Error bounds of finite difference schemes for two-dimensional scalar conservation laws with source term

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Abstract

This paper studies explicit and semi-implicit finite difference schemes approximating non-homogenous scalar conservation laws with both stiff and non-stiff source terms. Optimal error bounds are presented.

1 Introduction

In this paper, we study the following two-dimensional scalar conservation laws with source term

$$u_t + \mathrm{div}(f(u)) = g(u), \tag{1}$$

$$u_t + \mathrm{div}(f(u)) = \frac{1}{\delta} g(u), \tag{2}$$

associated with the initial data $u(x, y, 0) = u^0(x, y)$. Model (1) is referred to as the non-stiff model while (2) is the stiff model. Here $f(u) = (f_1(u), f_2(u))^T$ is a smooth vector-valued function, and δ is a small positive parameter which is often referred to as the relaxation time. Setting $\delta = 1$ in (2), we obtain (1).

Modified Lax-Friedrich's schemes approximating the models are analyzed, and the schemes will be fully described in Section 2. The main purpose of this paper is to study the global error bounds of the schemes. Modified Godunov methods approximating one-dimensional models are studied by Schroll and Winther in [3], where they proved that the error bound is $\mathcal{O}(\sqrt{\Delta t})$. It is also well-known that for homogenous, scalar conservation laws, the L^1-error for the Lax scheme is $\mathcal{O}(\sqrt{\Delta t})$, cf [2]. In this paper, we will prove the optimal error bound for the non-stiff case. In the stiff case, the same error bound independent of the relaxation time δ will be also proved under certain assumptions on g.

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A similar semi-explicit scheme approximating a stiff system of conservation laws in two space dimensions is studied by Shen, Tveito and Winther [4], and an error bound of $\mathcal{O}(\sqrt{\Delta t})$ independent of δ is proved. In this paper, the technique of generalizing one space dimension analysis into two space dimensions possesses the same principle as in that work. We also use some techniques in Schroll and Winther [3], and we will refer to these papers for some details in the proofs.

The paper is organized as follows: In Section 2, the precise description of the finite difference schemes and the assumptions on the models are given, and we also state the main results. The error bound for the non-stiff case is derived in Section 3, and in Section 4 we sketch the proof for the stiff case.

2 Preliminaries

In this section, we will give the detailed description of the finite different schemes, the assumptions on the models, and the main theorems. First, we introduce some notations that will be used throughout this paper. The discrete averaging operator $A(\cdot)_{i,j}$ on ω is defined as

$$A(\omega)_{i,j} := \frac{1}{4}\left(\omega_{i+1,j+1} + \omega_{i-1,j+1} + \omega_{i+1,j-1} + \omega_{i-1,j-1}\right),$$

and the discrete divergence operator $\mathrm{div}_\Delta(\cdot)_{i,j}$ on $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)^T$ is defined as

$$\begin{aligned} \mathrm{div}_\Delta(\mathcal{F})_{i,j} \quad := \quad & \frac{1}{4\Delta x}\left[(\mathcal{F}_1)_{i+1,j+1} + (\mathcal{F}_1)_{i+1,j-1} - (\mathcal{F}_1)_{i-1,j+1} - (\mathcal{F}_1)_{i-1,j-1}\right] \qquad (3) \\ & + \frac{1}{4\Delta y}\left[(\mathcal{F}_2)_{i+1,j+1} + (\mathcal{F}_2)_{i-1,j+1} - (\mathcal{F}_2)_{i+1,j-1} - (\mathcal{F}_2)_{i-1,j-1}\right]. \end{aligned}$$

The finite different schemes will be defined on a so-called "staggered grid". For notational clarity, we define the space W^n of grid indices (i, j) at a given time step n as

$$W^n := \{(i, j) : i + n = 0 (\mathrm{mod}\ 2), j + n = 0 (\mathrm{mod}\ 2)\}.$$

