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A system of conservation laws including a stiff relaxation term; the 2D case*

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Abstract.

We analyze a system of conservation laws in two space dimensions with stiff relaxation terms. A semi-implicit finite difference method approximating the system is studied and an error bound of order $\mathcal{O}(\sqrt{\Delta t})$ measured in L^1 is derived. This error bound is independent of the relaxation time $\delta > 0$. Furthermore, it is proved that the solutions of the system converge towards the solutions of the equilibrium model as the relaxation time δ tends to zero, and that the rate of convergence measured in L^1 is of order $\mathcal{O}(\delta^{1/3})$. Finally, we present some numerical illustrations.

AMS subject classification: 35L65, 65M15.

Key words: Hyperbolic conservation laws, two space dimensions, relaxation terms, non-equilibrium, error estimate, rate of convergence.

1 Introduction

In this paper, we study the following system of conservation laws in two space dimensions

$$
\begin{aligned}
u_t + \operatorname{div}(f(u)) &= -\frac{1}{\delta} r(u,v), \\
v_t + \operatorname{div}(g(v)) &= \frac{1}{\delta} r(u,v),
\end{aligned}
\tag{1.1}
$$

with initial data given by $u(x,y,0) = u^0(x,y)$ and $v(x,y,0) = v^0(x,y)$. Here, $f = (f_1, f_2)^T$, $g = (g_1, g_2)^T$ and $r(u,v)$ are given smooth functions, and $\delta > 0$ is referred to as the relaxation time. More assumptions on the model will be given in Section 2.

The *"reaction function"* $r(u,v)$ is assumed to have the property that for each u there is a unique $v = a(u)$ such that $r(u, a(u)) = 0$. We shall in particular be interested in the behavior of the solution of (1.1) as the relaxation time δ tends to zero. One can formally argue that a small value of δ will force the reaction term $r(u,v)$ to be small. Hence, in the limit we obtain the "equilibrium model"

$$
(w + a(w))_t + \operatorname{div}\left(f(w) + g\left(a(w)\right)\right) = 0,
\tag{1.2}
$$

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where w denotes the u-variable in the limit. The main purpose of this paper is to study this limit process rigorously.

The equilibrium model (1.2) represents a scalar, nonlinear, hyperbolic conservation law in two space dimensions. It is well known that such problems in general do not admit smooth solutions, even if the initial data is smooth. In fact, a natural function space for this problem is the set of functions with bounded total variation. Furthermore, additional entropy conditions are necessary in order to guarantee a unique solution in this space. Also, as a consequence of the lack of higher regularity of the solution, standard finite difference schemes will in general converge at a rate of at most $\mathcal{O}(\sqrt{\Delta t})$ in L^1, where Δt represents the mesh size(cf. e.g. Kuznetsov [7]).

Hyperbolic conservation laws including stiff relaxation terms have recently been studied by several authors, cf. Chen and Liu [2], Liu [8], Natalini [10], Pember [11, 12], Chen, Levermore and Liu [1], Jin and Xin [5] and Tveito and Winther [16] or the references given in these papers. An introduction to the models of this type is given in the book of Whitham [17], where models arising in e.g. chromatography, traffic modeling, water waves and gas dynamics are discussed.

The limit problem described above is studied by deriving proper δ-independent bounds for a finite difference scheme approximating the system (1.1). In particular, we will derive an $\mathcal{O}(\sqrt{\Delta t})$ estimate for the L^1-error uniformly in δ. Furthermore, we will verify that the limit, as Δt tends to zero, satisfies a proper "Kruzkov inequality"(or entropy inequality) associated the system (1.1).

The uniform error bound for the finite difference scheme will then be used to show that

$$\|u_\delta - w\|_1 = \mathcal{O}(\delta^{1/3}),$$

for initial data sufficiently close to equilibrium. Similar results have been derived for a one dimensional problem; convergence towards equilibrium is analyzed in [16] and an error estimate for the finite difference scheme is proved in [13].

The paper is organized as follows: Section 2 gives precise assumptions on the system (1.1) and the initial data. Furthermore, the main results of the paper is stated here. Section 3 analyses the solutions of the finite difference scheme approximating system (1.1). Then in Section 4, we use these results to prove the existence, uniqueness and the stability of entropy solutions of (1.1). In Section 5, we prove that the errors of the numerical solution generated by the finite difference scheme are bounded by $\mathcal{O}(\sqrt{\Delta t})$, and we take advantage of this results to prove the rate of convergence of the non-equilibrium model towards the equilibrium model in Section 6. And finally in Section 7, the theoretical results are illustrated by numerical examples.

2 Preliminaries and the main results

The precise statement of our results and the assumptions on model (1.1) will be given in this section. The vector-valued functions $f = f(u)$ and $g = g(v)$ are smooth and satisfy

$$f_1(0) = f_2(0) = g_1(0) = g_2(0) = 0.$$

The function $r = r(u,v)$ is also smooth, and in order to give a meaningful equilibrium, we assume that first-order partial derivatives r_u and r_v are bounded and satisfy

$$r_u(u,v) \geq \alpha > 0, \quad r_v(u,v) \leq -\alpha < 0 \quad \text{for all } (u,v) \in [0,1]^2. \tag{2.1}$$

We also assume that

$$r(0,0) = 0 \quad \text{and} \quad r(1,1) = 0. \tag{2.2}$$

These assumptions on $r(u,v)$ imply that the algebraic equation $r(u,v) = 0$ has a unique solution given by $v = a(u)$ where $a(\cdot)$ is a smooth function satisfying $a(0) = 0$, $a(1) = 1$ and $a'(u) \geq \gamma > 0$ for all u. In this way, r becomes a measure of the deviation from equilibrium in the non-equilibrium model. An alternative measure can be defined by the auxiliary variable $p(u,v)$ as

$$p(u,v) = a(u) - v.$$

As a consequence, we can write $r(u,v) = p(u,v)q(u,v)$ for some smooth and uniformly bounded function $q = q(u,v)$ that satisfies

$$0 < \alpha_1 \leq q(u,v) \leq \alpha_2 < \infty, \quad \forall (u,v) \in [0,1]^2,$$

for some positive constants α_1 and α_2. Therefore, measuring the deviation in r or p is equivalent in the sense that

$$M_1 \|p(u,v)\|_1 \leq \|r(u,v)\|_1 \leq M_2 \|p(u,v)\|_1, \tag{2.3}$$

where the constants M_1 and M_2 are independent of the relaxation parameter δ. Here the L^1-norm $\|\cdot\|_1$ for a function $w = w(x,y)$ is defined by

$$\|w\|_1 := \int_{\mathcal{R}} \int_{\mathcal{R}} |w(x,y)| \, dx \, dy.$$

We seek solutions of (1.1) in the state-space

$$\mathcal{S} = [0,1] \times [0,1]$$

and solutions of (1.2) in $[0,1]$.

The initial data (u^0, v^0) are supposed to satisfy the following requirements

$$\begin{aligned} &i) \quad (u^0(x,y), v^0(x,y)) \in \mathcal{S} \quad \forall (x,y) \in \mathcal{R}^2, \\ &ii) \quad TV(u^0) + TV(v^0) \leq M, \\ &iii) \quad \left\|p^0\right\|_1 \leq M\delta, \\ &iv) \quad u^0(\pm\infty, \pm\infty) = 0, v^0(\pm\infty, \pm\infty) = 0. \end{aligned} \tag{2.4}$$

Here, and in the rest of this paper, M denotes a generic finite constant independent of δ. The total variation $TV(\cdot)$ is defined as

$$\begin{aligned} TV(w) \quad := \quad & \sup_{h\neq 0}\int_{\mathcal{R}}\int_{\mathcal{R}} \frac{|w(x+h,y)-w(x,y)|}{|h|}\,dx\,dy \\ & +\sup_{k\neq 0}\int_{\mathcal{R}}\int_{\mathcal{R}} \frac{|w(x,y+k)-w(x,y)|}{|k|}\,dx\,dy. \end{aligned}$$

We notice that by (2.3) condition $iii)$ actually implies a similar property for r, i.e., $\left\|r^0\right\|_1 \leq M\delta$. We observe, in particular, that requirements (2.4) allow the initial functions u^0 and v^0 to be discontinuous.

Let $BV = BV(\mathcal{R}^2)$ denote the subspace of L^1_{loc} consisting of functions with bounded total variation. For any $T>0$, let $\mathcal{D}_+(T)$ be the set of all nonnegative $\mathcal{C}^\infty$-functions with compact support in $\mathcal{R}^2\times[0,T]$. Furthermore, let σ denote the standard sign function, we introduce the notation

$$\begin{aligned} F(u,k) &= (F_1(u,k),F_2(u,k))^T = \sigma(u-k)(f(u)-f(k)), \\ G(v,q) &= (G_1(v,q),G_2(v,q))^T = \sigma(v-q)(g(v)-g(q)). \end{aligned} \tag{2.5}$$

We note that if $f=(f_1,f_2)^T$ and $g=(g_1,g_2)^T$ are Lipschitz functions, so are F and G. Then an entropy solution of (1.1) is defined as follows:

Definition 2.1. *The pair (u,v) is called an entropy solution of (1.1) with initial data (u^0,v^0) satisfying (2.4) if the following requirements are satisfied:*

1. $(u,v)\in\mathcal{S}\quad \forall (x,y,t)\in\mathcal{R}\times\mathcal{R}\times\mathcal{R}_0^+$,
2. $TV(u(.,.,t)),\ TV(v(.,.,t))\leq M\quad \forall t\in\mathcal{R}_0^+$,
3. $\|u(.,.,t)-u(.,.,\tau)\|_1+\|v(.,.,t)-v(.,.,\tau)\|_1\leq M|t-\tau|\quad \forall t,\tau\in\mathcal{R}_0^+$,
4. *For any $(k,q)\in\mathcal{S}$ and any $\phi,\psi\in\mathcal{D}_+(T)$, where $T>0$ is arbitrary,*

$$\begin{aligned} &\int_0^T\int_{\mathcal{R}^2}[|u-k|\phi_t+F(u,k)\cdot grad(\phi) \\ &\qquad\qquad +|v-q|\psi_t+G(v,q)\cdot grad(\psi)]\,dx\,dy\,dt \\ &+\int_{\mathcal{R}^2}\left[\left|u^0-k\right|\phi(x,y,0)+\left|v^0-q\right|\psi(x,y,0)\right]\,dx\,dy \\ &-\int_{\mathcal{R}^2}[|u(x,y,T)-k|\phi(x,y,T)+|v(x,y,T)-q|\psi(x,y,T)]\,dx\,dy \\ \geq\ & \frac{1}{\delta}\int_0^T\int_{\mathcal{R}^2}[\sigma(u-k)\phi-\sigma(v-q)\psi]\,r(u,v)\,dx\,dy\,dt. \end{aligned} \tag{2.6}$$

In order to construct a family of approximate solutions for this system, we use the Lax-Friedrich's scheme combined with an implicit backward Euler method. In order to write the scheme on compact form, we introduce some notation. The discrete averaging operator $A(\cdot)$ on ω is defined as

$$A(\omega)_{i,j} := \frac{1}{4}\left(\omega_{i+1,j+1}+\omega_{i-1,j+1}+\omega_{i+1,j-1}+\omega_{i-1,j-1}\right),$$

and a discrete divergence operator $\mathrm{div}_\Delta(\cdot)$ on $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ is given by

$$\begin{aligned}\mathrm{div}_\Delta(\mathcal{F})_{i,j} \quad &:= \quad \frac{1}{4\Delta x}\left[(\mathcal{F}_1)_{i+1,j+1} + (\mathcal{F}_1)_{i+1,j-1} - (\mathcal{F}_1)_{i-1,j+1} - (\mathcal{F}_1)_{i-1,j-1}\right] \\ &\quad + \frac{1}{4\Delta y}\left[(\mathcal{F}_2)_{i+1,j+1} + (\mathcal{F}_2)_{i-1,j+1} - (\mathcal{F}_2)_{i+1,j-1} - (\mathcal{F}_2)_{i-1,j-1}\right].\end{aligned}$$

