)} + \int_{I\!R}\Big[\int_0^1 (\varphi \circ y_{\theta,s})(x)\mathrm{d}s\Big] v(x)\mathrm{D}u(x) \\
&= \|v - v^\theta\|_{\mathbf{L}^1(\mathrm{D}u)} - \int_{I\!R}\Big[\int_0^1 (\varphi \circ y_{\theta,s})(x)\mathrm{d}s\Big] w(x)\mathrm{d}x,
\end{aligned} \tag{2.24}$$

thanks to the identity $w\mathrm{d}x = -v\mathrm{D}u$. Using the change of variable $y = y_{\theta,s}(x)$ we now obtain

$$\begin{aligned}
\int_{I\!R} w(y)\varphi(y)\mathrm{d}y &= \int_{I\!R} w(y_{\theta,s}(x))\varphi(y_{\theta,s}(x))\frac{\mathrm{d}}{\mathrm{d}x}y_{\theta,s}(x)\mathrm{d}x \\
&= \int_{I\!R} (w \circ y_{\theta,s})(x)(\varphi \circ y_{\theta,s})(x)\Big[1 + s\theta\frac{\mathrm{d}}{\mathrm{d}x}v^\theta(x)\Big]\mathrm{d}x \\
&\le \int_{I\!R} (w \circ y_{\theta,s})(x)(\varphi \circ y_{\theta,s})(x)\mathrm{d}x + \Big\|s\theta\frac{\mathrm{d}}{\mathrm{d}x}v^\theta\Big\|_\infty \cdot \|w \circ y_{\theta,s}\|_{\mathbf{L}^1(I\!R)} \\
&\le \int_{I\!R} w(x)(\varphi \circ y_{\theta,s})(x)\mathrm{d}x + \|w \circ y_{\theta,s} - w\|_{\mathbf{L}^1(I\!R)} + 2\theta Lip(v^\theta)\|w\|_{\mathbf{L}^1(I\!R)},
\end{aligned} \tag{2.25}$$

because of (2.20). Combining (2.24) and (2.25) with (2.23) one obtains

$$\begin{aligned}
&\frac{1}{\theta}\int_{I\!R}\big[u(y) + \theta w(y) - (v^\theta \star u)(y)\big]\varphi(y)\mathrm{d}y \\
&\qquad \le \int_0^1 \mathrm{d}s\int_{I\!R} w(y)\varphi(y)\mathrm{d}y - \int_0^1 \mathrm{d}s\int_{I\!R}(\varphi \circ y_{\theta,s})(x)w(x)\mathrm{d}x + \|v - v^\theta\|_{\mathbf{L}^1(\mathrm{D}u)} \\
&\qquad \le \|v^\theta - v\|_{\mathbf{L}^1(\mathrm{D}u)} + 2\theta Lip(v^\theta)\|w\|_{\mathbf{L}^1(I\!R)} + \int_0^1 \|w \circ y_{\theta,s} - w\|_{\mathbf{L}^1(I\!R)}\mathrm{d}s.
\end{aligned} \tag{2.26}$$

Taking the supremum in (2.26) over all functions $\varphi \in \mathbf{C}^0_c(I\!R)$ with $\|\varphi\|_\infty \le 1$, we deduce

$$\begin{aligned}
&\frac{1}{\theta}\int_{I\!R}\big|u(y) + \theta w(y) - (v^\theta \star u)(y)\big|\mathrm{d}y \le \\
&\qquad \le \|v^\theta - v\|_{\mathbf{L}^1(\mathrm{D}u)} + 2\theta Lip(v^\theta)\|w\|_{\mathbf{L}^1(I\!R)} + \int_0^1 \|w \circ y_{\theta,s} - w\|\mathrm{d}s.
\end{aligned} \tag{2.27}$$

Finally, taking in (2.27) the upper limit as $\theta \to 0$, by (2.21) we obtain (2.22).

4. Let now $\theta \mapsto u^\theta \in X$ be a curve with $u^0 = u$ generating the shift tangent vector v, and assume that $-v\mathrm{D}u = w\mathrm{d}x$, for some $w \in \mathbf{L}^1(I\!R)$. Then (2.22) implies

$$\begin{aligned}
&\limsup_{\theta\to 0}\frac{1}{\theta}\|u^\theta - u - \theta w\|_{\mathbf{L}^1(I\!R)} \\
&\qquad \le \limsup_{\theta\to 0}\frac{1}{\theta}\|u^\theta - v^\theta \star u\|_{\mathbf{L}^1(I\!R)} + \limsup_{\theta\to 0}\frac{1}{\theta}\|v^\theta \star u - u - \theta w\|_{\mathbf{L}^1(I\!R)} = 0.
\end{aligned}$$

This shows that the map $\theta \mapsto u^\theta$ is differentiable at $\theta = 0$ in the usual sense, with $w \in \mathbf{L}^1(\mathbb{R})$ as differential.

5. Finally let (2.17) hold, with $w\mathrm{d}x = -v\mathrm{D}u$ for some $v \in \mathbf{L}^1(\mathrm{D}u)$. Then (2.22) implies

$$\begin{aligned}&\limsup_{\theta\to 0} \frac{1}{\theta}\|u^\theta - v^\theta \star u\|_{\mathbf{L}^1(\mathbb{R})}\\ &\qquad \le \limsup_{\theta\to 0} \frac{1}{\theta}\|u^\theta - u - \theta w\|_{\mathbf{L}^1(\mathbb{R})} + \limsup_{\theta\to 0}\frac{1}{\theta}\|u + \theta w - v^\theta \star u\|_{\mathbf{L}^1(\mathbb{R})} = 0.\end{aligned}$$

Hence u^θ generates the shift tangent vector v. $\square$

We conclude this section by proving one more lemma, for later use.

Lemma 4. *Let $\theta \mapsto u^\theta$ ($u^0 = u$) generate the shift tangent vector $v \in \mathbf{L}^1(\mathrm{D}u)$. Then*

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_{\mathbb{R}}\big[u^\theta(y) - u(y)\big]\,\mathrm{d}y = -\int_{\mathbb{R}} v(x)\mathrm{D}u(x). \tag{2.28}$$

Proof. Choose any $v^\theta \stackrel{\wedge}{\longrightarrow} v$. Since $v^\theta \in \mathbf{Lip}(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$, Lemma 2 holds. Introduce the function $y_\theta(x) \doteq x + \theta v^\theta(x)$ and denote its inverse by $x_\theta(y)$. We now have

$$\begin{aligned}\frac{1}{\theta}\int_{\mathbb{R}}\big[(v^\theta\star u)(y) - u(y)\big]\,\mathrm{d}y &= \frac{1}{\theta}\int_{\mathbb{R}}\Big[u\big(x_\theta(y_\theta(x))\big)\Big(1 + \theta\frac{\mathrm{d}}{\mathrm{d}x}v^\theta(x)\Big) - u(x)\Big]\,\mathrm{d}x\\ &= \int_{\mathbb{R}} u(x)\frac{\mathrm{d}}{\mathrm{d}x}v^\theta(x)\mathrm{d}x.\end{aligned}$$

Since

$$\lim_{\theta\to 0}\int_{\mathbb{R}} u(x)\frac{\mathrm{d}}{\mathrm{d}x}v^\theta(x)\mathrm{d}x = -\lim_{\theta\to 0}\int_{\mathbb{R}} v^\theta(x)\mathrm{D}u(x) = -\int_{\mathbb{R}} v(x)\mathrm{D}u(x),$$

we thus obtain

$$\begin{aligned}\lim_{\theta\to 0}\frac{1}{\theta}\int_{\mathbb{R}}\big[u^\theta(y) - u(y)\big]\,\mathrm{d}y &= \lim_{\theta\to 0}\left\{\frac{1}{\theta}\int_{\mathbb{R}}\big[u^\theta(y) - (v^\theta\star u)(y)\big]\,\mathrm{d}y + \frac{1}{\theta}\int_{\mathbb{R}}\big[(v^\theta\star u)(y) - u(y)\big]\,\mathrm{d}y\right\}\\ &= 0 - \int_{\mathbb{R}} v(x)\mathrm{D}u(x).\end{aligned}$$

$\square$

3 - Shift Differentials

Throughout the following, we consider a locally Lipschitz continuous operator Φ mapping the function space X into itself.

Definition 4. We say that Φ is *shift differentiable at the point $u \in X$ along the direction $v \in \mathbf{L}^1(\mathrm{D}u)$* if there exists $w \in \mathbf{L}^1(\mathrm{D}\Phi(u))$ such that

$$\lim_{\theta \to 0} \frac{1}{\theta} \left\| w^\theta \star \Phi(u) - \Phi(v^\theta \star u) \right\|_{\mathbf{L}^1(\mathbb{R})} = 0 \tag{3.1}$$

for some $v^\theta \xrightarrow{\wedge} v$, $w^\theta \xrightarrow{\wedge} w$. In this case, we call w the shift derivative of Φ at the point u along v and write $w = \overset{\leftrightarrow}{\nabla}_v \Phi(u)$.