For the non-stiff equation, we study the following explicit scheme

$$\frac{u^{n+1} - A(u^n)}{\Delta t} + \mathrm{div}_\Delta(f(u^n)) = A(g(u^n)), \tag{4}$$

at index $_{i,j}$ where $(i, j) \in W^{n+1}$. Next semi-implicit scheme is applied to analyze the stiff model

$$\frac{u^{n+1} - A(u^n)}{\Delta t} + \mathrm{div}_\Delta(f(u^n)) = \frac{1}{\delta} g(u^{n+1}), \tag{5}$$

at $_{i,j}$ for $(i, j) \in W^{n+1}$. The initial data for the schemes is defined to be the cell average of u^0

$$u^0_{i,j} := \frac{1}{4\Delta x \Delta y}\int_{I_{i,j}} u^0(x, y)\, dx\, dy, \quad (i, j) \in w^0,$$

where $I_{i,j}$ is the computational cell

$$I_{i,j} := [x_{i-1}, x_{i+1}) \times [y_{j-1}, y_{j+1}).$$

The CFL conditions for both schemes are

$$\frac{\Delta t}{\Delta x} \max |f_1'(u)|, \frac{\Delta t}{\Delta y} \max |f_2'(u)| \le \frac{1}{2}. \tag{6}$$

We assume that f and g are smooth, bounded functions, and $g(0) = 0$. We consider the solutions of the Cauchy problems with the initial data satisfying

$$u^0 \in L^1(\mathcal{R}^2) \cap BV(\mathcal{R}^2, \mathcal{R}),$$

where BV denotes the subspace of L^1_{loc} consisting of functions with bounded total variation. And the total variation TV is defined as

$$\begin{aligned} TV(w) \quad := \quad & \sup_{h \neq 0} \int_{\mathcal{R}^2} \frac{|w(x+h, y) - w(x, y)|}{|h|} \, dx \, dy \\ & + \sup_{k \neq 0} \int_{\mathcal{R}^2} \frac{|w(x, y+k) - w(x, y)|}{|k|} \, dx \, dy. \end{aligned}$$

Let $\sigma(\cdot)$ denote the usual sign function. The entropy solution of (2) satisfies the Kruzkov inequality(cf. [2])

$$\begin{aligned} & \int_0^T \int_{\mathcal{R}^2} [|u - k| \phi_t + F(u, k) \cdot \nabla \phi] \, dx \, dy \, dt \\ & + \int_{\mathcal{R}^2} \left[\left| u^0 - k \right| \phi(x, y, 0) - |u(x, y, T) - k| \phi(x, y, T) \right] \, dx \, dy \\ \ge \quad & -\frac{1}{\delta} \int_0^T \int_{\mathcal{R}^2} [\sigma(u - k) g(u) \phi] \, dx \, dy \, dt, \end{aligned} \tag{7}$$

where

$$F(u, k) = (F_1(u, k), F_2(u, k))^T = \sigma(u - k)(f(u) - f(k)). \tag{8}$$

We note that this formulation is valid independent of the choice of δ, i.e., by simply setting $\delta = 1$, we obtain the entropy inequality for (1). For a fixed final time $T > 0$, the solution of the non-stiff equation (1) is assumed to be located in the state-space

$$\mathcal{S} = \{u \in \mathcal{R} : |u| \le \alpha\},$$

for some constant α that satisfies $\exp(L_g T) \sup \left| u^0 \right| \le \alpha$, where $L_g = \sup_{u \in \mathcal{S}} |g'(u)|$. Furthermore, the total variation is bounded in the sense

$$TV(u(\cdot, \cdot, t)) \le M \, TV(u^0), \quad \forall t \in [0, T],$$

for some constant M, and the L^1 continuity in time is also valid

$$\|u(\cdot, \cdot, t) - u(\cdot, \cdot, \tau)\|_1 \le M |t - \tau|, \quad \forall t, \tau \in [0, T].$$