We also introduce a discrete gradient operator $\mathrm{grad}_\Delta(\cdot)$ on ω as

$$\mathrm{grad}_\Delta(\omega)_{i,j} := \begin{pmatrix} \dfrac{\omega_{i+1,j+1} + \omega_{i+1,j-1} - \omega_{i-1,j+1} - \omega_{i-1,j-1}}{4\Delta x} \\ \\ \dfrac{\omega_{i+1,j+1} + \omega_{i-1,j+1} - \omega_{i+1,j-1} - \omega_{i-1,j-1}}{4\Delta y} \end{pmatrix}. \tag{2.7}$$

Our difference scheme will be defined on a "staggered grid". For notational convenience we therefore define

$$W = \{(i,j,n) : i+n = 0(\mathrm{mod}\ 2), j+n = 0(\mathrm{mod}\ 2), i,j,n \in \mathcal{Z}, n \geq 0\}, \tag{2.8}$$

and

$$W^n = \{(i,j) : (i,j,n) \in W\}, \tag{2.9}$$

then our numerical scheme for the system (1.1) can be written on the following form where we suppress the subscript $_{i,j}$

$$u^{n+1} \quad = \quad A(u^n) - \Delta t \mathrm{div}_\Delta(f(u^n)) - \frac{\Delta t}{\delta} r(u^{n+1}, v^{n+1}), \tag{2.10}$$

$$v^{n+1} \quad = \quad A(v^n) - \Delta t \mathrm{div}_\Delta(g(v^n)) + \frac{\Delta t}{\delta} r(u^{n+1}, v^{n+1}), \tag{2.11}$$

for all $(i,j,n+1) \in W$. Here, and in the rest of this paper, the indices (i,j,n) of $u^n_{i,j}$ always satisfies $(i,j,n) \in W$, and $(u^n_{i,j},\ v^n_{i,j})$ denotes an approximation of $(u(x,y,t), v(x,y,t))$ over the grid block

$$B^n_{i,j} = [x_{i-1}, x_{i+1}) \times [y_{j-1}, y_{j+1}) \times [t_n, t_{n+1}).$$

In addition, $x_i = i\Delta x$, $y_j = j\Delta y$, $t_n = n\Delta t$, and Δx, Δy, Δt denote the step-lengths in the x, y and t directions, respectively. The computational molecule is given in Figure 2.1, where point '$\times$' at t_{n+1} is dependent on the points 'o' at t_n.

The scheme is initialized by

$$u^0_{i,j} \quad = \quad \frac{1}{4\Delta x \Delta y}\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}} u^0(x,y)\, dx\, dy, \tag{2.12}$$

$$v^0_{i,j} \quad = \quad \frac{1}{4\Delta x \Delta y}\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}} v^0(x,y)\, dx\, dy, \tag{2.13}$$

for all $(i,j) \in W^0$.

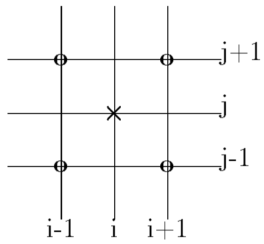

Figure 2.1: The computational molecule for Lax-Friedrich's scheme in 2D.

We assume that the discrete initial data satisfies

$$
\begin{aligned}
&i) \quad (u^0_{i,j}, v^0_{i,j}) \in \mathcal{S} \quad \forall (i,j) \in W^0, \\
&ii) \quad TV(u^0) + TV(v^0) \le M, \\
&iii) \quad \left\| p^0 \right\|_1 \le M\delta, \\
&iv) \quad u^0_{\pm\infty,\pm\infty} = v^0_{\pm\infty,\pm\infty} = 0,
\end{aligned}
\tag{2.14}
$$

and the grid parameters Δx, Δy and Δt satisfy $\Delta x/\Delta t = \mathcal{O}(1)$, $\Delta y/\Delta t = \mathcal{O}(1)$, and the CFL conditions

$$
\frac{\Delta t}{\Delta x}|f_1'(u)|, \frac{\Delta t}{\Delta y}|f_2'(u)|, \frac{\Delta t}{\Delta x}|g_1'(v)|, \frac{\Delta t}{\Delta y}|g_2'(v)| \le \frac{1}{2} \quad \text{for all } u, v \in \mathcal{S}. \tag{2.15}
$$

Here, the total variation of a grid function u^n in two space dimensions is defined by

$$
TV(u^n) = TV^x(u^n) + TV^y(u^n), \tag{2.16}
$$

where

$$
TV^x(u^n) = \Delta y \sum_{(i,j)\in W^n} \left| u^n_{i,j} - u^n_{i-2,j} \right| \text{ and } TV^y(u^n) = \Delta x \sum_{(i,j)\in W^n} \left| u^n_{i,j} - u^n_{i,j-2} \right|,
$$

and the discrete L^1-norm in two dimensions is defined by

$$
\|u^n\|_1 := 4\Delta x \Delta y \sum_{(i,j)\in W^n} \left| u^n_{i,j} \right|.
$$

In order to compare the numerical solutions with the solutions of system (1.1), we define a family of approximate solutions of piecewise constant functions by

$$
(u_\Delta, v_\Delta)(x,y,t) = (u^n_{i,j}, v^n_{i,j}) \quad \text{for } (x,y,t) \in B^n_{i,j}, \forall (i,j,z) \in W. \tag{2.17}
$$

Then, we have that

$$
TV(u_\Delta(\cdot,\cdot,t_n)) = TV(u^n) \quad \text{and} \quad \|u_\Delta(\cdot,\cdot,t_n)\|_1 = \|u^n\|_1, \tag{2.18}
$$

and the conditions in (2.14) hold also for $(u_\Delta, v_\Delta)(x,y,0)$.

Based on the properties of the numerical scheme (2.10-2.11), we will prove the existence of an entropy solution of the initial value problem. Moreover, we will show that the initial value problem is well-posed independently of δ.

THEOREM 2.1. *Let (u^0, v^0) and $(u^0_{i,j}, v^0_{i,j})$ be the initial data satisfying (2.4) and (2.14), respectively. Then, the family $\{(u_\Delta, v_\Delta)\}$ of approximate solutions generated by the finite difference scheme (2.10-2.11) converges in $(L^1_{loc}(\mathcal{R}^2 \times \mathcal{R}^+_0))^2$ towards a pair of functions (u, v) as Δx, Δy and Δt tend to zero. Furthermore, the limit (u, v) is a unique entropy solution satisfying the requirements of Definition 2.1 and the following bound*

$$\|p(.,.,t)\|_1 \leq M\delta \quad \text{for all } t \geq 0.$$

Moreover, if $(\bar{u}, \bar{v})$ is the entropy solution with initial data $(\bar{u}^0, \bar{v}^0)$ which satisfies (2.4), we have for $t \geq 0$,

$$\|u(.,.,t) - \bar{u}(.,.,t)\|_1 + \|v(.,.,t) - \bar{v}(.,.,t)\|_1 \leq \left\|u^0 - \bar{u}^0\right\|_1 + \left\|v^0 - \bar{v}^0\right\|_1.$$

The proof of this result is given in Section 4.

Our next result states that the error bound of the numerical scheme is of order $\sqrt{\Delta t}$ in L^1, independent of the relaxation time δ. This theorem will be proved in Section 5.

THEOREM 2.2. *Let (u, v) be the entropy solution of (1.1) with initial data (u^0, v^0), and let (u^N, v^N) be a piecewise constant representation of the data $\{u^N_{i,j}, v^N_{i,j}\}$ generated by (2.10-2.11). Then, for any fixed $T = N\Delta t > 0$, there is a finite constant M independent of Δt, Δx, Δy and δ such that*

$$\left\|u(\cdot,\cdot,T) - u^N\right\|_1 + \left\|v(\cdot,\cdot,T) - v^N\right\|_1 \leq M\sqrt{\Delta t}.$$

This error bound eventually leads to the following result.

THEOREM 2.3. *Let (u, v) and w be the solutions of (1.1) and (1.2) with initial data (u^0, v^0) satisfying (2.4) and $w^0 = u^0$, respectively. Then, for any $T > 0$, there is a finite constant M such that*

$$\|u(.,.,t) - w(.,.,t)\|_1 \leq M\delta^{1/3} \quad \text{for all } 0 \leq t \leq T.$$

This theorem generalizes the main result of [11] where a one-dimensional case is analyzed.

3 Bounds on the approximate solution

In this section, we will study the properties of the finite difference scheme (2.10-2.11) approximating the system (1.1). This finite difference approximation has the following properties:

LEMMA 3.1. *Let (u^n, v^n) and $(\bar{u}^n, \bar{v}^n)$ be the discrete solutions generated by (2.10-2.11) with initial data (u^0, v^0) and $(\bar{u}^0, \bar{v}^0)$ satisfying (2.14), and the grid* *parameters satisfying the CFL condition (2.15). Then, for any $\delta > 0$ and $t \geq 0$, there is a finite constant M independent of δ, Δt, Δx and Δy such that*

$$\begin{array}{rl}
I. & (u_\Delta, v_\Delta) \in \mathcal{S} \quad \textit{for all} \quad (x,y,t) \in \mathcal{R} \times \mathcal{R} \times \mathcal{R}^+ \\
II. & TV(u_\Delta(\cdot,\cdot,t)) + TV(v_\Delta(\cdot,\cdot,t)) \leq TV(u_\Delta(\cdot,\cdot,0)) + TV(v_\Delta(\cdot,\cdot,0)), \\
III. & \|p(u_\Delta(\cdot,\cdot,t), v_\Delta(\cdot,\cdot,t))\|_1 \leq M\delta, \\
IV. & \|u_\Delta(\cdot,\cdot,t_n) - u_\Delta(\cdot,\cdot,t_m)\|_1 + \|v_\Delta(\cdot,\cdot,t_n) - v_\Delta(\cdot,\cdot,t_m)\|_1 \\
& \qquad \leq M|n-m|\Delta t \\
V. & \|u_\Delta(\cdot,\cdot,t) - \bar{u}_\Delta(\cdot,\cdot,t)\|_1 + \|v_\Delta(\cdot,\cdot,t) - \bar{v}_\Delta(\cdot,\cdot,t)\|_1 \\
& \leq \|u_\Delta(\cdot,\cdot,0) - \bar{u}_\Delta(\cdot,\cdot,0)\|_1 + \|v_\Delta(\cdot,\cdot,0) - \bar{v}_\Delta(\cdot,\cdot,0)\|_1 .
\end{array}$$

We use the rest of this section to prove this lemma.

3.1 Proof of I; (The maximum principle)

We notice that by (2.17) and (2.18), it is sufficient to prove that

$$(u^n_{i,j}, v^n_{i,j}) \in \mathcal{S} \quad \text{for all} \quad (i,j,n) \in W.$$

For a fixed $(i,j,n) \in W$, we let $(u,v) = (u^{n+1}_{i,j}, v^{n+1}_{i,j})$ and (see Figure 3.1)

$$\begin{array}{ll}
(u^l, v^l) = (u^n_{i-1,j-1}, v^n_{i-1,j-1}), & (u^L, v^L) = (u^n_{i-1,j+1}, v^n_{i-1,j+1}), \\
(u^r, v^r) = (u^n_{i+1,j-1}, v^n_{i+1,j-1}), & (u^R, v^R) = (u^n_{i+1,j+1}, v^n_{i+1,j+1}).
\end{array}$$

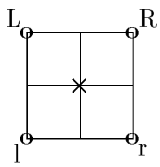

Figure 3.1: Illustration of the notation in the proof of the maximum principle.