Remark 4. The Lipschitz continuity of Φ and Remark 2 imply that, if Φ is shift differentiable at u along the direction v, then (3.1) holds for all $\tilde{v}^\theta$, $\tilde{w}^\theta$ hat-converging to v and $\overset{\leftrightarrow}{\nabla}_v \Phi(u)$ respectively. Moreover, Remark 3 shows that, if a shift derivative exists, it is necessarily unique.

Definition 5. We say that Φ is *shift-differentiable* at u if there exists a continuous linear map $\Lambda : \mathbf{L}^1(\mathrm{D}u) \to \mathbf{L}^1(\mathrm{D}\Phi(u))$ such that the following holds. For all $v \in \mathbf{L}^1(\mathrm{D}u)$ there exist $v^\theta \xrightarrow{\wedge} v$ and $w^\theta \xrightarrow{\wedge} \Lambda v$ such that

$$\lim_{\theta \to 0} \frac{1}{\theta} \left\| w^\theta \star \Phi(u) - \Phi(v^\theta \star u) \right\|_{\mathbf{L}^1(\mathbb{R})} = 0. \tag{3.2}$$

In this case we say that Λ is the *shift differential* of Φ at u and write $\Lambda = \overset{\leftrightarrow}{\nabla} \Phi(u)$.

In other words Φ is shift differentiable at u if it is shift differentiable along each direction $v \in \mathbf{L}^1(\mathrm{D}u)$ and the map $v \mapsto \overset{\leftrightarrow}{\nabla}_v \Phi(u)$ is linear.

Remark 5. By Theorem 1, the shift differential of Φ is a linear continuous map Λ whose norm is bounded by the local Lipschitz constant of Φ.

The next theorem shows that, in order to prove the shift differentiability of Φ at a point u, it suffices to check that it holds along a dense set of directions.

Theorem 3. *Let $\Phi : X \mapsto X$ be locally Lipschitz continuous. If Φ is shift differentible at $u \in X$ along all the directions $v \in Y$, where Y is a dence subspace of $\mathbf{L}^1(\mathrm{D}u)$, and if the map $\Lambda : Y \to \mathbf{L}^1(\mathrm{D}\Phi(u))$ that associates v to the shift derivative $\overset{\leftrightarrow}{\nabla}_v \Phi(u)$ is linear, then Φ is shift differentiable in u and $\overset{\leftrightarrow}{\nabla} \Phi(u) = \widetilde{\Lambda}$ where $\widetilde{\Lambda}$ is the continuous extension of Λ to the space $\mathbf{L}^1(\mathrm{D}u)$.*

Proof. Theorem 1 and the local Lipschitz continuity of Φ imply that Λ is continuous w.r.t. the norm of $\mathbf{L}^1(\mathrm{D}u)$, hence we can extend Λ by continuity to the whole space $\mathbf{L}^1(\mathrm{D}u)$. We call $\widetilde{\Lambda}$ this extension. Now consider any $v \in \mathbf{L}^1(\mathrm{D}u)$ and choose a sequence $v_k \in Y$ converging to v in $\mathbf{L}^1(\mathrm{D}u)$. If $v^\theta, v_k^\theta, w^\theta, w_k^\theta$ hat-converge respectively to $v, v_k, \widetilde{\Lambda} v, \widetilde{\Lambda} v_k$, for each fixed k we have

$$\begin{aligned} \frac{1}{\theta} \| w^\theta \star \Phi(u) - \Phi(v^\theta \star u) \|_{\mathbf{L}^1(\mathbb{R})} &\le \frac{1}{\theta} \| w^\theta \star \Phi(u) - w_k^\theta \star \Phi(u) \|_{\mathbf{L}^1(\mathbb{R})} \\ &+ \frac{1}{\theta} \| w_k^\theta \star \Phi(u) - \Phi(v_k^\theta \star u) \|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{\theta} \| \Phi(v_k^\theta \star u) - \Phi(v^\theta \star u) \|_{\mathbf{L}^1(\mathbb{R})}. \end{aligned} \tag{3.3}$$

Using Theorem 1 we obtain

$$\lim_{\theta\to 0}\frac{1}{\theta}\|w^\theta\star\Phi(u)-w_k^\theta\star\Phi(u)\|_{\mathbf{L}^1(I\!R)}=\|\widetilde{\Lambda}v-\widetilde{\Lambda}v_k\|_{\mathbf{L}^1(\mathrm{D}\Phi(u))},\tag{3.4}$$

$$\limsup_{\theta\to 0}\frac{1}{\theta}\|\Phi(v_k^\theta\star u)-\Phi(v^\theta\star u)\|_{\mathbf{L}^1(I\!R)}\leq L\cdot\lim_{\theta\to 0}\frac{1}{\theta}\|v_k^\theta\star u-v^\theta\star u\|_{\mathbf{L}^1(I\!R)}=L\cdot\|v_k-v\|_{\mathbf{L}^1(\mathrm{D}u)},\tag{3.5}$$

where L is a local Lipschitz constant for Φ. Since $v_k\in Y$, our hypotheses imply

$$\lim_{\theta\to 0}\frac{1}{\theta}\|w_k^\theta\star\Phi(u)-\Phi(v_k^\theta\star u)\|_{\mathbf{L}^1(I\!R)}=0.\tag{3.6}$$

Inserting (3.4)–(3.6) in (3.3), for each k we obtain

$$\limsup_{\theta\to 0}\frac{1}{\theta}\|w^\theta\star\Phi(u)-\Phi(v^\theta\star u)\|_{\mathbf{L}^1(I\!R)}\leq\|\widetilde{\Lambda}v-\widetilde{\Lambda}v_k\|_{\mathbf{L}^1(\mathrm{D}\Phi(u))}+L\cdot\|v_k-v\|_{\mathbf{L}^1(\mathrm{D}u)}.\tag{3.7}$$

Letting $k\to\infty$, the right hand side of (3.7) approaches zero and the theorem is proved. □

4 - Application to Conservation Laws

Throughout the following, by $\mathbf{S}:X\times[0,+\infty)\to X$ we denote the semigroup generated by the scalar conservation law (1.7). We assume that $f\in\mathcal{C}^2(I\!R)$ with $f''(s)\geq c>0$ for all $s\in I\!R$. By definition, $(t,x)\mapsto(\mathbf{S}_t u)(x)$ is the unique entropic solution of the Cauchy problem

$$\begin{cases} w_t+f(w)_x=0\\ w(0,\cdot)=u\in X.\end{cases}\tag{4.1}$$

It is well known that $\mathbf{S}$ is a contractive semigroup in the $\mathbf{L}^1(I\!R)$ norm [5,7,9]. An explicit representation of the function $w(t,\cdot)=\mathbf{S}_t u$ is provided by the theory of generalized characteristics [6] and by the integral formula of Lax [8]. Fix any time $t>0$. For $u\in X$, $x\in I\!R$, consider the right and left limits $\omega^\pm=\lim_{y\to x\pm}(\mathbf{S}_t u)(y)$, and define

$$\xi^\pm(u;x)\doteq x-f'(\omega^\pm)t.\tag{4.2}$$

Otherwise stated, the points $\xi^\pm(u;x)$ are the intersections with the x-axis of the maximal and the minimal backward characteristic from the point (t,x), respectively. Here we do not explicitly indicate the dependence on t, because the time $t>0$ will be kept fixed throughout the sequel. The maps $x\mapsto\xi^\pm(u;x)$ are nondecreasing [8]. If $x'<x''$ one has

$$\xi^-(u;x')\leq\xi^+(u;x')\leq\xi^-(u;x'')\leq\xi^+(u;x'').\tag{4.3}$$

If $\mathbf{S}_t u$ is continuous at a point x, then in (4.2) $\omega^+ = \omega^-$ and we simply write $\xi(u;x) \doteq \xi^+(u;x) = \xi^-(u;x)$. For $u \in X$ and $y \in \overline{I\!R}$, consider the function

$$F_u^y(\kappa, x) = t\big[f'(\kappa)\kappa - f(\kappa)\big] + \int_y^{x-f'(\kappa)t} u(\zeta)\mathrm{d}\zeta. \tag{4.4}$$

For any y, y', it is clear that the functions F_u^y and $F_u^{y'}$ differ only by a constant. Therefore, for a fixed x, the set of values κ where F_u^y is minimized does not depend on y. This set of values will be denoted by

$$\mathcal{S}_u(x) \doteq \Big\{\kappa;\quad F_u^y(\kappa, x) = \min_\zeta F_u^y(\zeta, x)\Big\} \tag{4.5}$$

For simplicity, we shall write $F_u(\kappa, x)$ instead of $F_u^{-\infty}(\kappa, x)$. The following results were proved in [8].

a) At time t, the left and right limits of the solution of (4.1) at a point x are given by

$$(\mathbf{S}_t u)(x-) = \max \mathcal{S}_u(x), \qquad (\mathbf{S}_t u)(x+) = \min \mathcal{S}_u(x).$$

b) $\mathbf{S}_t u$ is continuous at x if and only if the set $\mathcal{S}_u(x)$ contains a single point.