For the non-stiff model, the error bound is estimated in the following Theorem:

Theorem 2.1 *Suppose the given assumptions for the non-stiff problem are satisfied. Let u be the entropy solution of (1), $\{u_{i,j}^n\}$ be the data generated by the scheme (4) with the CFL condition (6) satisfied. Furthermore, we let u^n be the piecewise constant function representing the data $u_{i,j}^n$ in the way that u^n interpolates $u_{i,j}^n$ and is constant in each computational cell. Then there is a constant M independent of Δx, Δy and Δt, such that*

$$\|u(\cdot,\cdot,t_n) - u^n\|_1 \leq M\sqrt{\Delta t}, \quad \forall t_n \in [0,T].$$

This Theorem will be proved in Section 3.

For the stiff model, we assume that zero is the unique equilibrium. And a monotone property of g is also assumed to ensure that the semi-implicit scheme is well defined. Precisely, we assume

$$g(0) = 0, \quad g'(u) \leq -\gamma < 0 \quad \forall u \in [0,1].$$

We also assume that $u^0 \in \mathcal{S}_1 = [0,1]$. Then, the entropy solution of the stiff model will remain in the state-space $\mathcal{S}_1$. We also assume that the initial state is close to the equilibrium, i.e., $\left\|g(u^0)\right\|_1 \leq M_0\delta$ for some constant M_0 independent of δ. Next theorem, which will be proved in Section 4, gives the error bound for the stiff model.

Theorem 2.2 *Suppose all the assumptions mentioned above are satisfied for the stiff problem. And the initial data satisfies $u^0 \in BV(\mathcal{R}^2, \mathcal{S}_1)$. Then the semi-implicit scheme (5) has a unique solution $\{u_{i,j}^n\} \in \mathcal{S}_1$. Let u be the entropy solution of (2), and u^n be the piecewise constant function representing the data $u_{i,j}^n$. Then, there exists a constant M, independent of the relaxation time δ and the grid parameters Δx, Δy and Δt, such that*

$$\|u(\cdot,\cdot,t_n) - u^n\|_1 \leq M\sqrt{\Delta t}, \quad \forall t_n \in [0,T].$$

3 The non-stiff model

In this section, we prove Theorem 2.1. An a-priori estimates of the solution of the scheme (4) is first given in Section 3.1. Then we introduce the comparison function in Section 3.2, and the entropy inequality of the comparison function is derived in Section 3.3. Finally, in Section 3.4, the two entropy inequalities of the comparison function and the entropy solution of (1) will be combined by a mollifier function to derive the error bound in Theorem 2.1.

3.1 A-priori estimates

The solution of the explicit finite difference scheme (4) has the following properties:

Lemma 3.1 *Suppose the conditions in Theorem 2.1 are satisfied. Let $\{u_{i,j}^n\}$ be the discrete solution of the explicit scheme (4), and u^n the corresponding* *piecewise constant function representing* $\{u^n_{i,j}\}$. *Then, there exists a constant* M, *independent of the grid parameters* Δx, Δy *and* Δt, *such that the following estimates hold*

$$\begin{array}{rl} i) & u^n \in \mathcal{S}, \\ ii) & \|u^n\|_1 \leq M, \\ iii) & TV(u^n) \leq M, \\ iv) & \|u^n - u^m\|_1 \leq M\Delta t|n-m|, \quad 0 \leq n, m \leq N = T/\Delta t. \end{array}$$

Here, $TV(u^n)$ equals to the total variation of the grid function $u^n_{i,j}$

$$TV(u^n) = \Delta y \sum_{(i,j)\in W^n} \left|u^n_{i+2,j} - u^n_{i,j}\right| + \Delta x \sum_{(i,j)\in W^n} \left|u^n_{i,j+2} - u^n_{i,j}\right|,$$

and the L^1-norm for the picewise constant function u^n equals to

$$\|u^n\|_1 := 4\Delta x \Delta y \sum_{(i,j)\in W^n} \left|u^n_{i,j}\right|.$$

The proof of Lemma 3.1 is almost straightforward, and we simply refer to [3, 4]. We note that with these a-priori estimates, one can further prove that the finite difference solution converges to the entropy solution, cf. [4].