By using these notations, we can rewrite the scheme (2.10-2.11) on the following form

$$\begin{aligned}
u &= \frac{1}{4}(u^l + u^L + u^r + u^R) - \frac{\Delta t}{4\Delta x}\left[f_1(u^r) + f_1(u^R) - f_1(u^l) - f_1(u^L)\right] \\
&\quad - \frac{\Delta t}{4\Delta y}\left[f_2(u^L) + f_2(u^R) - f_2(u^l) - f_2(u^r)\right] - \frac{\Delta t}{\delta} r(u,v) \qquad (3.1) \\
v &= \frac{1}{4}(v^l + v^L + v^r + v^R) - \frac{\Delta t}{4\Delta x}\left[g_1(v^r) + g_1(v^R) - g_1(v^l) - g_1(v^L)\right] \\
&\quad - \frac{\Delta t}{4\Delta y}\left[g_2(v^R) + g_2(v^L) - g_2(v^r) - g_2(v^l)\right] + \frac{\Delta t}{\delta} r(u,v) \qquad (3.2)
\end{aligned}$$

We first claim that (u,v) is uniquely determined by (3.1-3.2). In order to prove this, we rewrite the scheme (3.1-3.2) as

$$u+\frac{\Delta t}{\delta}r(u,v)=k_1, \quad v-\frac{\Delta t}{\delta}r(u,v)=k_2,$$

where k_1 and k_2 denote the sum of the first three terms at the right hand sides of (3.1) and (3.2), respectively. Then $u+v=k_1+k_2$, and the single equation

$$K(u)=u+\frac{\Delta t}{\delta}r(u,k_1+k_2-u)-k_1=0$$

has a unique solution since the function $K(u)$ is monotone and $K(\pm\infty)=\pm\infty$.

Next, we assume that $0\le u^l,\ u^L,\ u^r,\ u^R,\ v^l,\ v^L,\ v^r,\ v^R\le 1$, and we want to prove that this implies $0\le u,v\le 1$. We consider u and v as functions of u^l,u^L,u^r,u^R,v^l,v^L,v^r,v^R, i.e.,

$$u=u(u^l,u^L,u^r,u^R,v^l,v^L,v^r,v^R) \text{ and } v=v(u^l,u^L,u^r,u^R,v^l,v^L,v^r,v^R),$$

and then show that u and v are monotone in all arguments. This is almost straightforward. We only prove this property for u since the arguments for v are analogous. We first consider u^l. From (3.1-3.2), we get:

$$\frac{\partial u}{\partial u^l}=\frac{1-r_v(u,v)\Delta t/\delta}{1-r_v(u,v)\Delta t/\delta+r_u(u,v)\Delta t/\delta}\left[\frac{1}{4}+\frac{\Delta t}{4\Delta x}f_1'(u^l)+\frac{\Delta t}{4\Delta y}f_2'(u^l)\right].$$

Then, by the CFL condition (2.15) and the fact that $r_u>0$ and $r_v<0$, it follows that

$$0\le\frac{\partial u}{\partial u^l}<1/2. \tag{3.3}$$

Similarly, we also get

$$0\le\frac{\partial u}{\partial u^L},\frac{\partial u}{\partial u^r},\frac{\partial u}{\partial u^R}<1/2, \quad \text{and} \quad 0<\frac{\partial u}{\partial v^l},\frac{\partial u}{\partial v^L},\frac{\partial u}{\partial v^r},\frac{\partial u}{\partial v^R}<1/2. \tag{3.4}$$

Now, by the monotonicity properties (3.3) and (3.4), we have

$$u(0,0,0,0,0,0,0,0)\le u(u^l,u^L,u^r,u^R,v^l,v^L,v^r,v^R)\le u(1,1,1,1,1,1,1,1).$$

By the uniqueness of u and v, we verify that

$$u(0,0,0,0,0,0,0,0)=0 \quad \text{and} \quad u(1,1,1,1,1,1,1,1)=1,$$

and the maximum principle I of the Lemma follows by induction.

3.2 Proof of II; (the total variation estimate)

Recall that the total variation is defined in (2.16). It is sufficient to prove the result for the x- and y-sections separately. We first consider the x-section of the total variation. Introducing the notations

$$U_{i,j}^n=u_{i+1,j}^n-u_{i-1,j}^n \quad \text{and} \quad V_{i,j}^n=v_{i+1,j}^n-v_{i-1,j}^n,$$

for $(i+1,j) \in W^n$, it follows from (2.10-2.11) that

$$
\begin{aligned}
(3.5)\qquad U_{i,j}^{n+1} &= A(U^n)_{i,j} - \Delta t \operatorname{div}_\Delta (f'(\tilde{u}^n)U^n)_{i,j} \\
&\quad -\frac{\Delta t}{\delta} r_u(\breve{u}_{i,j}^{n+1}, v_{i+1,j}^{n+1})U_{i,j}^{n+1} - \frac{\Delta t}{\delta} r_v(u_{i,j}^{n+1}, \breve{v}_{i,j}^{n+1})V_{i,j}^{n+1} \\
(3.6)\qquad V_{i,j}^{n+1} &= A(V^n)_{i,j} - \Delta t \operatorname{div}_\Delta (g'(\tilde{v}^n)V^n)_{i,j} \\
&\quad +\frac{\Delta t}{\delta} r_u(\breve{u}_{i,j}^{n+1}, v_{i+1,j}^{n+1})U_{i,j}^{n+1} + \frac{\Delta t}{\delta} r_v(u_{i,j}^{n+1}, \breve{v}_{i,j}^{n+1})V_{i,j}^{n+1}
\end{aligned}
$$

where

$$
\begin{aligned}
(f'(\tilde{u}^n)U^n)_{i,j} &= f(u_{i+1,j}^n) - f(u_{i-1,j}^n) \\
(g'(\tilde{v}^n)V^n)_{i,j} &= g(v_{i+1,j}^n) - g(v_{i-1,j}^n) \\
r_u(\breve{u}_{i,j}^{n+1}, v_{i+1,j}^{n+1})U_{i,j}^{n+1} &= r(u_{i+1,j}^{n+1}, v_{i+1,j}^{n+1}) - r(u_{i-1,j}^{n+1}, v_{i+1,j}^{n+1}) \\
r_v(u_{i,j}^{n+1}, \breve{v}_{i,j}^{n+1})V_{i,j}^{n+1} &= r(u_{i,j}^{n+1}, v_{i+1,j}^{n+1}) - r(u_{i,j}^{n+1}, v_{i-1,j}^{n+1})
\end{aligned}
$$

Now we multiply (3.5) and (3.6) by $\sigma(U_{i,j}^{n+1})$ and $\sigma(V_{i,j}^{n+1})$ respectively, and we add the inequalities and apply the CFL condition. This leads to the estimate

$$TV^x(u^{n+1}) + TV^x(v^{n+1}) \le TV^x(u^n) + TV^x(v^n),$$

for the x-variation. Analogously for the y-section, we get

$$TV^y(u^{n+1}) + TV^y(v^{n+1}) \le TV^y(u^n) + TV^y(v^n),$$

and hence

$$TV(u^{n+1}) + TV(v^{n+1}) \le TV(u^n) + TV(v^n).$$

Now property II follows by induction.

3.3 Proof of III; deviation from the equilibrium

We want to prove that $\|p^n\|_1 \le M\delta$. Since $p(u,v) = a(u) - v$, we have

$$p^{n+1} - A(p^n) = a'(\tilde{u})(u^{n+1} - A(u^n)) - (v^{n+1} - A(v^n)) + (a(A(u^n)) - A(a(u^n))),$$

where $A(\cdot)$ is the averaging operator, and $\tilde{u}$ satisfies

$$a'(\tilde{u})(u^{n+1} - A(u^n)) = a(u^{n+1}) - a(A(u^n)).$$

Using the scheme (2.10-2.11), we get

$$
\begin{aligned}
p^{n+1} &= A(p^n) + a'(\tilde{u})\left[-\Delta t \operatorname{div}_\Delta(f(u^n)) - \frac{\Delta t}{\delta} r^{n+1}\right] \\
(3.7)\qquad &\quad - \left[-\Delta t \operatorname{div}_\Delta(g(v^n)) + \frac{\Delta t}{\delta} r^{n+1}\right] + (a(A(u^n)) - A(a(u^n))) \\
&= A(p^n) - \frac{\Delta t}{\delta}(1 + a'(\tilde{u}^{n+1/2}))r^{n+1} - a'(\tilde{u})\Delta t \operatorname{div}_\Delta(f(u^n)) \\
&\quad + \Delta t \operatorname{div}_\Delta(g(v^n)) + (a(A(u^n)) - A(a(u^n))).
\end{aligned}
$$

Since $a(u)$ is a smooth function, then by Taylor expansion and the CFL condition (2.15), we have

$$\|a(A(u^n)) - A(a(u^n))\|_1 \leq M\Delta t TV(u^n).$$

Then, multiplying both sides of (3.7) with $\Delta x \Delta y \sigma(p^{n+1})$, and using the fact that $\sigma(p) = \sigma(r)$, we get

$$\left\|p^{n+1}\right\|_1 \leq \|p^n\|_1 + M_u \Delta t TV(u^n) + M_v \Delta t TV(v^n) - \frac{\Delta t}{\delta}(1+\gamma)\left\|r^{n+1}\right\|_1,$$

after summing over $(i,j) \in W^{n+1}$. Here, $\gamma = min_{u\in[0,1]}(a'(u))$, and M_u and M_v are constants independent of Δx, Δy, Δt and δ. Since $TV(u^n)$ and $TV(v^n)$ are bounded, $\left\|r^{n+1}\right\|_1 \geq M_1 \left\|p^{n+1}\right\|_1$, and $\gamma > 0$, we then get

$$\left\|p^{n+1}\right\|_1 \leq \|p^n\|_1 + M\Delta t - \frac{\Delta t}{\delta}(1+\gamma)M_1\left\|p^{n+1}\right\|_1,$$

and property III follows, cf. [16] for the details.

3.4 Proof of IV; L^1-continuity in time

The finite difference scheme (2.10-2.11) and property II gives

$$\|u_\Delta(\cdot,\cdot,t_{n+1}) - u_\Delta(\cdot,\cdot,t_n)\|_1 \leq M_u \Delta t + \frac{\Delta t}{\delta}\left\|r^{n+1}\right\|_1, \tag{3.8}$$

and

$$\|v_\Delta(\cdot,\cdot,t_{n+1}) - v_\Delta(\cdot,\cdot,t_n)\|_1 \leq M_v \Delta t + \frac{\Delta t}{\delta}\left\|r^{n+1}\right\|_1, \tag{3.9}$$

where M_u and M_v are finite constants independent of mesh size and δ. By using the bound on the deviation from equilibrium, we can add (3.8) and (3.9), and get

$$\|u_\Delta(\cdot,\cdot,t_{n+1}) - u_\Delta(\cdot,\cdot,t_n)\|_1 + \|v_\Delta(\cdot,\cdot,t_{n+1}) - v_\Delta(\cdot,\cdot,t_n)\|_1 = \mathcal{O}(\Delta t),$$

and property IV follows by the triangle inequality.