We now state the main theorem of the paper.

Theorem 4. *Assume that, at a given time $t > 0$, the entropic solution of* (4.1) *does not contain interacting shocks, nor centers of a compression wave. In other words, assume that for every $x \in I\!R$ the set $\mathcal{S}_u(x)$ contains at most two points. Then the map $\mathbf{S}_t$ is shift differentiable at the point u.*

The proof relies on a sequence of preliminary lemmas. First, we show that the multifunction $(u, x) \mapsto \mathcal{S}_u(x)$ is upper semicontinuous [1]. In the following, the distance of the point x to the compact set K is written $d(x, K)$.

Lemma 5. *Let $\theta \mapsto u^\theta \in X$ $(u^0 = u)$ be a continuous map defined on some interval $[0, \theta_0]$, with $\|u^\theta\|_\infty \le R$ for some $R > 0$ and all θ. Let A be an open set, and K be a compact set such that*

$$\mathcal{S}_u(x) \subset A \qquad \forall x \in K. \tag{4.6}$$

Then there exists $\theta^* \in (0, \theta_0]$ and $\delta > 0$ such that

$$\mathcal{S}_{u^\theta}(x) \subset A \qquad \textit{whenever} \quad \theta \in [0, \theta^*], \quad d(x, K) \le \delta. \tag{4.7}$$

Proof. If the conclusion does not hold, then there exists sequences x_j, θ_j and $\omega_j \in \mathcal{S}_{u^{\theta_j}}(x_j)$ such that $d(x_j, K) \to 0$, $\theta_j \to 0$ but $\omega_j \notin A$ for all j . Since the ω_j are uniformly bounded and A is open, by possibly taking a subsequence we can assume $\omega_j \to \omega_0 \notin A$ and $x_j \to x_0 \in K$. Therefore, from

$$F_{u^{\theta_j}}(\omega_j, x_j) \leq F_{u^{\theta_j}}(\kappa, x_j) \qquad \forall \kappa \in I\!R,$$

by the uniform convergence of $F_{u^{\theta_j}}$ to F_u it follows

$$F_u(\omega_0, x_0) \leq F_u(\kappa, x_0) \qquad \forall \kappa \in I\!R.$$

This implies $\omega_0 \in \mathcal{S}_u(x_0)$, hence $\mathcal{S}_u(x_0) \not\subset A$. This contradiction proves the lemma. $\square$

Remark 6. If $\hat{x}$ is a continuity point of $\mathbf{S}_t u$, then

$$\mathcal{S}_u(\hat{x}) = \big\{(\mathbf{S}_t u)(\hat{x})\big\} \subset \big((\mathbf{S}_t u)(\hat{x}) - \varepsilon, (\mathbf{S}_t u)(\hat{x}) + \varepsilon\big) \qquad \forall \varepsilon > 0.$$

Therefore if $\theta \mapsto u^\theta \in X$ ($u^0 = u$) is a continuous map, we can apply Lemma 5 with $K = \{\hat{x}\}$ and $A = \big((\mathbf{S}_t u)(\hat{x}) - \varepsilon, (\mathbf{S}_t u)(\hat{x}) + \varepsilon\big)$. As a consequence, for all $\varepsilon > 0$ there exists $\theta_\varepsilon > 0$ such that $\mathcal{S}_{u^\theta}(\hat{x}) \subset \big((\mathbf{S}_t u)(\hat{x}) - \varepsilon, (\mathbf{S}_t u)(\hat{x}) + \varepsilon\big)$ for $\theta \in [0, \theta_\varepsilon]$. In particular, one has

$$\lim_{\theta \to 0} (\mathbf{S}_t u^\theta)(\hat{x}\pm) = (\mathbf{S}_t u)(\hat{x}). \tag{4.8}$$

The next two lemmas show that perturbations in the initial condition propagate along characteristics.

Lemma 6. *Let $u_i \in X$ for $i = 1, 2$, choose any two points $z' < z''$, and define*

$$a = \min_{i=1,2} \xi^+(u_i; z'), \qquad b = \max_{i=1,2} \xi^-(u_i; z'').$$

If $u_1(x) = u_2(x)$ for all $x \in [a, b]$, then $(\mathbf{S}_t u_1)(x) = (\mathbf{S}_t u_2)(x)$ for every x in the open interval (z', z'').

Proof. Fix any $\hat{x} \in (z', z'')$ and define

$$\mathcal{U}_{\hat{x}} = \big\{\kappa \in I\!R \ : \ \hat{x} - tf'(\kappa) \in [a, b]\big\}.$$

From (4.3) it follows

$$a \leq \xi^-(u_i, \hat{x}) = \hat{x} - tf'\big((\mathbf{S}_t u_i)(\hat{x}-)\big) \leq \hat{x} - tf'\big((\mathbf{S}_t u_i)(\hat{x}+)\big) = \xi^+(u_i, \hat{x}) \leq b.$$

Therefore $\max \mathcal{S}_{u_i}(\hat{x}) = (\mathbf{S}_t u_i)(\hat{x}-) \in \mathcal{U}_{\hat{x}}$ and $\min \mathcal{S}_{u_i}(\hat{x}) = (\mathbf{S}_t u_i)(\hat{x}+) \in \mathcal{U}_{\hat{x}}$. Since f' is strictly increasing, $\mathcal{U}_{\hat{x}}$ is an interval, hence $\mathcal{S}_{u_i}(\hat{x}) \subset \mathcal{U}_{\hat{x}}$. Choosing any $y \in (a, b)$ we can represent $\mathcal{S}_{u_i}(\hat{x})$ as the set of solutions to the problem

$$\min_{\kappa \in I\!R} F^y_{u_i}(\kappa, \hat{x}). \tag{4.9}$$

Since $\mathcal{S}_{u_i}(\hat{x}) \subset \mathcal{U}_{\hat{x}}$, the minimization problem (4.9) is equivalent to

$$\min_{\kappa \in \mathcal{U}_{\hat{x}}} F^y_{u_i}(\kappa, \hat{x}).$$

Our assumptions clearly imply $F^y_{u_1}(\kappa, \hat{x}) = F^y_{u_2}(\kappa, \hat{x})$ for all $\kappa \in \mathcal{U}_{\hat{x}}$. Hence $\mathcal{S}_{u_1}(\hat{x}) = \mathcal{S}_{u_2}(\hat{x})$. Since $\hat{x} \in (z', z'')$ was arbitrary, the lemma is proved. $\square$

Lemma 7. *Let $\theta \mapsto u_i^\theta \in X$, $i = 1, 2$, be two continuous maps defined on some interval $[0, \theta_0]$, with $u_1^0 = u_2^0 = u$, and such that $\|u_i^\theta\|_\infty \le R$ for some constant R and all θ. Then, for any $z' < z''$ and $\eta > 0$ we have*

$$\int_{z'}^{z''} \Big|\big(\mathbf{S}_t u_1^\theta\big)(x) - \big(\mathbf{S}_t u_2^\theta\big)(x)\Big| \mathrm{d}x \le \int_{\xi^-(u;z')-\eta}^{\xi^+(u;z'')+\eta} \big|u_1^\theta(x) - u_2^\theta(x)\big| \mathrm{d}x \tag{4.10}$$

for all θ sufficiently small.