3.2 The comparison function

In this section, we define the comparison function. First, for a given n, we define the average operator P^n on $u \in L^1_{loc}$ as

$$P^n(u)(x,y) := \frac{1}{4\Delta x \Delta y} \int_{I_{i,j}} u(r,s)\, dr\, ds \quad \text{for } (x,y) \in I_{i,j}, (i,j) \in W^n.$$

The comparison function $\bar{u}^n$ on $\mathcal{R}^2 \times \mathcal{R}^+$ is defined iteratively. For notational convenience, we will use (r, s, τ) instead of (x, y, t) in the context below. The crux of the proof is a dimension splitting method, see Shen, Tveito and Winther [4]. This splitting method was inspired by the work of Crandall and Majda [1]. We introduce two auxiliary variables v and w, and let $\bar{u}$ be the average of v and w

$$\bar{u} = \frac{1}{2}(v + w).$$

The iteration is initialized by

$$v(r,s,0^+) = w(r,s,0^+) = v(r,s,0) = w(r,s,0) = P^0(u^0)(r,s).$$

Then, for each $n \in \mathcal{Z}^+$, we iterate the following three steps:

- In (t^+_{n-1}, t^-_n), v is the solution of

$$v_\tau + 2f_1(v)_r = 0, \tag{9}$$

and w is the solution of

$$w_\tau + 2f_2(w)_s = 0, \tag{10}$$

with initial data

$$v(r,s,t_{n-1}^+) = w(r,s,t_{n-1}^+).$$

- At t_n, we take cell average as

$$\begin{aligned} v(r,s,t_n) &= P^n(v(\cdot,\cdot,t_n^-))(r,s), \\ w(r,s,t_n) &= P^n(w(\cdot,\cdot,t_n^-))(r,s). \end{aligned}$$

- The initial data for the next iteration is generated by a pseudo-Euler step

$$\bar{u}(r,s,t_n^+) = \bar{u}(r,s,t_n) + \Delta t P^n(g(\bar{u}(\cdot,\cdot,t_{n-1}^+)))(r,s), \tag{11}$$

where $\bar{u}(r,s,t_n)$ is obtained by the relation $\bar{u} = (v+w)/2$. We then set

$$v(r,s,t_n^+) = w(r,s,t_n^+) = \bar{u}(r,s,t_n^+).$$

The integral forms of (9) and (10) over $I_{i,j}\times(t_n,t_{n+1})$ give the Lax-Friedrich's scheme for $v_{i,j}^n$ and $w_{i,j}^n$. By setting $\bar{u}_{i,j}^n = (v_{i,j}^n + w_{i,j}^n)/2$, we immediately get the Lax-Friedrich's scheme in two dimensions for $\bar{u}_{i,j}^n$ as

$$\bar{u}_{i,j}^n = A(\bar{u}^{n-1})_{i,j} - \Delta t \mathrm{div}_\Delta(f(\bar{u}^{n-1}))_{i,j} + \Delta t A(g(u^{n-1}))_{i,j},$$

and the following interpolating property follows by induction

$$\bar{u}(x,y,t_n^+) = \lim_{\tau\to t_n,\tau>t_n} \bar{u}(x,y,t) = u_{i,j}^n \quad \text{for } (x,y)\in I_{i,j}, (i,j)\in W^n.$$

