ON THE RATE OF CONVERGENCE TO EQUILIBRIUM FOR A SYSTEM OF CONSERVATION LAWS INCLUDING A RELAXATION TERM

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Abstract. We analyze a simple system of conservation laws with a strong relaxation term. Well-posedness of the Cauchy problem, in the framework of BV-solutions, is proved. Furthermore, we prove that the solutions converge towards the solution of an equilibrium model as the relaxation time $\delta > 0$ tends to zero. Finally, we show that the difference between an equilibrium solution ($\delta = 0$) and a non-equilibrium solution ($\delta > 0$), measured in L^1, is bounded by $O(\delta^{1/3})$.

Key words. Hyperbolic conservation laws, relaxation terms, non-equilibrium, rate of convergence.

AMS(MOS) subject classification: 35L65, 65M10.

1. Introduction. The purpose of this paper is to study the following system of conservation laws

$$
\begin{aligned}
&(u+v)_t + f(u)_x = 0 \\
&\delta v_t = A(u) - v.
\end{aligned}
\tag{1.1}
$$

Here f and A are given functions, u and v are the unknowns and $\delta > 0$ is referred to as the relaxation time. The function $A(u)$ will throughout this paper be assumed to be an increasing function of u. Further assumptions on the model will be given in Section 2.

We will in particular be concerned with the convergence of $(u, v) = (u_\delta, v_\delta)$ as δ tends to zero. In the limit of zero relaxation time, a scalar conservation law of the form

$$
(w + A(w))_t + f(w)_x = 0 \tag{1.2}
$$

is obtained. Systems of the form (1.1) are usually referred to as "non-equilibrium" models, whereas (1.2) is the "equilibrium" model. The main result of the present paper

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is that, for proper conditions on the initial data, the solutions of the non-equilibrium model tend to the solution of the equilibrium model in L^1 with a deviation bounded by $O(\delta^{1/3})$.

The system (1.1) arises in chromatography and is discussed in the books [19, 20, 28]. In this framework, u denotes the density of some specie contained in a fluid flowing through a fixed bed, and v denotes the density of the specie adsorbed on the material in the bed. The right hand side of the second equation models the adsorption. Different form of adsorption functions A are discussed by e.g. Bear and Bachmat [1] Ch.6.

The equilibrium assumption in chromatography states that the rate of the chemical reaction is so large that the reaction can be considered as instantaneous compared to the timescales of other effects. We prove that for initial data close to equilibrium and for small relaxation times δ, the equilibrium model provides good approximations of the non-equilibrium solutions. One should, however, be cautious. In [24], the system, in a radial geometry, is considered as a model for near-well reservoir simulation. Their results indicate the equilibrium model is inadequate for the analysis of this problem.

A nice introduction to systems of conservation laws including relaxation terms can be found in Whitham's book [28], where such models arising in e.g. chromatography, traffic modeling, water waves and gas dynamics are discussed. For applications in chromatography, a detailed discussion is provided in the books of Rhee, Aris and Amundson [19, 20]. Systems consisting of one conservation equation and one equation with a relaxation term have been studied by several authors, cf. [2, 5, 6, 7, 12, 13, 15, 17, 18, 21, 24, 25, 27]. Much of this research is motivated by combustion theory and especially by Majda's model [15]. The question of well-posedness of this system is discussed by Teng and Ying [25] and by Levy [12], and computational studies of the system are presented by Colella, Majda and Roytburd [5] and by Pember [17, 18].

The so-called sub-characteristic condition plays a central role in systems with relaxation terms. Consider the system (1.1) above and let $\lambda_1 = 0$ and $\lambda_2 = f'$ denote the characteristic speeds. Similarly, $\lambda^* = f'/(1 + A')$ denotes the characteristic speed of (1.2). Then the sub-characteristic condition states that $\lambda_1 \leq \lambda^* \leq \lambda_2$, which, due to the monotonicity of A, is satisfied for our models. It turns out that the sub-characteristic condition is necessary for the stability of the non-equilibrium model. This issue is analyzed, using linearization, by Whitham [28] Ch.10, for "small waves" by Liu [13], for a linear model by LeVeque and Wang [11], and finally by Chen and Liu [2]. In the latter paper, Chen and Liu actually prove convergence of solutions of the non-equilibrium towards the solution of an equilibrium model for two different systems. Their results are based on the theory of compensated compactness. Generalizations of their results are given in the recent paper by Chen, Levermore and Liu [3]. Also Schochet [21] proves convergence of a family of non-equilibrium solutions to the solution of an equilibrium model. He studies the refined traffic model, introduced by Whitham [28], which consists of a standard conservation law for the density of cars and a non-equilibrium model for the velocity.