Proof. Introduce the functions

$$\tilde{u}_i(x) = \begin{cases} u_i^\theta(x) & \text{if} \quad x \in \big(\xi^-(u;z') - \eta,\ \xi^+(u;z'') + \eta\big), \\ u(x) & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{u}_i^\theta \to u$ as $\theta \to 0$. Applying Lemma 5 with

$$K = \{z''\}, \qquad A = \big\{\kappa : \quad z'' - tf'(\kappa) < \xi^+(u;z'') + \eta\big\},$$

we obtain $\big(\mathbf{S}_t u_i^\theta\big)(z''+)$, $\big(\mathbf{S}_t \tilde{u}_i^\theta\big)(z''+) \in A$ for all θ small enough. An entirely similar argument holds for z'. Hence, for all θ sufficiently small, we have

$$\xi^-(u;z') - \eta < \xi^-(u_i^\theta;z') \le \xi^+(u_i^\theta;z'') < \xi^+(u;z'') + \eta,$$

$$\xi^-(u;z') - \eta < \xi^-(\tilde{u}_i^\theta;z') \le \xi^+(\tilde{u}_i^\theta;z'') < \xi^+(u;z'') + \eta.$$

Since u_i^θ and $\tilde{u}_i^\theta$ coincide on the open interval $\big(\xi^-(u;z') - \eta,\ \xi^+(u;z'') + \eta\big)$, by Lemma 6 the corresponding solutions $\mathbf{S}_t u_i^\theta$ and $\mathbf{S}_t \tilde{u}_i^\theta$ coincide on (z', z''). Therefore

$$\begin{aligned} \int_{z'}^{z''} \Big|\big(\mathbf{S}_t u_1^\theta\big)(x) - \big(\mathbf{S}_t u_2^\theta\big)(x)\Big| \mathrm{d}x &= \int_{z'}^{z''} \Big|\big(\mathbf{S}_t \tilde{u}_1^\theta\big)(x) - \big(\mathbf{S}_t \tilde{u}_2^\theta\big)(x)\Big| \mathrm{d}x \\ &\le \int_{I\!R} \Big|\big(\mathbf{S}_t \tilde{u}_1^\theta\big)(x) - \big(\mathbf{S}_t \tilde{u}_2^\theta\big)(x)\Big| \mathrm{d}x \\ &\le \int_{I\!R} \big|\tilde{u}_1^\theta(x) - \tilde{u}_2^\theta(x)\big| \mathrm{d}x \\ &= \int_{\xi^-(u;z')-\eta}^{\xi^+(u;z'')+\eta} \big|u_1^\theta(x) - u_2^\theta(x)\big| \mathrm{d}x. \end{aligned}$$

□

The following two lemmas describe the behaviour of $\mathbf{S}_t u$ in the neighborhood of a shock, when the initial condition u is slightly perturbed.

Lemma 8. *Let $\theta \mapsto u^\theta \in X$ $(u^0 = u)$ be a continuous map defined on some interval $[0, \theta_0]$, such that $\|u^\theta\|_\infty \le R$ for all θ. Assume that $\hat{x} \in I\!R$ is a point where $\mathbf{S}_t u$ is discontinuous, with*

$\mathcal{S}_u(\hat{x}) = \{\omega^-, \omega^+\}$ *consisting of exactly two points. Then, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\big(\mathbf{S}_t u^\theta\big)(x) \in \big(\omega^- - \varepsilon, \omega^- + \varepsilon\big) \cup \big(\omega^+ - \varepsilon, \omega^+ + \varepsilon\big) \qquad \forall x \in [\hat{x} - \delta,\ \hat{x} + \delta] \tag{4.11}$$

for all θ *small enough.*

Indeed, this follows immediately from Lemma 5, taking

$$K = \{\hat{x}\}, \qquad A = \big(\omega^- - \varepsilon,\ \omega^- + \varepsilon\big) \cup \big(\omega^+ - \varepsilon,\ \omega^+ + \varepsilon\big).$$

Remark 7. In the setting of Lemma 8, define $\Delta\omega = \omega^- - \omega^+$ and choose $\varepsilon = \Delta\omega/4$. Then, for some $\delta^* > 0$ and all θ sufficiently small ($\theta \in [0, \theta^*]$ for some $th^* > 0$) there holds

$$\big(\mathbf{S}_t u^\theta\big)(x) \in \Big(\omega^- - \frac{\Delta\omega}{4},\ \omega^- + \frac{\Delta\omega}{4}\Big) \cup \Big(\omega^+ - \frac{\Delta\omega}{4},\ \omega^+ + \frac{\Delta\omega}{4}\Big) \qquad \forall x \in [\hat{x} - \delta^*, \hat{x} + \delta^*]. \tag{4.12}$$

Since the two intervals around ω^-, ω^+ are disjoint, and since $\mathbf{S}_t u^\theta$ cannot have upward jumps, for any $\theta \in [0, \theta^*]$ the corresponding solution $\mathbf{S}_t u^\theta$ will have a downward jump at some point $\hat{x}_\theta$, with $\hat{x}_0 = \hat{x}$, $|\hat{x}_\theta - \hat{x}| \le \delta^*$, and such that

$$\begin{aligned} \big(\mathbf{S}_t u^\theta\big)(x) &\in \Big(\omega^- - \frac{\Delta\omega}{4},\ \omega^- + \frac{\Delta\omega}{4}\Big) \qquad \forall x \in [\hat{x} - \delta^*,\ \hat{x}_\theta), \\ \big(\mathbf{S}_t u^\theta\big)(x) &\in \Big(\omega^+ - \frac{\Delta\omega}{4},\ \omega^+ + \frac{\Delta\omega}{4}\Big) \qquad \forall x \in (\hat{x}_\theta,\ \hat{x} + \delta^*]. \end{aligned} \tag{4.13}$$

Moreover,

$$|\hat{x}_\theta - \hat{x}| \le \frac{2}{\Delta\omega}\left|(\hat{x}_\theta - \hat{x})\Big(\omega^- - \frac{\Delta\omega}{4} - \omega^+ - \frac{\Delta\omega}{4}\Big)\right| \le \frac{2}{\Delta\omega}\|\mathbf{S}_t u^\theta - \mathbf{S}_t u\|_{\mathbf{L}^1(I\!R)} \le \frac{2}{\Delta\omega}\|u^\theta - u\|_{\mathbf{L}^1(I\!R)}. \tag{4.14}$$

We now study in greater detail the case where $u^\theta = \tilde{v}^\theta \star u$, for a particular family of functions $\tilde{v}^\theta$.

Lemma 9. *In addition to the hypotheses of Lemma 8 and Remark 7, suppose that* $u^\theta = \tilde{v}^\theta \star u$ *with* $\tilde{v}^\theta \xrightarrow{\wedge} \tilde{v} \in \mathbf{L}^1(\mathrm{D}u)$. *Assume that*

$$\begin{cases} \tilde{v}^\theta(x) = \alpha & \text{if} \quad x \le a', \\ \tilde{v}^\theta(x) = \beta & \text{if} \quad x \ge b', \end{cases}$$

for all $\theta \in [0, \theta^*]$, *where*

$$a \doteq \xi^-(u; \hat{x}) < a' < b' < \xi^+(u; \hat{x}) \doteq b.$$

Then the displacement of the point where u^θ *has a jump satisfies*

$$\lim_{\theta \to 0} \frac{\hat{x}_\theta - \hat{x}}{\theta} = v_0 \doteq \frac{-1}{\Delta\omega}\left\{ \int_{(a,b)} \tilde{v}(z)\mathrm{D}u(z) + \alpha\big[u(a+) - \omega^-\big] + \beta\big[\omega^+ - u(b-)\big]\right\}. \tag{4.15}$$

Moreover there exists $M > 0$ such that, for θ sufficiently small, one has

$$|\hat{x}_\theta - \hat{x}| < M\theta, \qquad (\mathbf{S}_t u^\theta)(y) = \begin{cases} (\mathbf{S}_t u)(y - \theta\alpha) & \textit{if} \quad y \in (-\infty, \hat{x} - M\theta), \\ (\mathbf{S}_t u)(y - \theta\beta) & \textit{if} \quad y \in (\hat{x} + M\theta, +\infty). \end{cases} \tag{4.16}$$

Finally, the quantity

$$\begin{aligned} \Delta_{\theta,M} \doteq & \sup_{\hat{x} - M\theta \le x < \hat{x}_\theta} \Big|\omega^- - \big(\mathbf{S}_t u^\theta\big)(x)\Big| + \sup_{\hat{x}_\theta < x \le \hat{x} + M\theta} \Big|\omega^+ - \big(\mathbf{S}_t u^\theta\big)(x)\Big| \\ & + \sup_{\hat{x} - M\theta \le x < \hat{x}} \Big|\omega^- - \big(\mathbf{S}_t u\big)(x)\Big| + \sup_{\hat{x} < x \le \hat{x} + M\theta} \Big|\omega^+ - \big(\mathbf{S}_t u\big)(x)\Big|. \end{aligned} \tag{4.17}$$

approaches zero as $\theta \to 0$.