3.5 Proof of V; stability in L^1

We want to show that

$$\|u^n - \bar{u}^n\|_1 + \|v^n - \bar{v}^n\|_1 \leq \left\|u^0 - \bar{u}^0\right\|_1 + \left\|v^0 - \bar{v}^0\right\|_1.$$

To this end, we let $\{u^n_{i,j}, v^n_{i,j}\}$ and $\{\bar{u}^n_{i,j}, \bar{v}^n_{i,j}\}$ be generated by the scheme (2.10-2.11) with initial data $\{u^0_{i,j}, v^0_{i,j}\}$ and $\{\bar{u}^0_{i,j}, \bar{v}^0_{i,j}\}$ satisfying (2.14). By defining

$$U^n_{i,j} = u^n_{i,j} - \bar{u}^n_{i,j} \quad \text{and} \quad V^n_{i,j} = v^n_{i,j} - \bar{v}^n_{i,j},$$

we get

(3.10)

$$U^{n+1}_{i,j} = A(U^n)_{i,j} - \Delta t \operatorname{div}_\Delta(f'(\tilde{u}^n)U^n)_{i,j} - \frac{\Delta t}{\delta}(\check{r}_u U)^{n+1}_{i,j} - \frac{\Delta t}{\delta}(\check{r}_v V)^{n+1}_{i,j},$$

and

(3.11)
$$V_{i,j}^{n+1} = A(V^n)_{i,j} - \Delta t \operatorname{div}_\Delta (g'(\tilde{v}^n)V^n)_{i,j} + \frac{\Delta t}{\delta}(\check{r}_u U)_{i,j}^{n+1} + \frac{\Delta t}{\delta}(\check{r}_v V)_{i,j}^{n+1},$$

where

$$\begin{aligned}
(f'(\tilde{u}^n)U^n)_{i,j} &= (f(u^n) - f(\bar{u}^n))_{i,j},\\
(g'(\tilde{v}^n)U^n)_{i,j} &= (g(u^n) - g(\bar{u}^n))_{i,j},\\
(\check{r}_u U)_{i,j}^{n+1} &= r(u_{i,j}^{n+1}, v_{i,j}^{n+1}) - r(\bar{u}_{i,j}^{n+1}, v_{i,j}^{n+1}),\\
(\check{r}_v V)_{i,j}^{n+1} &= r(\bar{u}_{i,j}^{n+1}, v_{i,j}^{n+1}) - r(\bar{u}_{i,j}^{n+1}, \bar{v}_{i,j}^{n+1}).
\end{aligned}$$

By multiplying (3.10) with $\sigma(U_{i,j}^{n+1})$ and (3.11) with $\sigma(V_{i,j}^{n+1})$, and summing the inequalities for $i,j \in W^{n+1}$, we get

$$\left\|U^{n+1}\right\|_1 + \left\|V^{n+1}\right\|_1 \le \|U^n\|_1 + \|V^n\|_1,$$

and property V follows by induction.

3.6 The discrete entropy inequalities

The numerical solutions satisfy a discrete version of the entropy inequality.

LEMMA 3.2. *Suppose that $(u_{i,j}^n, v_{i,j}^n)$ is generated by the scheme (2.10-2.11) with initial data $(u_{i,j}^0, v_{i,j}^0)$ that satisfies (2.14), and let $\phi, \psi \in \mathcal{D}_+(T)$ for some $T > 0$ and let $\phi_{i,j}^n = \phi(x_i, y_j, t_n)$ and $\psi_{i,j}^n = \psi(x_i, y_j, t_n)$. Furthermore, let $\mathcal{E} : [0,1] \to \mathcal{R}$ be a convex $\mathcal{C}^\infty$ entropy function with the associated entropy fluxes $\mathcal{F}$ and $\mathcal{G}$, i.e., $\mathcal{F} : [0,1] \to \mathcal{R}^2$ satisfies $\mathcal{F}' = \mathcal{E}'f'$ and $\mathcal{G} : [0,1] \to \mathcal{R}^2$ satisfies $\mathcal{G}' = \mathcal{E}'g'$. Without loss of generality, we assume that $N = 0(mod\ 2)$, which implies $W^0 = W^N$. Then the following inequalities hold:*

(3.12)
$$\begin{aligned}
&\Delta t \sum_{n=0}^{N-1} \Delta x \Delta y \sum_{(i,j)\in W^{n+1}} \frac{A(\phi^{n+1})_{i,j} - \phi_{i,j}^n}{\Delta t} \mathcal{E}(u_{i,j}^{n+1})\\
&+\Delta t \sum_{n=0}^{N-1} \Delta x \Delta y \sum_{(i,j)\in W^n} grad_\Delta(\phi^n)_{i,j} \cdot \mathcal{F}(u_{i,j}^n)\\
&+\Delta x \Delta y \sum_{(i,j)\in W^0} \left[\mathcal{E}(u_{i,j}^0)\phi_{i,j}^0 - \mathcal{E}(u_{i,j}^N)\phi_{i,j}^N\right]\\
\ge\ &\frac{\Delta t}{\delta} \sum_{n=0}^{N-1} \Delta x \Delta y \sum_{(i,j)\in W^{n+1}} \mathcal{E}'(u_{i,j}^{n+1})\phi_{i,j}^n r(u_{i,j}^{n+1}, v_{i,j}^{n+1})
\end{aligned}$$

and

(3.13)
$$\Delta t \sum_{n=0}^{N-1} \Delta x \Delta y \sum_{(i,j)\in W^{n+1}} \frac{A(\psi^{n+1})_{i,j} - \psi_{i,j}^n}{\Delta t} \mathcal{E}(v_{i,j}^{n+1})$$

$$
\begin{aligned}
&+\Delta t \sum_{n=0}^{N-1} \Delta x \Delta y \sum_{(i,j)\in W^n} grad_\Delta(\psi^n)_{i,j} \cdot \mathcal{G}(v_{i,j}^n)\\
&+\Delta x \Delta y \sum_{(i,j)\in W^0} \left[\mathcal{E}(v_{i,j}^0)\psi_{i,j}^0 - \mathcal{E}(v_{i,j}^N)\psi_{i,j}^N\right]\\
\geq\ & -\frac{\Delta t}{\delta} \sum_{n=0}^{N-1} \Delta x \Delta y \sum_{(i,j)\in W^{n+1}} \mathcal{E}'(v_{i,j}^{n+1})\psi_{i,j}^n r(u_{i,j}^{n+1}, v_{i,j}^{n+1}),
\end{aligned}
$$

where $grad_\Delta(\cdot)$ is defined in (2.7) and W^n is defined in (2.9).

PROOF. We only prove the inequality (3.12) since (3.13) can be obtained by a similar argument. First, we observe that (2.10) can be written as

$$
u^{n+1} - k = A(u^n - k) - \Delta t \operatorname{div}_\Delta(f(u^n) - f(k)) - \frac{\Delta t}{\delta} r(u^{n+1}, v^{n+1}), \tag{3.14}
$$

for any given constant $k \in [0,1]$. Next, we note that the CFL-condition (2.15) implies that

$$
\left|\frac{1}{8}(u-k) \pm \frac{\Delta t}{4\Delta x}(f_1(u) - f_1(k))\right| = \frac{1}{8}|u-k| \pm \frac{\Delta t}{4\Delta x} F_1(u,k),
$$

for any $u \in [0,1]$, because

$$
\begin{aligned}
&\left|\frac{1}{2}(u-k) \pm \frac{\Delta t}{\Delta x}(f_1(u) - f_1(k))\right| \ = \ \left|\left(\frac{1}{2} \pm \frac{\Delta t}{\Delta x} f_1'(w)\right)(u-k)\right|\\
=\ & \left(\frac{1}{2} \pm \frac{\Delta t}{\Delta x} f_1'(w)\right)|u-k| \ = \ \frac{1}{2}|u-k| \pm \frac{\Delta t}{\Delta x} f_1'(w)\sigma(u-k)(u-k)\\
=\ & \frac{1}{2}|u-k| \pm \frac{\Delta t}{\Delta x}\sigma(u-k)(f_1(u) - f_1(k)) \ = \ \frac{1}{2}|u-k| \pm \frac{\Delta t}{\Delta x} F_1(u,k),
\end{aligned}
$$

where w satisfies $f_1'(w)(u-k) = f_1(u) - f_1(k)$. We recall that $F = (F_1, F_2)^T$ is defined in (2.5). Similar property holds for f_2. Multiplying (3.14) by $\sigma(u_{i,j}^{n+1} - k)$, we get

$$
\begin{aligned}
\left|u^{n+1} - k\right| \ \leq\ & A(|u^n - k|) - \Delta t \operatorname{div}_\Delta(F(u^n, k))\\
& -\frac{\Delta t}{\delta}\sigma(u_{i,j}^{n+1} - k) r(u^{n+1}, v^{n+1}).
\end{aligned}
$$

Multiplying this inequality by $\phi_{i,j}^n$ and summing over $0 \leq n \leq N-1$, and $(i,j) \in W^{n+1}$, we obtain the following inequality

$$
\begin{aligned}
&\Delta t \sum_{n=0}^{N-1} \Delta x \Delta y \sum_{(i,j)\in W^{n+1}} \frac{A(\phi^{n+1})_{i,j} - \phi_{i,j}^n}{\Delta t} \left|u_{i,j}^{n+1} - k\right|\\
&+\Delta t \sum_{n=0}^{N-1} \Delta x \Delta y \sum_{(i,j)\in W^n} \operatorname{grad}_\Delta(\phi^n)_{i,j} \cdot F(u_{i,j}^n, k)
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
&+\Delta x \Delta y \sum_{(i,j)\in W^0} \left[\left|u_{i,j}^0 - k\right| \phi_{i,j}^0 - \left|u_{i,j}^N - k\right| \phi_{i,j}^N \right] \\
\geq\ & \frac{\Delta t}{\delta} \sum_{n=0}^{N-1} \Delta x \Delta y \sum_{(i,j)\in W^{n+1}} \sigma(u_{i,j}^{n+1} - k)\phi_{i,j}^n r(u_{i,j}^{n+1}, v_{i,j}^{n+1}).
\end{aligned}
$$

Now we define a convex piecewise linear function $\mathcal{E}_m : I \to \mathcal{R}$ and a corresponding flux $\mathcal{F}_m$ by

$$
\begin{aligned}
\mathcal{E}_m(u) &= \beta_0(u - k_0) + \sum_{i=1}^{m} \beta_i |u - k_i|, \\
\mathcal{F}_m(u) &= \beta_0(f(u) - f(k_0)) + \sum_{i=1}^{m} \beta_i \sigma(u - k)(f(u) - f(k_i)),
\end{aligned}
$$

where $\beta_i \geq 0$ for $i = 1, 2, \cdots, m$. Then $\mathcal{F}_m'(u) = \mathcal{E}_m'(u) f'(u)$, and (3.12) holds for $\mathcal{E}_m$ and $\mathcal{F}_m$ due to the equality (3.14) and the inequality (3.15). Furthermore, for any smooth convex function $\mathcal{E}$, there exists a sequence of $\mathcal{E}_m$ such that $\mathcal{E}_m$(resp. $\mathcal{E}_m'$) converges uniformly to $\mathcal{E}$(resp. $\mathcal{E}'$) as $m \to \infty$. The inequality (3.12) thus follows. ∎

4 Proof of Theorem 2.1

The purpose of this section is to complete the proof of Theorem 2.1. First, the following lemma shows the existence of entropy solutions for (1.1).