The plan of the present paper is as follows: We start in Section 2 by giving precise assumptions on the system (1.1) and the initial data. Furthermore, we state the main results of the paper. Section 3 is devoted to the analysis of a finite difference scheme approximating the system (1.1). The results of Section 3 are used in Section 4 where we demonstrate existence, uniqueness and stability of solutions of (1.1). We should mention here that well-posedness of (1.1) was proved both by Levy [12] and by Teng and Ying [25]. Their estimates do, however, depend on the relaxation time, and since we will study the convergence of solutions of (1.1) as $\delta \to 0$, we need δ independent estimates.

In Section 5, we introduce an auxiliary system where small diffusion terms are present. This auxiliary system enables us to prove, in Section 6, that the solution of (1.1) tends to the solution of (1.2) in L^1 with a deviation bounded by $O(\delta^{1/3})$.

Remarks:

1) The results obtained in the present paper could, without too much difficulty, be generalized to yield slightly more general systems than those covered by (1.1). We have however, for clarity chosen to analyze a model that is as simple as possible in order to avoid messy details containing no essential new insights.

2) We do not know if the convergence estimate of the form $O(\delta^{1/3})$ is optimal. However, computational experiments we have done seem to indicate that if an estimate of the form $O(\delta^{\gamma})$ is sought then the optimal value of γ is somewhere in the interval $[1/4, 1/2]$, see also Section 6 of [22]. $\square$

2. Preliminaries and statement of the main results. The purpose of this paper is to study how well solutions of the scalar conservation law

$$(w + A(w))_t + f(w)_x = 0 \tag{2.1}$$

approximate the corresponding solutions of the system

$$\begin{aligned}(u+v)_t + f(u)_x &= 0\\ \delta v_t &= A(u) - v,\end{aligned} \tag{2.2}$$

for small positive values of the relaxation time δ, i.e. $\delta \in (0, 1]$. The explicit dependence of the solution (u, v) on δ will usually be suppressed. We will assume that $f = f(u)$ is a smooth function satisfying $f(0) = 0$, and

$$f'(u) \geq 0 \quad \text{for } u \in [0, 1].$$

The function $A = A(u)$ will be assumed to satisfy the following requirements.

$$\begin{aligned}&A(0) = 0, A(1) = 1,\\ &A'(u) \geq 0,\\ &|A''(u)| \leq \alpha < 1,\end{aligned} \tag{2.3}$$

for all relevant values of u. We shall consider solutions of (2.2) in the state-space

$$\mathcal{S} = [0, 1] \times [0, 1]$$

and solutions of (2.1) in $[0, 1]$.

In equilibrium we have $v = A(u)$, and the system (2.2) degenerates to the scalar equation (2.1). In order to study the deviation from equilibrium in the non-equilibrium model, we introduce an auxiliary variable defined by

$$p = A(u) - v.$$

The initial conditions (u^0, v^0) for the system (2.2) are supposed to satisfy:

$$
\begin{aligned}
&\text{i) } (u^0(x), v^0(x)) \in \mathcal{S} \quad \forall x \in \mathcal{R},\\
(2.4) \qquad &\text{ii) } TV(u^0) + TV(v^0) \le M,\\
&\text{iii) } \|p^0\|_1 = \|A(u^0) - v^0\|_1 \le M\delta,\\
&\text{iv) } u^0(\pm\infty), v^0(\pm\infty) = 0.
\end{aligned}
$$

Here, and in the rest of this paper, M denotes a generic finite constant independent of δ. The L^1-norm is denoted by $\|\cdot\|_1$ and $TV(\cdot)$ denotes the total-variation, defined by

$$TV(z) = \sup_{h \neq 0} \int_{\mathcal{R}} \frac{|z(x+h) - z(x)|}{|h|} dx.$$

Furthermore, $BV = BV(\mathcal{R})$ denotes the subspace of L^1_{loc} consisting of functions with bounded total variation.