Proof. We fix $a'' \in (a, a')$, $b'' \in (b', b)$ and choose $\varepsilon_0 \in \big(0, \Delta\omega/4\big)$ such that, for all $h \in (-\varepsilon_0, +\varepsilon_0)$, one has

$$\hat{x} - tf'(\omega^- + h) < a'', \qquad \hat{x} - tf'(\omega^+ + h) > b''. \tag{4.18}$$

Observe that this is possible because

$$\hat{x} - tf'(\omega^-) = a, \qquad \hat{x} - tf'(\omega^+) = b.$$

Next, define

$$M \doteq 1 + \max\left\{ \frac{2}{\Delta\omega} \|\tilde{v}\|_{\mathbf{L}^1(\mathrm{D}u)}, |\alpha|, |\beta| \right\}. \tag{4.19}$$

By Remark 7 and Theorem 1 it follows

$$\limsup_{\theta\to 0} \frac{|\hat{x}_\theta - \hat{x}|}{\theta} \le \frac{2}{\Delta\omega} \limsup_{\theta \to 0} \|u^\theta - u\|_{\mathbf{L}^1(\mathbb{R})} = \frac{2}{\Delta\omega} \|\tilde{v}\|_{\mathbf{L}^1(\mathrm{D}u)} < M.$$

Hence, if θ is sufficiently small, one has $|\hat{x}_\theta - \hat{x}| < M\theta$. Choosing $\varepsilon \in (0, \varepsilon_0)$, by Lemma 8 there exists $\delta > 0$ such that, for θ small enough,

$$|\hat{x}_\theta - \hat{x}| < M\theta < \delta, \tag{4.20}$$

$$\big(\mathbf{S}_t u^\theta\big)(x) \in (\omega^- - \varepsilon,\ \omega^- + \varepsilon) \quad \text{if } x \in [\hat{x} - \delta,\ \hat{x}_\theta), \tag{4.21}$$

$$\big(\mathbf{S}_t u^\theta\big)(x) \in (\omega^+ - \varepsilon,\ \omega^+ + \varepsilon) \quad \text{if } x \in (\hat{x}_\theta,\ \hat{x} + \delta]. \tag{4.22}$$

For θ sufficiently small, we also have

$$(v^\theta \star u)(y) = \begin{cases} u(y - \theta\alpha) & \text{if } y \le a'', \\ u(y - \theta\beta) & \text{if } y \ge b''. \end{cases}$$

Since $\hat{x} - M\theta \in [\hat{x} - \delta,\ \hat{x}_\theta)$, by (4.21) one has $\big|\big(\mathbf{S}_t u^\theta\big)(\hat{x} - M\theta+) - \omega^-\big| \le \varepsilon < \varepsilon_0$, hence (4.18) implies

$$\xi^+(u^\theta, \hat{x} - M\theta) = \hat{x} - M\theta - tf'\big(\big(\mathbf{S}_t u^\theta\big)(\hat{x} - M\theta+)\big) < a''.$$

Setting $u_{\theta\alpha}(y) \doteq u(y-\theta\alpha)$ and observing that $M+\alpha>0$, for θ small we have

$$\xi^+\big(u_{\theta\alpha};\hat{x}-M\theta\big)=\xi^+\big(u;\hat{x}-(M+\alpha)\theta\big)+\theta\alpha\le\xi^-(u;\hat{x})+\theta\alpha<a''.$$

Since $u_{\theta\alpha}(y)=u^\theta(y)$ for all $y\le a''$, applying Lemma 6 we obtain

$$\big(\mathbf{S}_t u^\theta\big)(y)=\big(\mathbf{S}_t u_{\theta\alpha}\big)(y)=(\mathbf{S}_t u)(y-\theta\alpha)\qquad\forall y<\hat{x}-M\theta. \tag{4.23}$$

In a similar way one proves

$$\big(\mathbf{S}_t u^\theta\big)(y)=(\mathbf{S}_t u)(y-\theta\beta)\qquad\forall y>\hat{x}+M\theta. \tag{4.24}$$

The three relations (4.20)–(4.22) together imply $\Delta_{\theta,M}\le 4\varepsilon$ for all θ sufficiently small. Since $\varepsilon>0$ was arbitrary, the last statement of the lemma is proved.

Next, from the identity

$$\int_{I\!R}\Big[\big(\mathbf{S}_t u^\theta\big)(y)-\big(\mathbf{S}_t u\big)(y)\Big]\,\mathrm{d}y=\int_{I\!R}\big[u^\theta(y)-u(y)\big]\,\mathrm{d}y$$

we obtain

$$\begin{aligned}\frac{1}{\theta}\int_{\hat{x}-M\theta}^{\hat{x}+M\theta}\Big[\big(\mathbf{S}_t u^\theta\big)(y)-\big(\mathbf{S}_t u\big)(y)\Big]\,\mathrm{d}y&=\frac{1}{\theta}\int_{I\!R}\big[u^\theta(y)-u(y)\big]\,\mathrm{d}y\\&\quad-\frac{1}{\theta}\int_{-\infty}^{\hat{x}-M\theta}\Big[\big(\mathbf{S}_t u^\theta\big)(y)-\big(\mathbf{S}_t u\big)(y)\Big]\,\mathrm{d}y\\&\quad-\frac{1}{\theta}\int_{\hat{x}+M\theta}^{\infty}\Big[\big(\mathbf{S}_t u^\theta\big)(y)-\big(\mathbf{S}_t u\big)(y)\Big]\,\mathrm{d}y.\end{aligned} \tag{4.25}$$

With the help of (4.23) we now compute

$$\begin{aligned}\frac{1}{\theta}\int_{-\infty}^{\hat{x}-M\theta}\Big[\big(\mathbf{S}_t u^\theta\big)(y)-\big(\mathbf{S}_t u\big)(y)\Big]\,\mathrm{d}y&=\frac{1}{\theta}\int_{-\infty}^{\hat{x}-M\theta}\Big[\big(\mathbf{S}_t u\big)(y-\theta\alpha)-\big(\mathbf{S}_t u\big)(y)\Big]\,\mathrm{d}y\\&=\frac{1}{\theta}\int_{\hat{x}-M\theta}^{\hat{x}-(M+\alpha)\theta}\big(\mathbf{S}_t u\big)(y)\mathrm{d}y.\end{aligned}$$

Since $M+\alpha>0$, this yields

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_{-\infty}^{\hat{x}-M\theta}\Big[\big(\mathbf{S}_t u^\theta\big)(y)-\big(\mathbf{S}_t u\big)(y)\Big]\,\mathrm{d}y=-\alpha\omega^-. \tag{4.26}$$

An entirely similar computation yields

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_{\hat{x}+M\theta}^{\infty}\Big[\big(\mathbf{S}_t u^\theta\big)(y)-\big(\mathbf{S}_t u\big)(y)\Big]\,\mathrm{d}y=\beta\omega^+. \tag{4.27}$$

Using (4.26) and (4.27) in (4.25) and recalling Lemma 4 we obtain

$$
\begin{aligned}
\lim_{\theta\to 0}\frac{1}{\theta}\int_{\hat{x}-M\theta}^{\hat{x}+M\theta}\Big[\big(\mathbf{S}_t u^\theta\big)(y)-\big(\mathbf{S}_t u\big)(y)\Big]\,\mathrm{d}y &= -\int_{I\!R}\tilde{v}(x)\mathrm{D}u(x)+\alpha\omega^- -\beta\omega^+ \\
&= -\int_{(a,b)}\tilde{v}(x)\mathrm{D}u(x)-\alpha\big[u(a+)-\omega^-\big]-\beta\big[\omega^+ - u(b-)\big] \\
&= v_0\Delta\omega.
\end{aligned}
\tag{4.28}
$$

Call I_θ the interval enclosed by the points $\hat{x}$ and $\hat{x}_\theta$. Observe that

$$
\big|\mathbf{S}_t u^\theta(x)-\mathbf{S}_t u(x)\big|\le\Delta_{\theta,M}\qquad\forall x\in[\hat{x}-M\theta,\hat{x}+M\theta]\setminus I_\theta,
\tag{4.29}
$$

$$
\Delta\omega-\Delta_{\theta,M}\le\Big|\big(\mathbf{S}_t u^\theta\big)(y)-\big(\mathbf{S}_t u\big)(y)\Big|\le\Delta\omega+\Delta_{\theta,M}\qquad\forall x\in I_\theta.
\tag{4.30}
$$

From (4.29) and (4.30) it follows

$$
\frac{1}{\theta}\int_{\hat{x}-M\theta}^{\hat{x}+M\theta}\Big[\big(\mathbf{S}_t u^\theta\big)(y)-\big(\mathbf{S}_t u\big)(y)\Big]\mathrm{d}y
\begin{cases}
\le\frac{1}{\theta}\Big\{2M\theta\Delta_{\theta,M}+(\hat{x}_\theta-\hat{x})\big[\Delta\omega+\Delta_{\theta,M}\mathrm{sgn}(\hat{x}_\theta-\hat{x})\big]\Big\}\\
\ge\frac{1}{\theta}\Big\{-2M\theta\Delta_{\theta,M}+(\hat{x}_\theta-\hat{x})\big[\Delta\omega-\Delta_{\theta,M}\mathrm{sgn}(\hat{x}_\theta-\hat{x})\big]\Big\}
\end{cases}
\tag{4.31}
$$

Taking the limit in (4.31) and recalling that $\Delta_{\theta,M}\to 0$ as $\theta\to 0$, we finally obtain

$$
\Delta\omega\cdot\limsup_{\theta\to 0}\frac{\hat{x}_\theta-\hat{x}}{\theta}\le v_0\Delta\omega\le\Delta\omega\cdot\liminf_{\theta\to 0}\frac{\hat{x}_\theta-\hat{x}}{\theta},
$$

proving (4.15). □

5 - Proof of Theorem 4.