LEMMA 4.1. *Let $\{(u_\Delta, v_\Delta)\}$ be the sequence of the numerical solutions generated by the scheme (2.10-2.11). Then, the limit solution of the sequence $\{(u_\Delta, v_\Delta)\}$ as $\Delta \to 0$ is an entropy solution of (1.1).*

PROOF. We recall that the definition of entropy solutions is given in Definition 2.1. The existence of entropy solutions for (1.1) is established by a limit argument. First, we note that properties I–III of the finite difference scheme (2.10-2.11) imply that there is a subsequence of $\{(u_\Delta, v_\Delta)\}$ which converges in $(L_{loc}^1(\mathcal{R}^2 \times \mathcal{R}_0^+))^2$ to a pair of functions (u, v) as $\Delta \to 0$, and the limit functions (u, v) satisfy the requirements 1-3 in Definition 2.1. This can be proved by a standard compactness argument, see for example Conway and Smoller [3]. Furthermore, we need to prove that (u, v) satisfies the variational inequality (2.6) for any $\phi, \psi \in \mathcal{D}_+(T)$ and $(k, q) \in \mathcal{S}$. Let $\Delta x, \Delta y, \Delta t \to 0$ in (3.12), we get

$$
\begin{aligned}
&\int_0^T \int_{\mathcal{R}^2} [\mathcal{E}(u)\phi_t + \mathcal{F}(u) \cdot \mathrm{grad}(\phi)]\, dx\, dy\, dt \\
&+ \int_{\mathcal{R}^2} \left[\mathcal{E}(u^0)\phi(x, y, 0) - \mathcal{E}(u^N)\phi(x, y, T)\right] dx\, dy\, dt \\
\geq\ & \frac{1}{\delta} \int_0^T \int_{\mathcal{R}^2} \mathcal{E}'(u)\phi(x, y, t) r(u, v)\, dx\, dy\, dt.
\end{aligned}
$$

Let $\mathcal{E}_\theta$, $\mathcal{F}_\theta$ and $\mathcal{G}_\theta$ be sequences of functions such that

$$
\mathcal{E}_\theta(u) \to |u - k| \quad , \quad \mathcal{E}_\theta'(u) \to \sigma(u - k)
$$

$$\mathcal{F}_\theta(u) \to F(u,k) \quad , \quad \mathcal{G}_\theta(v) \to G(v,q),$$

pointwise as $\theta \to 0$. Then by the dominated convergence theorem, we get

$$\begin{aligned} &\int_0^T \int_{\mathcal{R}^2} [|u-k|\phi_t + F(u,k)\cdot \mathrm{grad}(\phi)]\, dx\, dy\, dt \\ &\quad + \int_{\mathcal{R}^2} \left[\left|u^0 - k\right| \phi(x,y,0) - |u(x,y,T) - k| \phi(x,y,T) \right] \, dx\, dy \\ &\geq \frac{1}{\delta} \int_0^T \int_{\mathcal{R}^2} \sigma(u-k)\phi(x,y,t) r(u,v)\, dx\, dy\, dt. \end{aligned} \tag{4.1}$$

By using the same argument on (3.13), we get

$$\begin{aligned} &\int_0^T \int_{\mathcal{R}^2} [|v-q|\psi_t + G(v,q)\cdot \mathrm{grad}(\psi)]\, dx\, dy\, dt \\ &\quad + \int_{\mathcal{R}^2} \left[\left|v^0 - q\right| \psi(x,y,0) - |v(x,y,T) - q| \psi(x,y,T) \right] \, dx\, dy \\ &\geq -\frac{1}{\delta} \int_0^T \int_{\mathcal{R}^2} \sigma(v-q)\psi(x,y,t) r(u,v)\, dx\, dy\, dt. \end{aligned} \tag{4.2}$$

Adding (4.1) and (4.2) we get (2.6). And the limit solution is the entropy solution of (1.1). $\square$

Furthermore, the entropy solutions are unique, with bounded deviation from equilibrium, and depend continuously with respect to the initial data in L^1.

LEMMA 4.2. *Let u, v and $\bar{u}, \bar{v}$ be two entropy solutions of (1.1), with initial data (u^0, v^0) and $(\bar{u}^0, \bar{v}^0)$, respectively, satisfying the requirements (2.14). Then for all $t \geq 0$, we have*

$$\|p(\cdot,\cdot,t)\|_1 \leq M\delta,$$

and

$$\|u(.,.,t) - \bar{u}(.,.,t)\|_1 + \|v(.,.,t) - \bar{v}(.,.,t)\|_1 \leq \left\|u^0 - \bar{u}^0\right\|_1 + \left\|v^0 - \bar{v}^0\right\|_1.$$

A rigorous proof of the one-dimensional case is given in [16]. We therefore omit the details here. The proof of Theorem 2.1 is complete.

5 Proof of Theorem 2.2

The purpose of this section is to derive the error estimate given in Theorem 2.2. First, we define the computation cell as

$$I_{i,j} = [x_{i-1}, x_{i+1}) \times [y_{j-1}, y_{j+1}),$$

and recall that the initial data for (2.10-2.11) are defined by (2.12-2.13). We will prove that the error bound for the above numerical scheme is of order $\sqrt{\Delta t}$ in L^1, independent of the relaxation time δ.

5.1 The comparison functions

We introduce the averaging operator P which computes a piecewise constant approximation of a L^1_{loc} function u by

$$P(u)(x,y) := \frac{1}{4\Delta x \Delta y} \int_{I_{i,j}} u(\zeta,\eta)\, d\zeta\, d\eta \quad \text{for } (x,y) \in I_{i,j}.$$

We note that P is the L^2-projection into the space of piecewise constant functions.

The crux of the proof is a dimension splitting method that is inspired by the work of Crandall and Majda [4](cf. also Teng [15]). In the context below we use (ζ, η, τ) instead of (x, y, t). The comparison functions $(\bar{u}, \bar{v})$ are defined iteratively. We introduce two pairs of new variables $(\tilde{u}, \tilde{v})$ and $(\hat{u}, \hat{v})$, and let $(\bar{u}, \bar{v})$ be the average of $(\tilde{u}, \tilde{v})$ and $(\hat{u}, \hat{v})$ in the sense

$$\bar{u} = \frac{1}{2}(\tilde{u} + \hat{u}) \quad \text{and} \quad \bar{v} = \frac{1}{2}(\tilde{v} + \hat{v}).$$

We initialize the iteration by

$$\begin{aligned} \bar{u}(\zeta,\eta,0) = \hat{u}(\zeta,\eta,0) = \tilde{u}(\zeta,\eta,0) = P(u^0)(\zeta,\eta) \\ \bar{v}(\zeta,\eta,0) = \hat{v}(\zeta,\eta,0) = \tilde{v}(\zeta,\eta,0) = P(v^0)(\zeta,\eta). \end{aligned}$$

Then, for each $n \in \mathcal{Z}_0^+$, we iterate the following three steps:

1). In (t_n, t_{n+1}), $(\tilde{u}, \tilde{v})$ is the entropy solution of

$$\begin{aligned} \tilde{u}_\tau + 2f_1(\tilde{u})_\zeta &= 0 \\ \tilde{v}_\tau + 2g_1(\tilde{v})_\zeta &= 0, \end{aligned} \tag{5.1}$$

and $(\hat{u}, \hat{v})$ is the entropy solution of

$$\begin{aligned} \hat{u}_\tau + 2f_2(\hat{u})_\eta &= 0 \\ \hat{v}_\tau + 2g_2(\hat{v})_\eta &= 0, \end{aligned} \tag{5.2}$$

with initial data

$$(\tilde{u}, \tilde{v})(\zeta, \eta, t_n^+) = (\hat{u}, \hat{v})(\zeta, \eta, t_n^+).$$

2). At t_{n+1}, we take cell averages as

$$\begin{aligned} \tilde{u}(\zeta,\eta,t_{n+1}) &= P(\tilde{u}(.,.,t_{n+1}^-))(\zeta,\eta) \\ \tilde{v}(\zeta,\eta,t_{n+1}) &= P(\tilde{v}(.,.,t_{n+1}^-))(\zeta,\eta) \\ \hat{u}(\zeta,\eta,t_{n+1}) &= P(\hat{u}(.,.,t_{n+1}^-))(\zeta,\eta) \\ \hat{v}(\zeta,\eta,t_{n+1}) &= P(\hat{v}(.,.,t_{n+1}^-))(\zeta,\eta). \end{aligned} \tag{5.3}$$

3). The initial data for the next iteration are generated by the backward Euler step:

$$\begin{aligned}(5.4)\qquad \bar u(\zeta,\eta,t_{n+1}^+) &= \bar u(\zeta,\eta,t_{n+1}) - \frac{\Delta t}{\delta} r(\bar u(\zeta,\eta,t_{n+1}^+), \bar v(\zeta,\eta,t_{n+1}^+))\\ \bar v(\zeta,\eta,t_{n+1}^+) &= \bar v(\zeta,\eta,t_{n+1}) + \frac{\Delta t}{\delta} r(\bar u(\zeta,\eta,t_{n+1}^+), \bar v(\zeta,\eta,t_{n+1}^+)),\end{aligned}$$

where $(\bar u(\zeta,\eta,t_{n+1}), \bar v(\zeta,\eta,t_{n+1}))$ is obtained by the relation $(\bar u,\bar v) = ((\tilde u + \tilde v)/2, (\hat u + \hat v)/2)$. Then, we simply set

$$\tilde u(\zeta,\eta,t_{n+1}^+) = \hat u(\zeta,\eta,t_{n+1}^+) = \bar u(\zeta,\eta,t_{n+1}^+)$$

and

$$\tilde v(\zeta,\eta,t_{n+1}^+) = \hat v(\zeta,\eta,t_{n+1}^+) = \bar v(\zeta,\eta,t_{n+1}^+).$$

The integral form of (5.1) over the cell $I_{i,j}\times(t_n,t_{n+1})$ gives the Lax-Friedrich's scheme for $(\tilde u,\tilde v)$:

$$\begin{aligned}\tilde u_{i,j}^{n+1} &= A(\tilde u^n)_{i,j} - 2\Delta t D_x(f_1(\tilde u))_{i,j}^n\\ \tilde v_{i,j}^{n+1} &= A(\tilde v^n)_{i,j} - 2\Delta t D_x(g_1(\tilde v))_{i,j}^n,\end{aligned}$$

and the integral form of (5.2) gives

$$\begin{aligned}\hat u_{i,j}^{n+1} &= A(\hat u^n)_{i,j} - 2\Delta t D_y(f_2(\hat u))_{i,j}^n\\ \hat v_{i,j}^{n+1} &= A(\hat v^n)_{i,j} - 2\Delta t D_y(g_2(\hat v))_{i,j}^n,\end{aligned}$$

where $D_x(.)_{i,j}^n$ and $D_y(.)_{i,j}^n$ are defined as

$$\begin{aligned}D_x(f)_{i,j}^n &= \frac{1}{4\Delta x}\left[f_{i+1,j+1}^n + f_{i+1,j-1}^n - f_{i-1,j+1}^n - f_{i-1,j-1}^n\right]\\ D_y(f)_{i,j}^n &= \frac{1}{4\Delta y}\left[f_{i+1,j+1}^n + f_{i-1,j+1}^n - f_{i+1,j-1}^n - f_{i-1,j-1}^n\right].\end{aligned}$$

It is easy to see that $\bar u_{i,j}^{n+1} = (\tilde u_{i,j}^{n+1} + \hat u_{i,j}^{n+1})/2$ and $\bar v_{i,j}^{n+1} = (\tilde v_{i,j}^{n+1} + \hat v_{i,j}^{n+1})/2$ gives the scheme for the system (2.10-2.11). And by induction, we get the following interpolating property

$$(\bar u,\bar v)(\zeta,\eta,t_n^+) = \lim_{\substack{\tau\to t_n\\ \tau>t_n}} (\bar u,\bar v)(\zeta,\eta,\tau) = (u_{i,j}^n, v_{i,j}^n),$$

for $(\zeta,\eta)\in I_{i,j}$ and $(i,j,n)\in W$.