Note that iii) above assures that the initial conditions are close to equilibrium. This is a natural assumption as we primarily want to analyze the convergence towards the solutions of the equilibrium model (2.1). But, we shall also indicate that if iii) is not satisfied initially, an initial layer will appear and $\|p(t)\|_1 = O(\delta)$ will be reached at exponential rate.

It is well known what we mean by an entropy solution of the scalar equation (2.1), but let us define a similar concept for (2.2). For any $T > 0$ we let $\mathcal{D}_+(T)$ denote the set of all non-negative C^∞-functions with compact support in $\mathcal{R} \times [0, T]$.

DEFINITION 1. *Let (u^0, v^0), satisfying (2.4), be the given initial data. Then (u, v) is called an entropy solution of the system (2.2) if the following requirements are satisfied:*

1. $(u, v) \in \mathcal{S} \quad \forall (x,t) \in \mathcal{R} \times \mathcal{R}_0^+$,

2. $TV(u(\cdot,t)), TV(v(\cdot,t)) \le M \quad \forall t \in \mathcal{R}_0^+$,

3. $\|u(\cdot,t) - u(\cdot,\tau)\|_1 + \|v(\cdot,t) - v(\cdot,\tau)\|_1 \le M|t - \tau| \quad \forall t, \tau \in \mathcal{R}_0^+$,

4. For any $(k,q) \in \mathcal{S}$ and any $\varphi, \psi \in \mathcal{D}_+(T)$, where $T > 0$ is arbitrary,

$$
\begin{aligned}
&\int_0^T \int_{\mathcal{R}} [|u - k|\varphi_t + |f(u) - f(k)|\varphi_x + |v - q|\psi_t] dx\, dt\\
(2.5) \qquad &+ \int_{\mathcal{R}} [|u^0 - k|\varphi(x,0) + |v^0 - q|\psi(x,0)] dx
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\mathcal{R}}[|u(x,T)-k|\varphi(x,T)+|v(x,T)-q|\psi(x,T)]dx \\
&\geq \frac{1}{\delta}\int_0^T\int_{\mathcal{R}}[\sigma(u-k)\varphi-\sigma(v-q)\psi](A(u)-v)dx\,dt,
\end{aligned}
$$

where σ denotes the sign function.

Remark

Let us again mention that the existence of entropy solutions of the systems of the form (2.2) has been proved previously by Levy [12] and by Teng and Ying [25]. The reason for us to redo the existence part here, is that we shall need δ-independent estimates when we consider the convergence of (u_δ, v_δ) as $\delta \to 0$. Such estimates are not provided by [12, 25]. $\square$

Existence of an entropy solution satisfying the requirements of Definition 1 will be proved using a finite difference scheme. We note that existence could also be proved using the parabolic regularizations discussed below, but the problem of computing solutions of systems of the form (2.2) is important, and hence, we find it desirable to prove existence by demonstrating convergence of a numerical scheme. An error estimate for the scheme is derived in the subsequent paper [22].

Our scheme for analyzing the system (1.1) is a straightforward semi-implicit one:

$$
\begin{aligned}
&\frac{(u_j^{n+1}+v_j^{n+1})-(u_j^n+v_j^n)}{\Delta t}+\frac{f(u_j^n)-f(u_{j-1}^n)}{\Delta x}=0 \\
&\frac{\delta(v_j^{n+1}-v_j^n)}{\Delta t}=A(u_j^{n+1})-v_j^{n+1}
\end{aligned}
\tag{2.6}
$$