Let $u\in X$ and $t>0$ be given, according to the assumptions of Theorem 4. Because of Theorem 3, it suffices to prove the existence of a dense subspace $Y\subset\mathbf{L}^1(\mathrm{D}u)$ and a linear operator $\Lambda: Y\to\mathbf{L}^1\big(\mathrm{D}(\mathbf{S}_t u)\big)$ such that, for every path $\theta\to u^\theta$ ($u^0=u$) which generates a shift tangent vector $v\in Y$, the corresponding path $\theta\to\mathbf{S}_t u^\theta$ generates the shift tangent vector Λv. A careful choice of the space Y will considerably simplify our computations. Define the sets

$$
B_1=\Big\{x\in I\!R;\ (\mathbf{S}_t u)(x-)>(\mathbf{S}_t u)(x+)\Big\},
$$

$$
A_1=\Big\{x\in I\!R;\ u(x+)\neq u(x-)\Big\},\ A_2=\Big\{x\in I\!R;\ x=\xi^-(u;y)\text{ or }x=\xi^+(u;y)\text{ for some }y\in B_1\Big\}.
$$

As Y we take the subspace

$$
Y=\Big\{v\in\mathbf{L}^1(\mathrm{D}u):\quad v=\sum_{i=1}^{n-1}\alpha_i\chi_{[y_i,y_{i+1})},\quad n\ge 2,\alpha_i\in I\!R,\quad y_i\notin A_1\cup A_2\ \ \forall i=1,....,n,\Big\}.
$$

In other words, Y is the space of piecewise constant functions with bounded support, whose jumps do not occur at points where u is discontinuous, nor at points where a maximal or minimal backward characteristic through a jump of $\mathbf{S}_t u$ meets the x-axis. Clearly, Y is dense in $\mathbf{L}^1(\mathrm{D}u)$.

The linear operator Λ is defined as follows.

$$(\Lambda v)(x) = \begin{cases} v\big(\xi^+(u;x)\big) & \text{if } x \notin B_1, \\ \Psi(x) & \text{if } x \in B_1, \end{cases} \tag{5.1}$$

where

$$\begin{aligned} \Psi(x) = \frac{1}{(\mathbf{S}_t u)(x+) - (\mathbf{S}_t u)(x-)} \bigg\{ & \int_{\left(\xi^-(u;x),\xi^+(u,x)\right)} v(\zeta)\mathrm{D}u(\zeta) \\ & + v\big(\xi^-(u;x)\big)\Big[u\big(\xi^-(u;x)+\big) - (\mathbf{S}_t u)(x-)\Big] \\ & + v\big(\xi^+(u;x)\big)\Big[(\mathbf{S}_t u)(x+) - u\big(\xi^+(u;x)-\big)\Big]\bigg\}. \end{aligned} \tag{5.2}$$

One easily checks that the function Λv is piecewise constant, hence Borel measurable. To show that $\Lambda v \in \mathbf{L}^1\big(\mathrm{D}(\mathbf{S}_t u)\big)$, we compute

$$\int_{I\!R} \big|\Lambda v(x)\big|\big|\mathrm{D}(\mathbf{S}_t u)\big|(x) = \int_{I\!R\setminus B_1} \big|v\big(\xi^+(u;x)\big)\big|\big|\mathrm{D}(\mathbf{S}_t u)\big|(x) + \int_{B_1} \big|\Psi(x)\big|\big|\mathrm{D}(\mathbf{S}_t u)\big|(x) < \infty. \tag{5.3}$$

Indeed,

$$\int_{I\!R\setminus B_1} \big|v\big(\xi^+(u;x)\big)\big|\big|\mathrm{D}(\mathbf{S}_t u)\big|(x) \le \|v\|_\infty \big|\mathrm{D}(\mathbf{S}_t u)\big|(I\!R) < \infty,$$

$$\begin{aligned} \int_{B_1} \big|\Psi(x)\big|\big|\mathrm{D}(\mathbf{S}_t u)\big|(x) &\le \|v\|_{\mathbf{L}^1(\mathrm{D}u)} + \|v\|_\infty \sum_{x\in B_1} \Big[|\mathrm{D}u|\big(\{\xi^-(u;x)\}\big) + |\mathrm{D}u|\big(\{\xi^+(u;x)\}\big)\Big] \\ &\le \|v\|_{\mathbf{L}^1(\mathrm{D}u)} + 2\|v\|_\infty |\mathrm{D}u|(I\!R) < \infty. \end{aligned}$$

By (5.3), Λ is a well defined linear operator from Y into $\mathbf{L}^1\big(\mathrm{D}(\mathbf{S}_t u)\big)$. Observe that, if $v(x) = \alpha$ for all $x \in \big[\xi^-(u;\hat{x}), \xi^+(u,\hat{x})\big]$, then

$$\begin{aligned} (\Lambda v)(\hat{x}) = \frac{\alpha}{(\mathbf{S}_t u)(\hat{x}+) - (\mathbf{S}_t u)(\hat{x}-)} \bigg\{ & \mathrm{D}u\Big(\big(\xi^-(u;\hat{x}), \xi^+(u,\hat{x})\big)\Big) + u\big(\xi^-(u;\hat{x})+\big) - (\mathbf{S}_t u)(\hat{x}-) \\ & + (\mathbf{S}_t u)(\hat{x}+) - u\big(\xi^+(u;\hat{x})+\big)\bigg\} = \alpha. \end{aligned} \tag{5.4}$$

Now consider any piecewise constant function of the form

$$v = \sum_{i=1}^{n-1} \alpha_i \chi_{[y_i, y_{i+1})} \in Y \tag{5.5}$$

and set $\alpha_0 = \alpha_n = 0$. Choose the following family of functions $v^\theta \xrightarrow{\wedge} v$:

$$v^\theta(x) = \begin{cases} \frac{\mathrm{sgn}(\alpha_{i+1}-\alpha_i)}{2\theta}(x - y_{i+1}) + \frac{\alpha_i+\alpha_{i+1}}{2} & \text{if } x \in \big[y_{i+1} - \theta|\alpha_i - \alpha_{i+1}|,\ y_{i+1} + \theta|\alpha_i - \alpha_{i+1}|\big], \\ v(x) & \text{otherwise.} \end{cases} \tag{5.6}$$

By the above definitions, it follows

$$(v^\theta \star u)(y) = u(y - \theta\alpha_i) \qquad \forall y \in \big[y_i + 4\theta\|v\|_\infty,\ y_{i+1} - 4\theta\|v\|_\infty\big]. \tag{5.7}$$

We shall consider separately the discontinuity points and two types of continuity points of $\mathbf{S}_t u$. For v as in (5.5), consider the sets

$$B_1 = \Big\{x \in I\!\!R :\ (\mathbf{S}_t u)(x-) > (\mathbf{S}_t u)(x+)\Big\},$$

$$B_2(v) = \Big\{x \in I\!\!R :\ x \notin B_1,\ \xi(u;x) \neq y_i \quad i = 1, ..., n\Big\},$$

$$B_3(v) = \Big\{x \in I\!\!R :\ x \notin B_1,\ \xi(u;x) = y_i \text{ for some } i = 1, ..., n\Big\}.$$

The next three lemmas describe the behavior of $\mathbf{S}_t u^\theta$ in a neighborhood of a point in B_2, B_3, B_1, respectively.

Lemma 10. *If $\hat{x} \in B_2(v)$, then there exists $\delta > 0$ such that*

$$\limsup_{\theta\to 0} \frac{1}{\theta} \int_{\hat{x}-\delta}^{\hat{x}+\delta} \big|(\mathbf{S}_t u^\theta)(x) - \bar{u}^\theta(x)\big| \mathrm{d}x = 0, \tag{5.8}$$

where $\theta \mapsto u^\theta$ and $\theta \mapsto \bar{u}^\theta$ are any paths generating the shift tangent vectors v and Λv, respectively.