3.3 The discrete entropy inequality

In this section, we will derive the entropy inequality of the comparison function, by which we called the discrete entropy inequality. Since v and w are the entropy solution of (9) and (10) on the open interval (t_{n-1},t_n), we can apply the usual Kruzkov inequality. Adding these two inequalities, and divided by a factor of 2, we get

$$\begin{aligned} &\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}^2}\left[\frac{|v-k|+|w-k|}{2}\phi_\tau + F_1(v,k)\phi_r + F_2(w,k)\phi_s\right] dr\,ds\,d\tau \\ &\quad+\int_{\mathcal{R}^2}\left[\frac{|v(r,s,t_{n-1}^+)-k|+|w(r,s,t_{n-1}^+)-k|}{2}\phi(r,s,t_{n-1})\right] dr\,ds \\ &\quad-\int_{\mathcal{R}^2}\left[\frac{|v(r,s,t_n^-)-k|+|w(r,s,t_n^-)-k|}{2}\phi(r,s,t_n)\right] dr\,ds \geq 0. \end{aligned}$$

We recall that $F = (F_1, F_2)^T$ is defined in (8). Then, summing from $n = 1$ to N, and using the fact that

$$v(r,s,t_n^+) = w(r,s,t_n^+) = \bar{u}(r,s,t_n^+),$$

and

$$-|v-k| - |w-k| \leq -2\left|\frac{v+w}{2} - k\right| = -2|\bar{u} - k|,$$

we get

$$\begin{aligned}
&\int_0^T \int_{\mathcal{R}^2} \left[\frac{|v-k| + |w-k|}{2}\phi_\tau + F_1(v,k)\phi_r + F_2(w,k)\phi_s \right] dr\,ds\,d\tau \\
&\quad + \sum_{n=1}^N \int_{\mathcal{R}^2} \left[\left|\bar{u}(r,s,t_n^+) - k\right| - \left|\bar{u}(r,s,t_n^-) - k\right| \right] \phi(r,s,t_n)\,dr\,ds \\
&\quad + \int_{\mathcal{R}^2} \left[|\bar{u}(r,s,0) - k|\phi(r,s,0) - \left|\bar{u}(r,s,t_N^+) - k\right|\phi(r,s,T) \right] dr\,ds \geq 0.
\end{aligned}$$

Finally, by setting in next inequality which comes from the pseudo Euler step (11)

$$\left|\bar{u}(r,s,t_n^+) - k\right| \leq |\bar{u}(r,s,t_n) - k| + \Delta t \sigma(\bar{u}(r,s,t_n^+) - k) P^n(g(\bar{u}(\cdot,\cdot,t_{n-1}^+)))(r,s),$$

we obtain the discrete entropy inequality

$$\begin{aligned}
&\int_0^T \int_{\mathcal{R}^2} \left[\frac{|v-k| + |w-k|}{2}\phi_\tau + F_1(v,k)\phi_r + F_2(w,k)\phi_s \right] dr\,ds\,d\tau \\
&\quad + \sum_{n=1}^N \int_{\mathcal{R}^2} \left[|\bar{u}(r,s,t_n) - k| - \left|\bar{u}(r,s,t_n^-) - k\right| \right] \phi(r,s,t_n)\,dr\,ds \qquad (12) \\
&\quad + \int_{\mathcal{R}^2} \left[|\bar{u}(r,s,0) - k|\phi(r,s,0) - \left|\bar{u}(r,s,t_N^+) - k\right|\phi(r,s,T) \right] dr\,ds \\
&\geq \; -\Delta t \sum_{n=1}^N \int_{\mathcal{R}^2} \sigma(\bar{u}(r,s,t_n^+) - k) P^n(g(\bar{u}(\cdot,\cdot,t_{n-1}^+)))(r,s)\phi(r,s,t_n)\,dr\,ds.
\end{aligned}$$

3.4 The error bound; proof of Theorem 2.1

In this section, we will finish the prove of Theorem 2.1. The two entropy inequalities (7) and (12) will be combined by using a mollifier function which is defined as

$$\omega_\theta(x) := \frac{1}{\theta}\Omega(\frac{x}{\theta}),$$

where $\Omega : \mathcal{R} \to \mathcal{R}$ is smooth, nonnegative, symmetric with support in $[-1,1]$ and $\int_{\mathcal{R}} \Omega(x)\,dx = 1$.