5.2 The entropy inequality for the comparison functions

We now derive the entropy inequality for the comparison function. From step i), $(\tilde u,\tilde v)$ and $(\hat u,\hat v)$ are the entropy solutions of (5.1) and (5.2) in the time interval (t_n,t_{n+1}), thus we can apply the usual Kruzkov entropy formulation, cf [6]. Adding these two inequalities, and summing from $n=0$ to $N-1$, we get

$$
\begin{aligned}
(5.5)\quad & \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}^2} \Big[\frac{|\tilde{u}-k| + |\hat{u}-k|}{2} \phi_\tau + F_1(\tilde{u},k)\phi_\zeta + F_2(\hat{u},k)\phi_\eta \\
& + \frac{|\tilde{v}-q| + |\hat{v}-q|}{2} \psi_\tau + G_1(\tilde{v},q)\psi_\zeta + G_2(\hat{v},q)\psi_\eta \Big] \, d\zeta \, d\eta \, d\tau \\
& + \sum_{n=1}^{N} \int_{\mathcal{R}^2} \big[\big|\bar{u}(\zeta,\eta,t_n^+) - k\big| - \big|\bar{u}(\zeta,\eta,t_n^-) - k\big| \big] \, \phi(\zeta,\eta,t_n) \, d\zeta \, d\eta \\
& + \sum_{n=1}^{N} \int_{\mathcal{R}^2} \big[\big|\bar{v}(\zeta,\eta,t_n^+) - q\big| - \big|\bar{v}(\zeta,\eta,t_n^-) - q\big| \big] \, \psi(\zeta,\eta,t_n) \, d\zeta \, d\eta \\
& + \int_{\mathcal{R}^2} \big[\big|\bar{u}^0(\zeta,\eta) - k\big| \phi(\zeta,\eta,0) + \big|\bar{v}^0(\zeta,\eta) - q\big| \psi(\zeta,\eta,0) \big] \, d\zeta \, d\eta \\
& - \int_{\mathcal{R}^2} \big[\big|\bar{u}(\zeta,\eta,t_N^+) - k\big| \phi(\zeta,\eta,T) + \big|\bar{v}(\zeta,\eta,t_N^+) - q\big| \psi(\zeta,\eta,T) \big] \, d\zeta \, d\eta \geq 0.
\end{aligned}
$$

In addition, the Euler step in (5.4) gives

$$
\begin{aligned}
\big|\bar{u}(\zeta,\eta,t_n^+) - k\big| &\leq |\bar{u}(\zeta,\eta,t_n) - k| - \frac{\Delta t}{\delta} \sigma(\bar{u}(\zeta,\eta,t_n^+) - k) r(\bar{u}(\zeta,\eta,t_n^+), \bar{v}(\zeta,\eta,t_n^+)) \\
\big|\bar{v}(\zeta,\eta,t_n^+) - q\big| &\leq |\bar{v}(\zeta,\eta,t_n) - q| + \frac{\Delta t}{\delta} \sigma(\bar{v}(\zeta,\eta,t_n^+) - q) r(\bar{u}(\zeta,\eta,t_n^+), \bar{v}(\zeta,\eta,t_n^+))
\end{aligned}
$$

for any $(k,q) \in \mathcal{S}$. Setting these inequalities into (5.5), we get

$$
\begin{aligned}
(5.6)\quad & \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}^2} \Big[\frac{|\tilde{u}-k| + |\hat{u}-k|}{2} \phi_\tau + F_1(\tilde{u},k)\phi_\zeta + F_2(\hat{u},k)\phi_\eta \\
& + \frac{|\tilde{v}-q| + |\hat{v}-q|}{2} \psi_\tau + G_1(\tilde{v},q)\psi_\zeta + G_2(\hat{v},q)\psi_\eta \Big] \, d\zeta \, d\eta \, d\tau \\
& + \sum_{n=1}^{N} \int_{\mathcal{R}^2} \big[|\bar{u}(\zeta,\eta,t_n) - k| - \big|\bar{u}(\zeta,\eta,t_n^-) - k\big| \big] \, \phi(\zeta,\eta,t_n) \, d\zeta \, d\eta \\
& + \sum_{n=1}^{N} \int_{\mathcal{R}^2} \big[|\bar{v}(\zeta,\eta,t_n) - q| - \big|\bar{v}(\zeta,\eta,t_n^-) - q\big| \big] \, \psi(\zeta,\eta,t_n) \, d\zeta \, d\eta \\
& + \int_{\mathcal{R}^2} \big[\big|\bar{u}^0 - k\big| \phi(\zeta,\eta,0) + \big|\bar{v}^0 - q\big| \psi(\zeta,\eta,0) \big] \, d\zeta \, d\eta \\
& - \int_{\mathcal{R}^2} \big[\big|\bar{u}(\zeta,\eta,t_N^+) - k\big| \phi(\zeta,\eta,T) + \big|\bar{v}(\zeta,\eta,t_N^+) - q\big| \psi(\zeta,\eta,T) \big] \, d\zeta \, d\eta \\
& \geq \frac{\Delta t}{\delta} \sum_{n=1}^{N} \int_{\mathcal{R}^2} \big[\sigma(\bar{u}(\zeta,\eta,t_n^+) - k) \phi(\zeta,\eta,t_n) - \sigma(\bar{v}(\zeta,\eta,t_n^+) - q) \psi(\zeta,\eta,t_n) \big] \\
& \qquad\qquad\qquad\qquad \times r(\bar{u}(\zeta,\eta,t_n^+), \bar{v}(\zeta,\eta,t_n^+)) \, d\zeta \, d\eta
\end{aligned}
$$

for all $(k,q) \in \mathcal{S}$ and any $\phi, \psi \in \mathcal{D}_+(T)$.

5.3 Proof of the error bound

We use the arguments inspired by Kruzkov [6] and Kuznetsov [7] to prove the error bound. For any $\theta \in (0,1]$, we introduce the mollifier function ω_θ on $\mathcal{R}$ as

$$\omega_\theta(x) = \frac{1}{\theta}\Omega(x/\theta)$$

where $\Omega : \mathcal{R} \to \mathcal{R}$ is a nonnegative, symmetric $\mathcal{C}^\infty$-function with support in $[-1,1]$ and satisfying

$$\int_{\mathcal{R}} \Omega(x)\, dx = 1.$$

In the inequality (5.6), we set $(k,q) = (u,v)(x,y,t)$ and

$$\phi(\zeta,\eta,\tau) = \psi(\zeta,\eta,\tau) = \omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t-\tau).$$

We integrate in x, y and t:

$$\begin{aligned}
(5.7)\quad & -\int_0^T \int_{\mathcal{R}^2} \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}^2} \Big\{ \Big[\frac{|\tilde u - u| + |\hat u - u|}{2} + \frac{|\tilde v - v| + |\hat v - v|}{2} \Big] \\
&\qquad \times \omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega'_\theta(t-\tau) \\
&\quad + [F_1(\tilde u, u) + G_1(\tilde v, v)]\, \omega'_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t-\tau) \\
&\quad + [F_2(\hat u, u) + G_2(\hat v, v)]\, \omega_\theta(x-\zeta)\omega'_\theta(y-\eta)\omega_\theta(t-\tau) \Big\} \\
&\qquad d\zeta\, d\eta\, d\tau\, dx\, dy\, dt \\
&\quad + \int_0^T \int_{\mathcal{R}^2} \sum_{n=1}^N \int_{\mathcal{R}^2} \big[|\bar u(\zeta,\eta,t_n) - u| - \big|\bar u(\zeta,\eta,t_n^-) - u\big| + |\bar v(\zeta,\eta,t_n) - v| \\
&\qquad - \big|\bar v(\zeta,\eta,t_n^-) - v\big| \big]\, \omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t-t_n)\, d\zeta\, d\eta\, dx\, dy\, dt \\
&\quad + \int_0^T \int_{\mathcal{R}^2} \int_{\mathcal{R}^2} [|\bar u_0 - u| + |\bar v - v|]\, \omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t)\, d\zeta\, d\eta\, dx\, dy\, dt \\
&\quad - \int_0^T \int_{\mathcal{R}^2} \int_{\mathcal{R}^2} \big[\big|\bar u(\zeta,\eta,t_N^+) - u\big| + \big|\bar v(\zeta,\eta,t_N^+) - v\big| \big] \\
&\qquad \times \omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t-T)\, d\zeta\, d\eta\, dx\, dy\, dt \\
\geq\ & \frac{1}{\delta} \int_0^T \int_{\mathcal{R}^2} \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}^2} \big[\sigma(\bar u(\zeta,\eta,t_n^+) - k) - \sigma(\bar v(\zeta,\eta,t_n^+) - q) \big]\, \bar r(\zeta,\eta,t_n^+) \\
&\qquad \times \omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t-t_n)\, d\zeta\, d\eta\, d\tau\, dx\, dy\, dt,
\end{aligned}$$

where

$$u = u(x,y,t), \quad v = v(x,y,t), \quad \bar u = \bar u(\zeta,\eta,\tau), \quad \bar v = \bar v(\zeta,\eta,\tau),$$

and

$$\bar r(\zeta,\eta,\tau) = r(\bar u(\zeta,\eta,\tau), \bar v(\zeta,\eta,\tau)).$$

Later we will use

$$r(x,y,t) = r(u(x,y,t), v(x,y,t)).$$

In the entropy inequality (2.6) for the system (1.1), we choose

$$(k,q) = (\bar{u},\bar{v})(\zeta,\eta,t_n^+)$$

and

$$\phi(x,y,t) = \psi(x,y,t) = \omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t-\tau).$$

Integrating (2.6) over $\mathcal{R}\times\mathcal{R}\times[t_n,t_{n+1}]$ with respect to ζ,η,τ, and sum in n, and add this inequality to the inequality (5.7), we obtain

$$L(\theta) \le R_1(\theta) + R_2(\theta) + R_3(\theta) + R_4(\theta), \tag{5.8}$$

where

$$\begin{aligned}
L(\theta) \quad = \quad & \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}^2}\int_{\mathcal{R}^2}\left[\left|u(x,y,T)-\bar{u}(\zeta,\eta,t_n^+)\right| + \left|v(x,y,T)-\bar{v}(\zeta,\eta,t_n^+)\right|\right] \\
& \qquad\qquad \times\omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(T-\tau)\,dx\,dy\,d\zeta\,d\eta\,d\tau \\
& + \int_0^T\int_{\mathcal{R}^2}\int_{\mathcal{R}^2}\left[\left|\bar{u}(\zeta,\eta,t_N^+)-u\right| + \left|\bar{v}(\zeta,\eta,t_N^+)-v\right|\right] \\
& \qquad\qquad \times\omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t-T)\,d\zeta\,d\eta\,dx\,dy\,dt,
\end{aligned}$$

$$\begin{aligned}
R_1(\theta) \quad = \quad & \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}^2}\int_{\mathcal{R}^2}\left[\left|u^0-\bar{u}(\zeta,\eta,t_n^+)\right| + \left|v^0-\bar{v}(\zeta,\eta,t_n^+)\right|\right] \\
& \qquad\qquad \times\omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(\tau)\,dx\,dy\,d\zeta\,d\eta\,d\tau \\
& + \int_0^T\int_{\mathcal{R}^2}\int_{\mathcal{R}^2}\left[\left|\bar{u}^0-u\right| + \left|\bar{v}^0-v\right|\right]\omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t)\,d\zeta\,d\eta\,dx\,dy\,dt,
\end{aligned}$$