Proof. We set $y_0 = -\infty$, $y_{n+1} = +\infty$ and take j such that $\xi(u;\hat{x}) \in (y_j, y_{j+1})$. Since the maps $x \mapsto \xi^\pm(u;x)$ are continuous at $\hat{x}$, there exists $\delta \in (0,1)$ such that

$$y_j < \xi^-(u;\hat{x} - 2\delta) < \xi^+(u;\hat{x} + 2\delta) < y_{j+1}.$$

From (5.4), the definition of Λv and the fact that $v(x) = \alpha_j$ for all $x \in (y_j, y_{j+1})$, it follows that $\big(\Lambda v\big)(x) = \alpha_j$ for all $x \in [\hat{x}-2\delta, \hat{x}+2\delta]$. We now choose $w^\theta \xrightarrow{\wedge} \Lambda v$ in such a way that $\big(w^\theta \star \mathbf{S}_t u\big)(y) = (\mathbf{S}_t u)(y - \theta\alpha_j)$ for all $y \in [\hat{x} - \delta, \hat{x} + \delta]$ and θ sufficiently small. Defining $u_{\theta\alpha_j}(y) = u(y - \theta\alpha_j)$, an easy application of Lemma 7 shows that

$$\big(\mathbf{S}_t(v^\theta \star u)\big)(y) = (\mathbf{S}_t u_{\theta\alpha_j})(y) = (\mathbf{S}_t u)(y - \theta\alpha_j) \qquad \forall y \in (\hat{x} - 2\delta, \hat{x} + 2\delta)$$

because $(v^\theta \star u)(y) = u_{\theta\alpha_j}(y)$ for some $\eta > 0$ and all $y \in [\xi^-(u;\hat{x} - 2\delta) - \eta, \xi^+(u;\hat{x} + 2\delta) + \eta]$. This yields

$$\begin{aligned} &\limsup_{\theta\to 0} \frac{1}{\theta} \int_{\hat{x}-\delta}^{\hat{x}+\delta} \big|(\mathbf{S}_t u^\theta)(y) - \bar{u}^\theta(y)\big| \mathrm{d}x \\ &\quad \le \limsup_{\theta\to 0} \left\{ \frac{\|\mathbf{S}_t u^\theta - \mathbf{S}_t(v^\theta \star u)\|_{\mathbf{L}^1(I\!\!R)}}{\theta} + \frac{\|w^\theta \star \mathbf{S}_t u - \bar{u}^\theta\|_{\mathbf{L}^1(I\!\!R)}}{\theta} \right\} \\ &\quad \le 0. \end{aligned}$$

□

Lemma 11. *If $\hat{x} \in B_3(v)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\limsup_{\theta \to 0} \frac{1}{\theta} \int_{\hat{x}-\delta}^{\hat{x}+\delta} \left|\left(\mathbf{S}_t u^\theta\right)(x) - \bar{u}^\theta(x)\right| \mathrm{d}x \leq \varepsilon, \tag{5.9}$$

where $\theta \mapsto u^\theta$ and $\theta \mapsto \bar{u}^\theta$ are any paths generating the shift tangent vectors v and Λv, respectively.

Proof. By assumption, $\xi(u; \hat{x}) = y_j$ for some j. Since $\hat{x}, y_j$ are continuity points for $\mathbf{S}_t u$ and u respectively, there exists $\delta' > 0$ such that $\hat{x} + \delta'$ is a continuity point for $\mathbf{S}_t u$ and there holds:

$$\left|\mathrm{D}(\mathbf{S}_t u)\right|\left([\hat{x}, \hat{x}+\delta']\right) < \frac{\varepsilon}{7\|v\|_\infty}, \qquad \left|\mathrm{D}u\right|\left([y_j, \xi^+(u; \hat{x}+\delta')]\right) < \frac{\varepsilon}{7\|v\|_\infty}. \tag{5.10}$$

Moreover, we can choose δ' small enough so that $y_j < \xi^+(u; \hat{x}+\delta') < y_{j+1}$. We now take $y' \in \left(y_j, \xi^+(u; \hat{x}+\delta')\right) \setminus A$ and define

$$\tilde{v}(x) = \begin{cases} \alpha_{j-1} & \text{if } x \in [y_j, y') \\ v(x) & \text{otherwise.} \end{cases}$$

One easily checks that $\tilde{v} \in Y$,

$$\left(\Lambda \tilde{v}\right)(x) = (\Lambda v)(x) \quad \text{if} \quad x \notin [\hat{x},\ \hat{x}+\delta']$$

and $\hat{x} \in B_2(\tilde{v})$. If $w^\theta \xrightarrow{\wedge} \Lambda \tilde{v}$, we can thus apply Lemma 10 and Theorem 1 and obtain the existence of some $\delta \in (0, \delta')$ such that

$$\begin{aligned}
\limsup_{\theta \to 0} \frac{1}{\theta} \int_{\hat{x}-\delta}^{\hat{x}+\delta} \left|\left(\mathbf{S}_t u^\theta\right)(y) - \bar{u}^\theta(y)\right| \mathrm{d}y &\leq \limsup_{\theta \to 0} \frac{1}{\theta} \int_{\hat{x}-\delta}^{\hat{x}+\delta} \left|\left(\mathbf{S}_t(\tilde{v}^\theta \star u)\right)(y) - \left(w^\theta \star (\mathbf{S}_t u)\right)(y)\right| \mathrm{d}y \\
&\quad + \limsup_{\theta \to 0} \left\{ \frac{\|\mathbf{S}_t u^\theta - \mathbf{S}_t(\tilde{v}^\theta \star u)\|_{\mathbf{L}^1(\mathbb{R})}}{\theta} + \frac{\|w^\theta \star \mathbf{S}_t u - \bar{u}^\theta\|_{\mathbf{L}^1(\mathbb{R})}}{\theta} \right\} \\
&\leq \|v - \tilde{v}\|_{\mathbf{L}^1(\mathrm{D}u)} + \|\Lambda v - \Lambda \tilde{v}\|_{\mathbf{L}^1\left(\mathrm{D}(\mathbf{S}_t u)\right)} \\
&\leq 2\|v\|_\infty |\mathrm{D}u|\left([y_j, \xi^+(u; \hat{x}+\delta')]\right) + \int_{[\hat{x}, \hat{x}+\delta']} \Big[\left|(\Lambda v)(x)\right| + \left|(\Lambda \tilde{v})(x)\right|\Big] \left|\mathrm{D}(\mathbf{S}_t u)\right|(x) \\
&\leq \frac{2\varepsilon}{7} + 2\|v\|_\infty \left|\mathrm{D}(\mathbf{S}_t u)\right|\left([\hat{x}, \hat{x}+\delta']\right) + \sum_{x \in B_1 \cap [\hat{x}, \hat{x}+\delta']} \left|(\Lambda \tilde{v})(x)\right| \left|\mathrm{D}(\mathbf{S}_t u)\right|(\{x\}) \\
&\leq \frac{4\varepsilon}{7} + 3\|v\|_\infty |\mathrm{D}u|\left([y_j, \xi^+(u; \hat{x}+\delta')]\right) \\
&\leq \varepsilon.
\end{aligned}$$

□

Lemma 12. *If $\hat{x} \in B_1(v)$, then there exists $\delta \in (0,1)$ such that*

$$\limsup_{\theta \to 0} \frac{1}{\theta} \int_{\hat{x}-\delta}^{\hat{x}+\delta} \big| (\mathbf{S}_t u^\theta)(x) - \bar{u}^\theta(x) \big| \mathrm{d}x = 0,$$

where $\theta \mapsto u^\theta$ and $\theta \mapsto \bar{u}^\theta$ are any paths generating the shift tangent vectors v and Λv, respectively.

Proof. Set $y_0 = -\infty$, $y_{n+1} = +\infty$, $\alpha_0 = \alpha_n = 0$ and choose $i \le j$ such that

$$\xi^-(u;\hat{x}) \in (y_i, y_{i+1}), \qquad \xi^+(u;\hat{x}) \in (y_j, y_{j+1}).$$

Introduce the function

$$\tilde{v}(x) = \begin{cases} \alpha_i & \text{if } x < y_{i+1}, \\ v(x) & \text{if } x \in [y_{i+1}, y_j], \\ \alpha_j & \text{if } x > y_j. \end{cases}$$

We now define $\tilde{v}^\theta$ and v^θ as in (5.6) and choose $\eta > 0$, $\delta > 0$ small enough so that

$$y_i < \xi^-(u;\hat{x}-2\delta) - \eta < \xi^+(u;\hat{x}+2\delta) + \eta < y_{j+1}\,. \tag{5.11}$$

Therefore, for θ sufficiently small, there holds

$$\big(\tilde{v}^\theta \star u\big)(y) = \big(v^\theta \star u\big)(y) \qquad \forall y \in \big[\xi^-(u;\hat{x}-2\delta) - \eta,\ \xi^+(u;\hat{x}+2\delta) + \eta\big].$$

Applying Lemma 7 we obtain

$$\big(\mathbf{S}_t(\tilde{v}^\theta \star u)\big)(y) = \big(\mathbf{S}_t(v^\theta \star u)\big)(y) \qquad \forall y \in (\hat{x}-2\delta, \hat{x}+2\delta).$$