Let $T > 0$ be given. In inequality (12), we set $k = u(x,y,t)$, $\phi(r,s,\tau) = \omega_\theta(x-r)\omega_\theta(y-s)\omega_\theta(t-\tau)$ and integrate in x, y and t, and we get an inequality.

In (7), we set $k = \bar{u}(r,s,t_n^+)$, $\phi(x,y,t) = \omega_\theta(x-r)\omega_\theta(y-s)\omega_\theta(t-\tau)$, integrate in r, s and τ over $\mathcal{R}^2 \times [t_{n-1}, t_n]$, and sum in n from 1 to N, we get another inequality. Summing these two inequalities together, we obtain the following inequality

$$L(\theta) \le R_1(\theta) + R_2(\theta) + R_3(\theta) + R_4(\theta), \tag{13}$$

where

$$\begin{aligned} L(\theta) \quad := \quad & \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}^2} \int_{\mathcal{R}^2} \left| u(x,y,T) - \bar{u}(r,s,t_n^+) \right| \\ & \qquad \omega_\theta(x-r)\omega_\theta(y-s)\omega_\theta(T-\tau)\, dx\, dy\, dr\, ds\, d\tau \\ & + \int_0^T \int_{\mathcal{R}^2} \int_{\mathcal{R}^2} \left| \bar{u}(r,s,t_N^+) - u \right| \omega_\theta(x-r)\omega_\theta(y-s)\omega_\theta(t-T)\, dx\, dy\, dr\, ds\, d\tau, \end{aligned}$$

$$\begin{aligned} R_1(\theta) := & \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}^2} \int_{\mathcal{R}^2} \left| u^0 - \bar{u}(r,s,t_n^+) \right| \omega_\theta(x-r)\omega_\theta(y-s)\omega_\theta(\tau)\, dx\, dy\, dr\, ds\, d\tau \\ & + \int_0^T \int_{\mathcal{R}^2} \int_{\mathcal{R}^2} |\bar{u}(r,s,0) - u| \omega_\theta(x-r)\omega_\theta(y-s)\omega_\theta(t)\, dr\, ds\, dx\, dy\, dt, \end{aligned}$$

$$\begin{aligned} R_2(\theta) \quad := \quad & \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}^2} \int_0^T \int_{\mathcal{R}^2} \left\{ \left[\left| u - \bar{u}(r,s,t_n^+) \right| - \frac{|v-u| + |w-u|}{2} \right] \right. \\ & \qquad \omega_\theta(x-r)\omega_\theta(y-s){\omega_\theta}'(t-\tau) \\ & + \left[F_1(u, \bar{u}(r,s,t_n^+)) - F_1(v,u) \right] {\omega_\theta}'(x-r)\omega_\theta(y-s)\omega_\theta(t-\tau) \\ & + \left. \left[F_2(u, \bar{u}(r,s,t_n^+)) - F_2(w,u) \right] \omega_\theta(x-r){\omega_\theta}'(y-s)\omega_\theta(t-\tau) \right\} \\ & \qquad dx\, dy\, dt\, dr\, ds\, d\tau, \end{aligned}$$

$$\begin{aligned} R_3(\theta) \quad := \quad & \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}^2} \int_0^T \int_{\mathcal{R}^2} \sigma(u - \bar{u}(r,s,t_n^+)) \omega_\theta(x-r)\omega_\theta(y-s) \\ & \times \left[g(u)\omega_\theta(t-\tau) - P^n(g(\bar{u}(\cdot,\cdot,t_{n-1}^+)))(r,s)\omega_\theta(t-t_n) \right]\, dx\, dy\, dt\, dr\, ds\, d\tau \end{aligned}$$

and

$$\begin{aligned} R_4(\theta) \quad := \quad & \int_0^T \int_{\mathcal{R}^2} \sum_{n=1}^{N} \int_{\mathcal{R}^2} \left[|\bar{u}(r,s,t_n) - u| - \left| \bar{u}(r,s,t_n^-) - u \right| \right] \\ & \qquad \omega_\theta(x-r)\omega_\theta(y-s)\omega_\theta(t-\tau)\, dr\, ds\, dx\, dy\, dt. \end{aligned}$$