$$\begin{aligned}
R_2(\theta) \quad = \quad & \int_0^T\int_{\mathcal{R}^2}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}^2}\left\{\left[\left|u-\bar{u}(\zeta,\eta,t_n^+)\right| - \frac{|\tilde{u}-u|+|\hat{u}-u|}{2}\right]\right. \\
& \qquad\qquad \times\omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega'_\theta(t-\tau) \\
& + \left[F_1(u,\tilde{u}(\zeta,\eta,t_n^+)) - F_1(\tilde{u},u)\right]\omega'_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t-\tau) \\
& + \left[F_2(u,\hat{u}(\zeta,\eta,t_n^+)) - F_2(\hat{u},u)\right]\omega_\theta(x-\zeta)\omega'_\theta(y-\eta)\omega_\theta(t-\tau) \\
& + \left[\left|v-\bar{v}(\zeta,\eta,t_n^+)\right| - \frac{|\tilde{v}-v|+|\hat{v}-v|}{2}\right]\omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega'_\theta(t-\tau) \\
& + \left[G_1(v,\tilde{v}(\zeta,\eta,t_n^+)) - G_1(\tilde{v},v)\right]\omega'_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t-\tau) \\
& \left. + \left[G_2(v,\hat{v}(\zeta,\eta,t_n^+)) - G_2(\hat{v},v)\right]\omega_\theta(x-\zeta)\omega'_\theta(y-\eta)\omega_\theta(t-\tau)\right\} \\
& \qquad\qquad\qquad\qquad d\zeta\,d\eta\,d\tau\,dx\,dy\,dt,
\end{aligned}$$

$$R_3(\theta) = \frac{1}{\delta}\int_0^T \int_{\mathcal{R}^2} \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}^2} \left[\sigma(\bar{u}(\zeta,\eta,t_n^+) - u) - \sigma(\bar{v}(\zeta,\eta,t_n^+) - v)\right] \cdot$$
$$\left[r(x,y,t)\omega_\theta(t-\tau) - \bar{r}(\zeta,\eta,t_n^+)\omega_\theta(t-t_n)\right] \omega_\theta(x-\zeta)\omega_\theta(y-\eta)\, d\zeta d\eta\, d\tau\, dx\, dy\, dt,$$

and

$$R_4(\theta) = \int_0^T \int_{\mathcal{R}^2} \sum_{n=1}^N \int_{\mathcal{R}^2} \left[|\bar{u}(\zeta,\eta,t_n) - u| - \left|\bar{u}(\zeta,\eta,t_n^-) - u\right| + |\bar{v}(\zeta,\eta,t_n) - v|\right.$$
$$\left. -\left|\bar{v}(\zeta,\eta,t_n^-) - v\right|\right] \omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t-t_n)\, d\zeta\, d\eta\, dx\, dy\, dt.$$

The bounds of there terms are estimated as following:

LEMMA 5.1. *For any $T > 0$, there is a constant M independent of step sizes, mesh size, the relaxation time δ and θ such that*

i) $\left|L(\theta) - \left\|u(.,.,T) - \bar{u}(.,.,t_N^+)\right\|_1 - \left\|v(.,.,T) - \bar{v}(.,.,t_N^+)\right\|_1\right| \le M\theta;$

ii) $\left|R_1(\theta) - \left\|u^0 - \bar{u}^0\right\|_1 - \left\|v^0 - \bar{v}^0\right\|_1\right| \le M(\theta + \Delta t);$

iii) $|R_2(\theta)| \le M\dfrac{\Delta t}{\theta};$

iv) $R_3(\theta) \le M\dfrac{\Delta t}{\theta};$

v) $|R_4(\theta)| \le M\dfrac{\Delta t}{\theta}.$

This Lemma leads to next inequality

$$\left\|u^T - \bar{u}^{t_N^+}\right\|_1 + \left\|v^T - \bar{v}^{t_N^+}\right\|_1 \le \left\|u^0 - \bar{u}^0\right\|_1 + \left\|v^0 - \bar{v}^0\right\|_1 + M(\theta + \Delta t + \frac{\Delta t}{\theta}).$$

Choosing $\theta = \sqrt{\Delta t}$, we get the error estimate in Theorem 2.2. However, in order to complete the proof we need to establish Lemma 5.1.

5.4 Proof of Lemma 5.1

The proof of *i*), *ii*) and *iii*) can be done by the similar arguments as in the proof for the one-dimension case, cf [13]. The source error $R_3(\theta)$ can be written as

$$R_3(\theta) = \frac{1}{\delta}\int_0^T \int_{\mathcal{R}^2} \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}^2} \left[\sigma(\bar{u}(\zeta,\eta,t_n^+) - u) - \sigma(\bar{v}(\zeta,\eta,t_n^+) - v)\right]$$
$$\times \left[r(x,y,t) - \bar{r}(\zeta,\eta,t_n^+)\right] \omega_\theta(x-\zeta)\omega_\theta(y-\eta)\omega_\theta(t-\tau)\, d\zeta\, d\eta\, d\tau\, dx\, dy\, dt$$
$$+\frac{1}{\delta}\int_0^T \int_{\mathcal{R}^2} \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}^2} \left[\sigma(\bar{u}(\zeta,\eta,t_n^+) - u) - \sigma(\bar{v}(\zeta,\eta,t_n^+) - v)\right]$$
$$\times \bar{r}(\zeta,\eta,t_n^+)\left[\omega_\theta(t-\tau) - \omega_\theta(t-t_n)\right]\omega_\theta(x-\zeta)\omega_\theta(y-\eta)\, d\zeta\, d\eta\, d\tau\, dx\, dy\, dt.$$

Note that the first term is always negative, then

$$\begin{aligned} R_3(\theta) &\le \frac{2}{\delta}\int_0^T \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left\|\bar r(.,.,t_n^+)\right\|_1 |\omega_\theta(t-t_n)-\omega_\theta(t-\tau)|\,d\tau\,dt \\ &\le M\Delta t \int_0^T\int_0^T |\omega_\theta'(\eta-t)|\,d\eta\,dt \le M\frac{\Delta t}{\theta}, \end{aligned}$$

which is iv). Finally, $R_4(\theta)$ is the Godunov-type error introduced by the averaging (5.3). For notational convenience, we let

$$U^n(\zeta,\eta)=\bar u(\zeta,\eta,t_n^-) \quad \text{and} \quad V^n(\zeta,\eta)=\bar v(\zeta,\eta,t_n^-).$$

Then $\bar u(\zeta,\eta,t)=P(U)(\zeta,\eta)$ and $\bar v(\zeta,\eta,t)=P(V)(\zeta,\eta)$, and

$$R_4(\theta)=\int_0^T\sum_{n=1}^N \left[r_{4a}^n(t,\theta)+r_{4b}^n(t,\theta)\right]\omega_\theta(t-t_n)\,dt,$$

where

$$r_{4a}^n(t,\theta)=\int_{\mathcal{R}^2}\int_{\mathcal{R}^2}\left[|P(U^n)-u|-|U^n-u|\right]\omega_\theta(x-\zeta)\omega_\theta(y-\eta)\,d\zeta\,d\eta\,dx\,dy,$$

and

$$r_{4b}^n(t,\theta)=\int_{\mathcal{R}^2}\int_{\mathcal{R}^2}\left[|P(V^n)-v|-|V^n-v|\right]\omega_\theta(x-\zeta)\omega_\theta(y-\eta)\,d\zeta\,d\eta\,dx\,dy,$$

and the proof can be executed in a similar way as for the one-dimensional case, and we refer to [13] for the details. The proof of Lemma 5.1 is finished.

6 Proof of theorem 2.3

In this section we will prove that the solution of the non-equilibrium model converges towards the solution of the equilibrium model as δ goes to zero. In order to do this, we invoke the Lax-Friedrich's scheme for the equilibrium model (1.2)

$$w^{n+1}+a(w^{n+1})=A(w^n+a(w^n))-\Delta t\,\mathrm{div}_\Delta(f(w^n)+g(a(w^n))), \tag{6.1}$$

where $\mathrm{div}_\Delta(\cdot)$ is defined above. We assume that the initial data satisfies

$$w(x,y,0)=u(x,y,0).$$

The Lax-Friedrich's scheme has an error bound of $\mathcal{O}(\sqrt{\Delta t})$ approximating the scalar conservation laws, cf. [7]. Then, by the triangle inequality, we get

$$\begin{aligned} &\|u(.,.,T)-w(.,.,T)\|_1 \\ \le\ &\left\|u(.,.,T)-u^N\right\|_1+\left\|u^N-w^N\right\|_1+\left\|w(.,.,T)-w^N\right\|_1 \\ =\ &\mathcal{O}(\sqrt{\Delta t})+\left\|u^N-w^N\right\|_1, \end{aligned} \tag{6.2}$$

where w^N is the piecewise constant function representing the data $w^N_{i,j}$ generated by the scheme (6.1). Thus, we only need to investigate the discrete deviation, i.e., $u^N - w^N$. Adding (2.10) to (2.11), we get

$$u^{n+1} + v^{n+1} = A(u^n + v^n) - \Delta t \mathrm{div}_\Delta (f(u^n) + g(v^n)).$$

Setting $v = a(u) - p(u, v)$ in the previous equation, we get

$$\begin{aligned} u^{n+1} + a(u^{n+1}) &= A(u^n + a(u^n)) + p^{n+1} - A(p^n) \\ &\quad -\Delta t \mathrm{div}_\Delta (f(u^n)) - \Delta t \mathrm{div}_\Delta (g(a(u^n) - p^n)) \end{aligned} \tag{6.3}$$

Then, subtracting (6.1) from (6.3), we obtain

$$\begin{aligned} u^{n+1} - w^{n+1} + a(u^{n+1}) - a(w^{n+1}) &= A(u^n - w^n + a(u^n) - a(w^n)) \\ +p^{n+1} - A(p^n) - \Delta t \mathrm{div}_\Delta & \left[f(u^n) - f(w^n) + g(a(u^n) - p^n) - g(a(w^n)) \right]. \end{aligned}$$

Using the CFL conditions (2.15), we obtain

$$\begin{aligned} & \Delta x \Delta y \sum_{i,j} \left| u^{n+1}_{i,j} + a(u^{n+1}_{i,j}) - w^{n+1}_{i,j} - a(w^{n+1}_{i,j}) \right| \\ \leq \quad & \Delta x \Delta y \sum_{i,j} \left| u^n_{i,j} + a(u^n_{i,j}) - w^n_{i,j} - a(w^n_{i,j}) \right| + M_1 \|p^n\|_1 + M_2 \left\| p^{n+1} \right\|_1. \end{aligned}$$

Recall that $\|p^n\|_1 \leq M\delta$, and thus we have

$$\Delta x \Delta y \sum_{i,j} \left| u^N_{i,j} + a(u^N_{i,j}) - w^N_{i,j} - a(w^N_{i,j}) \right| \leq MN\delta = MT\delta/\Delta t,$$

for some constant M independent of Δt and δ. And the monotonicity property of $a(\cdot)$ gives

$$\left\| u^N - w^N \right\|_1 \leq MT\delta/\Delta t. \tag{6.4}$$

Finally, by choosing $\Delta t = \delta^{2/3}$, it follows from (6.2) that

$$\|u(.,.,T) - w(.,.,T)\|_1 = \mathcal{O}(\delta^{1/3}),$$

for any finite time T.

7 Numerical experiments

The purpose of this section is to provide some illustrations of the theory derived above. The result of Theorem 2.3 shows that the rate of convergence towards the equilibrium solution is at least $\mathcal{O}(\delta^{1/3})$ measured in L^1-norm. This result should be compared with the formal result, obtained from an asymptotic expansion (cf. Liu [8]), which indicates that this error is no larger than $\mathcal{O}(\delta^{1/2})$. Below we will present numerical experiments where the rate of convergence of this error is estimated. In particular, we will present a linear example where the estimated convergence rate is about 0.3-0.4. This indicates that the bound $\mathcal{O}(\delta^{1/3})$ is fairly sharp. We also give an example which illustrates that if the initial data is far from equilibrium, then we can not expect convergence towards the solution of the equilibrium problem. This issue is also discussed by considering a simple system of ordinary differential equations.