If $w^\theta \xrightarrow{\wedge} \Lambda v$, to prove the lemma is thus suffices to show that

$$\limsup_{\theta \to 0} \frac{1}{\theta} \int_{\hat{x}-\delta}^{\hat{x}+\delta} \big| \big(\mathbf{S}_t(\tilde{v}^\theta \star u)\big)(x) - (w^\theta \star \mathbf{S}_t u)(x) \big| \mathrm{d}x = 0. \tag{5.12}$$

By (5.11) and the definition of Λv it follows

$$(\Lambda v)(x) = \begin{cases} \alpha_i & \text{if } x \in (\hat{x}-2\delta, \hat{x}) \\ v_0 & \text{if } x = \hat{x} \\ \alpha_j & \text{if } x \in (\hat{x}, \hat{x}+2\delta), \end{cases}$$

where v_0 is defined as in (4.15), with $\alpha_i = \alpha$ and $\alpha_j = \beta$. Now to the functions $\tilde{v}^\theta \star u$ we can apply Remark 7, and Lemma 9. In analogy with (4.19), we thus take

$$M \doteq 1 + \max\left\{ \frac{2}{\Delta\omega} \|\tilde{v}\|_{\mathbf{L}^1(\mathrm{D}u)}, |\alpha_j| + |\alpha_i| + |v_0| \right\}$$

and choose $w^\theta \xrightarrow{\wedge} \Lambda v$ such that

$$w^\theta(x) = \begin{cases} v_0 & \text{if} \quad x = \hat{x}, \\ \alpha_i & \text{if} \quad x \in [\hat{x}-\delta,\ \hat{x}-M\theta], \\ \alpha_j & \text{if} \quad x \in [\hat{x}+\delta,\ \hat{x}+M\theta], \end{cases}$$

with w^θ linear on each of the intervals $[\hat{x} - M\theta,\ \hat{x}]$ and $[\hat{x},\ \hat{x} + M\theta]$. If $\Delta_{\theta,M}$ is defined as in (4.17), calling J_θ the interval enclosed by the points $\hat{x}_\theta$ and $\hat{x} + \theta v_0$, we now have

$$\Big|\mathbf{S}_t(\tilde{v}^\theta \star u)(x) - w^\theta \star \mathbf{S}_t u(x)\Big| \quad \begin{cases} \le 2\|u\|_\infty & \text{if} \quad x \in J_\theta, \\ \le \Delta_{\theta,M} & \text{if} \quad x \in [\hat{x} - \theta M,\ \hat{x} + \theta M] \setminus J_\theta, \\ = 0 & \text{if} \quad x \in [\hat{x} - \delta,\ \hat{x} - \theta M] \cup [\hat{x} + \theta M,\ \hat{x} + \delta]. \end{cases}$$

Applying Lemma 9 we now obtain

$$\begin{aligned} &\limsup_{\theta \to 0} \frac{1}{\theta} \int_{\hat{x}-\delta}^{\hat{x}+\delta} \big|\big(\mathbf{S}_t(\tilde{v}^\theta \star u)\big)(x) - (w^\theta \star \mathbf{S}_t u)(x)\big| \mathrm{d}x \\ &\qquad \le \limsup_{\theta \to 0} \frac{1}{\theta} \Big\{ 2M\theta \cdot \Delta_{\theta,M} + 2\big|\hat{x}_\theta - \hat{x} - v_0\theta\big| \cdot \|u\|_\infty \Big\} \\ &\qquad = 0. \end{aligned}$$

□

We can now complete the proof of Theorem 4. Let $v \in Y$ and $v^\theta \xrightarrow{\wedge} v$ as in (5.6). Since the supports of the functions v^θ are uniformly bounded, we can choose a constant R large enough so that $\mathbf{S}_t(v^\theta \star u) = \mathbf{S}_t u$ outside the interval $[-R, R]$. Moreover since Λv also has compact support, it is not restrictive to assume that $w^\theta \star \mathbf{S}_t u = \mathbf{S}_t u$ outside $[-R, R]$. Theorem 4 will thus be proved if we show that

$$L \doteq \limsup_{\theta \to 0} \frac{1}{\theta} \int_{-R}^{R} \Big|\big(\mathbf{S}_t(v^\theta \star u)\big)(y) - \big(w^\theta \star (\mathbf{S}_t u)\big)(y)\Big| \mathrm{d}y = 0. \tag{5.14}$$

Let $\varepsilon > 0$ be given. By Lemmas 10, 11 and 12, for each $x \in [-R, R]$ there exists $\delta_x > 0$ such that

$$\limsup_{\theta \to 0} \frac{1}{\theta} \int_{x-\delta_x}^{x+\delta_x} \Big|\big(\mathbf{S}_t(v^\theta \star u)\big)(y) - \big(w^\theta \star (\mathbf{S}_t u)\big)(y)\Big| \mathrm{d}y \le \begin{cases} 0 & \text{if } x \in B_1 \cup B_2(v), \\ \varepsilon & \text{if } x \in B_3(v). \end{cases}$$

Covering the compact interval $[-R, R]$ with finitely many neighborhoods, say $[x_i - \delta_i,\ x_i + \delta_i]$, $i = 1, \ldots, N$, we obtain

$$\frac{1}{\theta} \int_{-R}^{R} \big|\big(\mathbf{S}_t(\tilde{v}^\theta \star u)\big)(x) - (w^\theta \star \mathbf{S}_t u)(x)\big| \mathrm{d}x \le \sum_{i=1}^{N} \frac{1}{\theta} \int_{x_i-\delta_i}^{x_i+\delta_i} \big|\big(\mathbf{S}_t(\tilde{v}^\theta \star u)\big)(x) - (w^\theta \star \mathbf{S}_t u)(x)\big| \mathrm{d}x. \tag{5.15}$$

Taking the lim sup of (5.15) as $\theta \to 0$, one finds

$$L \le \sum_{x \in B_3(v)} \varepsilon \le n\varepsilon,$$

where n is the number of points where v has a jump. Observing that n is independent of ε while $\varepsilon > 0$ is arbitrary, this yields (5.14). □

Corollary 2. *Let $u \in X$ be given. Then the flow map $\mathbf{S}_t$ generated by the conservation law* (1.6) *is shift differentiable at u for all except countably many times $t > 0$.*

Proof. It suffices to show that, for any $\tau > 0$, the set of times $t > \tau$ at which the assumptions of Theorem 4 do not hold is at most countable. For each rational numbers p, call $x_p(\cdot)$, the generalized characteristic for the solution of (4.1), with $x_p(\tau) = p$. It is well known that this characteristic is uniquely defined for $t \geq \tau$. For p, q rationals, call t_{pq} the first time at which the characteristics through p and q meet, i.e.

$$t_{pq} \doteq \min \bigl\{t; \quad x_p(t) = x_q(t)\bigr\}.$$

Set $t_{pq} \doteq \infty$ if the two characteristics remain disjoint for all $t > \tau$.

Now suppose that the assumptions in Theorem 4 do not hold at some time $t > \tau$. Then, for some $\bar{x} \in \mathbb{R}$, there exist three distinct characteristics through the point $(t, \bar{x})$, of the form

$$x(s) = \bar{x} + \lambda_i(s - t) \qquad\qquad i = 1, 2, 3, \quad s \in [0, t],$$

with $\lambda_1 > \lambda_2 > \lambda_3$. Choose rational numbers p, q such that

$$\bar{x} - \lambda_1(t - \tau) < p < \bar{x} - \lambda_2(t - \tau) < q < \bar{x} - \lambda_3(t - \tau).$$

We then have $t = t_{p,q}$. Since the set $\{t_{pq}\}$ is countable, the corollary is proved.

References

[1] J P. Aubin and A. Cellina, "Differential Inclusions", Springer-Verlag, New York, 1984.

[2] A. Bressan and R. M. Colombo, The semigroup generated by 2×2 conservation laws, *Arch. Rational Mech. Anal.* **133** (1995), 1-75.

[3] A. Bressan and A. Marson, A variational calculus for shock solutions of systems of conservation laws, *Comm. Part. Diff. Eq.* **20** (1995), 1491-1553.

[4] A. Bressan and A. Marson, A maximum principle for optimally controlled systems of conservation laws, *Rend. Sem. Univ. Padova,* **94** (1995), 79-94.

[5] M. Crandall, The semigroup approach to first order quasilinear equations in several space variables, *Israel J. Math.* **26** (1977), 1097-1119.

[6] C. Dafermos, Generalized characteristics and the structure of solutions of hyperbolic conservation laws, *Indiana Univ. Math. J.* **12** (1972), 108-132.

[7] S. Kruzkov, First order quasilinear equations with several space variables, *Math. USSR Sbornik* **10** (1970), 217-243.

[8] P. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* **10** (1957), 537-566.

[9] J. Smoller, "Shock waves and reaction diffusion equations", Springer-Verlag, New York, 1982.