The following bounds hold for these terms:

Lemma 3.2 *For a given $T > 0$, there is a constant M independent of θ and the grid parameters Δx, Δy and Δt, such that*

$$\begin{array}{rl} i) & \left|L(\theta) - \left\|u(\cdot,\cdot,T) - \bar{u}(\cdot,\cdot,t_N^+)\right\|_1\right| \leq M\theta, \\ ii) & \left|R_1(\theta) - \left\|u^0 - \bar{u}(\cdot,\cdot,0)\right\|_1\right| \leq M(\theta + \Delta t), \\ iii) & |R_2(\theta)| \leq M\dfrac{\Delta t}{\theta}, \\ iv) & |R_3(\theta)| \leq M\left(\theta + \Delta t + \dfrac{\Delta t}{\theta} + \Delta t \displaystyle\sum_{n=0}^{N-1} \left\|u(\cdot,\cdot,t_n) - \bar{u}(\cdot,\cdot,t_n^+)\right\|_1\right), \\ v) & |R_4(\theta)| \leq M\dfrac{\Delta t}{\theta}. \end{array}$$

Proof: The estimates $i)$, $ii)$, $iii)$ and $v)$ can be carried out by the similar analysis in e g. [3] and [4]. Consider $iv)$, we have

$$\begin{aligned} |R_3(\theta)| \leq & \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}^2}\int_0^T\int_{\mathcal{R}^2}\left|g(u) - g(\bar{u}(r,s,t_{n-1}^+))\right| \\ & \qquad \times\omega_\theta(x-r)\omega_\theta(y-s)\omega_\theta(t-\tau)\,dx\,dy\,dt\,dr\,ds\,d\tau \\ & +\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}^2}\int_0^T\int_{\mathcal{R}^2}\left|g(\bar{u}(r,s,t_{n-1}^+)) - P^n(g(\bar{u}(\cdot,\cdot,t_{n-1}^+)))(r,s)\right| \\ & \qquad \times\omega_\theta(x-r)\omega_\theta(y-s)\omega_\theta(t-\tau)\,dx\,dy\,dt\,dr\,ds\,d\tau \\ & +\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_0^T\int_{\mathcal{R}^2}\left|P^n(g(\bar{u}(\cdot,\cdot,t_{n-1}^+)))(r,s)\right|\left|\omega_\theta(t-t_n) - \omega_\theta(t-\tau)\right| \\ & \qquad dr\,ds\,dt\,d\tau \\ = & R_{3,1}(\theta) + R_{3,2}(\theta) + R_{3,3}(\theta). \end{aligned}$$

By the similar arguments in [3], we have

$$\begin{aligned} R_{3,1}(\theta) &\leq M(\theta + \Delta t) + M\Delta t\sum_{n=1}^{N} e^{n-1}, \\ R_{3,3}(\theta) &\leq M\frac{\Delta t}{\theta}, \end{aligned}$$

where $e^n := \|u(\cdot,\cdot,t_n) - \bar{u}(\cdot,\cdot,t_n^+)\|_1$. And by the well-known property of the averaging operator P^n

$$\|w - P^n(w)\|_1 \leq \Delta t TV(w),$$

we get the estimate for $G_2(\theta)$ as

$$R_{3,2}(\theta) \leq M\Delta t,$$

and $iv)$ follows. $\diamond$

Now, by setting the estimates in Lemma 3.2 into inequality (13), we get a recursive relation for the error

$$e^N \leq e^0 + M(\theta + \Delta t + \frac{\Delta t}{\theta}) + \Delta t M \sum_{n=1}^{N} e^{n-1}.$$

Since $e^0 \leq M\Delta t$, the by choosing $\theta = \sqrt{\Delta t}$, we get

$$e^N \leq \Delta t M \sum_{n=1}^{N} e^{n-1} + M\sqrt{\Delta t},$$

and induction gives the error bound in Theorem 2.1.