7.1 *Example 1*

We first consider a linear problem with the functions defined as

$$r(u,v) = a(u) - v, \quad a(u) = u, \quad f(u) = (u,u)^T, \quad g(v) = (-\tfrac{1}{2}v, -\tfrac{3}{2}v)^T.$$

Thus, the non-equilibrium system becomes

$$\begin{aligned} u_t + u_x + u_y &= -\frac{1}{\delta}(u-v) \\ v_t - \tfrac{1}{2}v_x - \tfrac{3}{2}v_y &= \frac{1}{\delta}(u-v), \end{aligned}$$

and the scalar equilibrium equation is

$$w_t + \tfrac{1}{4}w_x - \tfrac{1}{4}w_y = 0.$$

The initial data for u is given as

$$u^0(x,y) = \begin{cases} 0, & |x| > \frac{1}{2}, \quad |y| > \frac{1}{2} \\ 1.0, & -\frac{1}{2} < x < 0, \quad -\frac{1}{2} < y < 0 \\ 0.5, & 0 < x < \frac{1}{2}, \quad -\frac{1}{2} < y < 0 \\ 0.5, & -\frac{1}{2} < x < 0, \quad 0 < y < \frac{1}{2} \\ 0, & 0 < x < \frac{1}{2}, \quad 0 < y < \frac{1}{2} \end{cases},$$

and we set $w^0(x,y) = u^0(x,y)$ and $v^0(x,y) = a(u^0(x,y)) = u^0(x,y)$. Since the equilibrium is simply a scalar linear equation, we get the exact solution for a given $t > 0$ as

$$w(x,y,t) = w^0(x - \tfrac{1}{4}t, y + \tfrac{1}{4}t).$$

The grid sizes are chosen to be

$$\Delta x = \Delta y = 0.001, \quad \Delta t = 0.0005,$$

and the computed data at time $t = 0.1$ are listed in Table 7.1. There $u^N - w$ is the difference between the numerical solution of the non-equilibrium system and the exact solution of the equilibrium equation, and $r(u^N, v^N)$ is the discrete deviation from equilibrium of the non-equilibrium system. We see that the rate of convergence for the difference between two solutions is around 0.3 and 0.4. The rate of convergence for the discrete deviation from equilibrium in the non-equilibrium model is about 1.0.

7.2 *Example 2*

Next, we consider a non-linear example with

$$r(u,v) = a(u) - v, \; a(u) = \frac{2u}{u+1}, \quad f(u) = (\tfrac{1}{2}u^2, \tfrac{1}{2}u^2)^T, \; g(v) = (\tfrac{1}{2}v^2, \tfrac{1}{2}v^2)^T.$$

Table 7.1: Errors and rates of convergence for Example 1. ($\Delta x = \Delta y = 0.001, \Delta t = 0.0005, t = 0.1$.)

δ	$\|u^N - w\|_1$	rate	$\|r(u^N, v^N)\|_1$	rate
$1/2^3$	0.20078	—	0.259510	—
$1/2^4$	0.16137	0.315194	0.156990	0.725087
$1/2^5$	0.12361	0.384632	0.083124	0.917359
$1/2^6$	0.09350	0.402729	0.042692	0.961315
$1/2^7$	0.071485	0.387327	0.021822	0.968167
$1/2^8$	0.056305	0.344371	0.011087	0.976862

The initial data are

$$u^0(x,y) = \begin{cases} \sin(xy), & x^2 + y^2 < 1/4 \\ 0, & \text{otherwise} \end{cases},$$

and $w^0(x,y) = u^0(x,y)$ and $v^0(x,y) = a(u^0(x,y))$.

In this example, we compute the difference between two numerical solutions and the discrete deviation from equilibrium of the system. These data are listed in Table 7.2, together with the estimated rate of convergence. We observe that the rate of convergence for the difference between the two numerical solutions is very close to 1, which fit our estimate in (6.4). We also note that the rate of convergence towards equilibrium is close to 1 in this case.

Table 7.2: Errors and rates of convergence for Example 2. ($\Delta x = \Delta y = 0.002$, $\Delta t = 0.001$, $t = 0.25$.)

δ	$\|u^N - w^N\|_1$	rate	$\|r(u,v)\|_1$	rate
$1/2^3$	0.00032551	—	0.00095065	—
$1/2^4$	0.00016331	0.995135	0.00047537	0.999870
$1/2^5$	8.1793e-05	0.997536	0.00023754	1.000860
$1/2^6$	4.0951e-05	0.998068	0.00011876	1.000090
$1/2^7$	2.0492e-05	0.998816	5.9384e-05	0.999955
$1/2^8$	1.0251e-05	0.999336	2.9693e-05	0.999955

We plot the solutions of u and w at three different times $t = 0$, $t = 0.1$ and $t = 0.5$ in Figure 7.1. Here the u solutions are computed with $\delta = 0.1$. Notice that the two solutions at the same time are very close to each other, which again means that the equilibrium model is a good approximation to the non-equilibrium model when δ is small.

7.3 Example 3

Finally, we consider the same problem as in Example 2 with another choice of initial data. The data $u^0(x, y)$ and $w^0(x, y)$ are the same as in Example 2, but we choose $v^0(x, y) = 0$, i.e., the initial data for the non-equilibrium model is not at the equilibrium, and thus the initial deviation from the equilibrium is $\mathcal{O}(1)$. We compute the same data as in Example 2, and list them in Table 7.3. We see that the solutions of the non-equilibrium model no longer converge to the solutions of the equilibrium model. We also notice that even though we do not start from the equilibrium, the deviation from equilibrium $r(u^N, v^N)$ for the non-equilibrium model is bounded by $\mathcal{O}(\delta)$ after a short time.

Table 7.3: Errors and rates of convergence for Example 3. ($\Delta x = \Delta y = 0.002, \Delta t = 0.001, T = 0.25$.)

δ	$\left\|u^N - w^N\right\|_1$	$\|\|r(u,v)\|\|_1$	rate
$1/2^3$	0.020446	0.00017991	—
$1/2^4$	0.020501	5.5600e-05	1.69410
$1/2^5$	0.020501	2.7759e-05	1.00213
$1/2^6$	0.020501	1.3871e-05	1.00092
$1/2^7$	0.020501	6.9334e-06	1.00041
$1/2^8$	0.020501	3.4663e-06	1.00019

7.4 An ordinary differential equation

The phenomenon encountered in Example 3 can also be explained analytically by looking at a simple example. Consider the case with $u^0 \equiv 1$, $w^0 \equiv 1$, $v^0 \equiv 0$, $r(u, v) = a(u) - v$ and $a(u) = u$. Then, the non-equilibrium system becomes a system of ordinary differential equations

$$\begin{aligned} \delta u_t &= -(u - v) \\ \delta v_t &= u - v, \end{aligned}$$

for which the analytical solution is

$$u(t) = \frac{1}{2}(e^{-2t/\delta} + 1).$$

As δ turns to 0, the limit is $u = 1/2$. But the equilibrium equation is simply $w_t = 0$ and the exact solution is $w(t) = 1$, which clearly is not the equilibrium of u.

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REFERENCES

1. G.-Q. Chen, C.D. Levermore and T.-P. Liu. *Hyperbolic conservation laws with stiff relaxation terms and entropy.* Comm. Pure Appl. Math. 47, 787–830, 1994.
2. G.-Q. Chen and T.-P. Liu. *Zero relaxation and dissipation limits for hyperbolic conservation laws.* Communications on Pure and Applied Mathematics, Vol. XLVI, 755–781 (1993).
3. E. Conway and J. Smoller. *Global solutions of the Cauchy problem of quasi-linear first order equations in several space variables.* Comm. pure & appl. math. (1966) pp. 95-105.
4. M. Crandall and A. Majda. *Monotone difference approximations for scalar conservation laws.* Math. of Comp., 34(149):1–21, January 1980.
5. S. Jin and Z.-P. Xin. *The relaxation schemes for systems of conservation laws in arbitrary space dimensions.* To appear, 1995.
6. S. N. Kruzkov. *First order quasi linear equations with several space variables.* Math. USSR. Sb., 10:217–243, 1970.
7. N. N. Kuznetsov. *Accuracy of some approximate methods for computing the weak solutions of a first order quasi linear equations.* USSR. Comp. Math and Math Phys., 16:105–119, 1976.
8. T.P. Liu. *Hyperbolic conservation laws with relaxation.* Commun. Math. Phys., 108:153–175, 1987.
9. B. J. Lucier. *Error bounds for the methods of Glimm, Godunov and Leveque.* SIAM J. Numer. Anal., 22:1074–1081, 1985.
10. R. Natalini. *Convergence to equilibrium for the relaxation approximations of conservation laws.* Preprint, 1995.
11. R.B. Pember. *Numerical methods for hyperbolic conservation laws with stiff relaxation I. Spurious solutions.* Siam. J. Appl. Math., Vol. 53, No. 5, pp. 1293–1330, 1993.
12. R.B. Pember. *Numerical methods for hyperbolic conservation laws with stiff relaxation II. Higher-order Godunov methods.* Siam. J. Sci. Comput 14. No. 4, pp. 824–859, 1993.
13. H. J. Schroll, A. Tveito, and R. Winther. *An L^1-error bound for a semiimplicit scheme applied to a stiff system of conservation laws.* To appear in Siam J. Num. Anal..
14. J. Smoller. *Shock waves and reaction-diffusion equations, second edition* Springer-Verlag, 1994.
15. Z.-H. Teng. *On the accuracy of fractional step methods for conservation laws in two dimensions.* SIAM J. Numer. Anal., Vol. 31, No. 1, pp. 43–63, 1994.
16. A. Tveito and R. Winther. *On the rate of convergence to equilibrium for a system of conservation laws including a relaxation term.* To appear in Siam J. Math. Anal..
17. G.B. Whitham. *Linear and non-linear waves.* New York: Wiley, 1974.

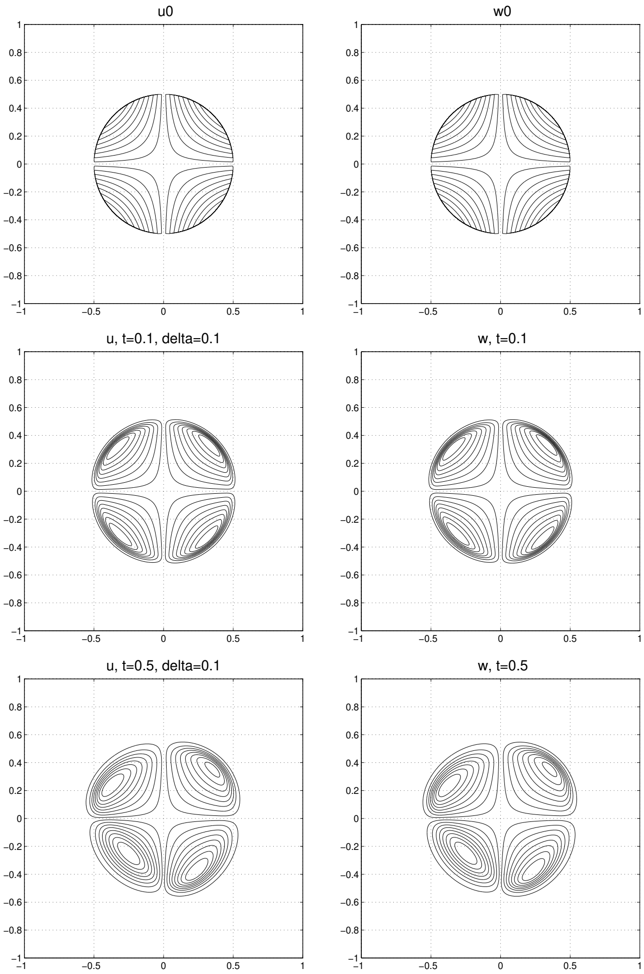

Figure 7.1: Plots of solutions w and u at time $t = 0, 0.1, 0.5$. Here, u is the solution with $\delta = 0.1$, and the grid parameters are $\Delta x = \Delta y = 0.002$ and $\Delta t = 0.001$.