Here u_j^n denotes an approximation of $u(x,t)$ over the grid-block

$$B_j^n=[x_{j-1/2},x_{j+1/2})\times[t_n,t_{n+1}),$$

where $x_j = j\Delta x$ and $t_n = n\Delta t$. Furthermore, Δt and Δx denote the step-lengths in t and x directions, respectively. Similarly, v_j^n approximates v on B_j^n. The scheme is initialized by putting

$$u_j^0=\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}}u^0(x)\,dx \quad\text{and}\quad v_j^0=\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}}v^0(x)\,dx. \tag{2.7}$$

Throughout the paper we will assume that the grid parameters satisfy the CFL-condition

$$\mu f'(u)\leq 1 \tag{2.8}$$

for all $u \in [0,1]$, where $\mu = \Delta t/\Delta x$. It is interesting, from a computational point of view, to note that this condition is independent of the relaxation time δ, thus very small relaxation-times do not force us to use equally small time-steps.

For a grid function u, the discrete total-variation is defined by

$$TV(u) = \sum_{j \in \mathcal{Z}} |u_j - u_{j-1}|,$$

and $\|\cdot\|_1$ denotes the discrete L^1-norm defined by

$$\|u\|_1 = \Delta x \sum_{j \in \mathcal{Z}} |u_j|.$$

In order to study the deviation from equilibrium in the non-equilibrium system, we define

$$p_j^n = A(u_j^n) - v_j^n.$$

We consider initial data satisfying

$$\begin{aligned} &i) \quad (u_j^0, v_j^0) \in S \text{ for all } j \in \mathcal{Z}, \\ &ii) \quad TV(u^0) + TV(v^0) \le M, \\ &iii) \quad \|p^0\|_1 \le M\delta, \\ &iv) \quad (u_{-\infty}^0, v_{-\infty}^0) = (u_\infty^0, v_\infty^0) = 0, \end{aligned} \tag{2.9}$$

where M is a finite constant independent of the grid parameters and the relaxation time. Note that (2.9) is simply a discrete version of (2.4).

Based on the results of the scheme, a family of approximate solutions is defined by putting

$$(u_\Delta, v_\Delta)(x,t) = (u_j^n, v_j^n) \quad \text{for } (x,t) \in B_j^n,\ \forall (j,n) \in \mathcal{Z} \times \mathcal{Z}^+.$$

By using the properties of the scheme presented above, the following theorem will be proved in Section 4.

THEOREM 2.1. *Let (u^0, v^0) be initial data satisfying (2.4) and let (u_j^0, v_j^0) be the corresponding discrete initial values generated by (2.7). Then, as Δx and Δt tends to* *zero, the family* $\{(u_\Delta, v_\Delta)\}$ *of approximate solutions generated by the finite difference scheme described above, converges in* $(L^1_{loc}(\mathcal{R}\times\mathcal{R}_0^+))^2$ *towards a pair of functions* (u,v). *Furthermore, the limit* (u,v) *is a unique entropy solution satisfying the requirements of Definition 1.*

The entropy solution (u,v) *satisfies*

$$\|p(\cdot,t)\|_1 = \|A(u)-v\|_1 \le M\delta \quad \textit{for} \;\; t\ge 0,$$

where M *is a finite constant not depending on* δ. *If* $(\bar{u}^0,\bar{v}^0)$ *is another pair of initial data satisfying (2.4), there is a unique entropy solution* $(\bar{u},\bar{v})$ *such that*

$$\|u(\cdot,t)-\bar{u}(\cdot,t)\|_1 + \|v(\cdot,t)-\bar{v}(\cdot,t)\|_1 \le \|u^0-\bar{u}^0\|_1 + \|v^0-\bar{v}^0\|_1 \quad \textit{for} \;\; t\ge 0. \qquad \square$$

Since our existence theorem is based on estimates independent of δ, we could, by appealing to Helly's theorem, prove convergence of a subsequence in L^1_{loc} of $(u,v) = (u_\delta, v_\delta)$ as $\delta\to 0$. But we want more than convergence - we are also interested in the rate of convergence. In order to analyze this issue, we are lead to study a parabolic regularization of the system:

$$\begin{aligned} u^\epsilon_t + f(u^\epsilon)_x &= \frac{1}{\delta}(v^\epsilon - A(u^\epsilon)) + \epsilon u^\epsilon_{xx}, \\ v^\epsilon_t &= \frac{1}{\delta}(A(u^\epsilon)-v^\epsilon) + \epsilon v^\epsilon_{xx}. \end{aligned} \tag{2.10}$$