4 The stiff model

For the stiff case, we apply the similar arguments as for the non-stiff case. Here, the state-space where the solution resides is $\mathcal{S}_1 = [0, 1]$, and the genetic constant M is independent of the relaxation time δ. A-prior estimate of the numerical solution generated by the scheme (5) reads:

Lemma 4.1 *Let the assumptions in Theorem 2.2 and the CFL condition (6) be satisfied. Then, the solution of the scheme (5) is unique, and there exists a constant M independent of the relaxation time δ and the grid parameters Δx, Δy and Δt, such that*

$$\begin{array}{rl} i) & u^n \in \mathcal{S}_1, \\ ii) & TV(u^n) \leq TV(u^0) \leq M, \\ iii) & \|g(u^n)\|_1 \leq M\delta, \\ iv) & \|n^n - u^m\|_1 \leq M\Delta t|n - m|, \quad 0 \leq n, m \leq N = T/\Delta t. \end{array}$$

Here u^n is the piecewise constant representation of the grid data $u^n_{i,j}$.

Again, we refer to [3] and [4] for the proof of Lemma 4.1, and mention that these a-priori estimates carry over to the entropy solution of (2).

The proof of Theorem 2.2 can be carried out in a similar way as for the non-stiff case. Only a few points are different. First, we use the backward-Euler step instead of the pseudo-Euler step in (11)

$$\bar{u}(r, s, t_n^+) = \bar{u}(r, s, t_n) + \frac{\Delta t}{\delta} g(\bar{u}(r, s, t_n^+)),$$

and then the discrete entropy inequality reads

$$\int_0^T \int_{\mathcal{R}^2} \left[\frac{|v - k| + |w - k|}{2} \phi_\tau + F_1(v, k)\phi_r + F_2(w, k)\phi_s \right] dr\, ds\, d\tau$$

$$
\begin{aligned}
&+\sum_{n=1}^{N}\int_{\mathcal{R}^2}\left[|\bar{u}(r,s,t_n)-k|-\left|\bar{u}(r,s,t_n^-)-k\right|\right]\phi(r,s,t_n)\,dr\,ds\\
&+\int_{\mathcal{R}^2}\left[|\bar{u}(r,s,0)-k|\phi(r,s,0)-\left|\bar{u}(r,s,t_N^+)-k\right|\phi(r,s,T)\right]\,dr\,ds\\
\geq\quad &-\frac{\Delta t}{\delta}\sum_{n=1}^{N}\int_{\mathcal{R}^2}\sigma(\bar{u}(r,s,t_n^+)-k)g(\bar{u}(r,s,t_n^+))\phi(r,s,t_n)\,dr\,ds.
\end{aligned}
$$

And $R_3(\theta)$ in the inequality (13) now becomes

$$
\begin{aligned}
R_3(\theta)\quad :=\quad &\frac{1}{\delta}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}^2}\int_0^T\int_{\mathcal{R}^2}\left[g(u)\omega_\theta(t-\tau)-g(\bar{u}(r,s,t_n^+))\omega_\theta(t-t_n)\right]\\
&\times\sigma(u-\bar{u}(r,s,t_n^+))\omega_\theta(x-r)\omega_\theta(y-s)\,dx\,dy\,dt\,dr\,ds\,d\tau.
\end{aligned}
$$

This term is estimated by the similar arguments in [3] as

$$
R_3(\theta)\leq M\frac{\Delta t}{\theta},
$$

and Theorem 2.2 follows by choosing $\theta=\sqrt{\Delta t}$.

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