The properties of this system are analyzed in Section 5. In the last section, we start by proving that (cf. Lemma 6.1)

$$\|u^\epsilon(\cdot,t)-u(\cdot,t)\|_1 + \|v^\epsilon(\cdot,t)-v(\cdot,t)\|_1 \le M\epsilon^{1/2}, \tag{2.11}$$

where (u^ϵ, v^ϵ) solves the regularized problem and (u,v) solves (2.2). Then we introduce a regularized equilibrium model of the form

$$(w^\epsilon + A(w^\epsilon))_t + f(w^\epsilon)_x = \epsilon(w^\epsilon + A(w^\epsilon))_{xx}. \tag{2.12}$$

For this scalar equation, it is well known, cf. Kuznetsov [10], that

$$\|w-w^\epsilon\|_1 \le M\epsilon^{1/2}, \tag{2.13}$$

where w solves (2.1) and w^ϵ solves (2.12). Furthermore, we prove (cf. Lemma 6.2) that

$$\|u^\epsilon - w^\epsilon\|_1 \leq M\delta/\epsilon, \tag{2.14}$$

where δ is the relaxation time, and ϵ is the diffusion coefficient. Collecting these results we observe that

$$\|u - w\|_1 \leq \|u - u^\epsilon\|_1 + \|u^\epsilon - w^\epsilon\|_1 + \|w^\epsilon - w\|_1 \leq M(\epsilon^{1/2} + \delta/\epsilon + \epsilon^{1/2}).$$

Hence by choosing $\epsilon = \delta^{2/3}$, we obtain the following result.

THEOREM 2.2. *Let (u^0, v^0) be a pair of initial data satisfying (2.4), and let $w^0 = u^0$. Then, for any finite $T > 0$, there is a finite constant M such that*

$$\|u(\cdot,t) - w(\cdot,t)\|_1 \leq M\delta^{1/3} \quad \textit{for all } \ 0 \leq t \leq T.$$

Here (u, v) and w are solutions of (2.2) and (2.1) respectively. □

The rest of this paper is devoted to the proof of Theorem 2.1 and 2.2.

Remark:

Heuristic arguments, e.g. by a Chapman-Enskog expansion, indicate a rate of 1/2 rather than 1/3 in Theorem 2.2, but we have not been able to give rigorous arguments for such an improved rate. Neither have we been able to prove that 1/3 is optimal. In the estimates (2.11),(2.13) and (2.14) leading to this rate, the first and second results are sharp, but the estimate (2.14) is probably not sharp. Our attempts at improving this estimate have not been successful.

3. Bounds on the approximate solutions. In this section, we will study the properties of the finite difference scheme (2.6) approximating the solutions of (2.2). In fact, existence of an entropy solution of the system is proved by demonstrating convergence of the family of approximate solutions generated by the finite difference scheme. We start by considering the standard estimates needed to prove existence of a BV-solution; an L^∞-bound, a bound on the total variation, and finally an L^1-continuity estimate in time. Having these bounds, independent of the mesh-size and independent of the relaxations time δ, existence of a weak solution can be demonstrated, cf. e.g. Smoller [23].

We also show that, if the initial data is close to equilibrium, the discrete solution remains close to equilibrium for all time. Finally we show that the finite difference solutions satisfy a discrete entropy inequality. This latter property enables us to prove the existence of an entropy solution satisfying the requirements of Definition 1.

The finite-difference approximations have the following properties.

LEMMA 3.1. *Suppose that the initial data (u^0, v^0) satisfy (2.9), and that the grid-parameters Δt and Δx satisfies the CFL-condition (2.8). Then, for any $\delta > 0$, there is a finite constant M independent of δ, Δt and Δx such that:*

I. $(u_j^n, v_j^n) \in \mathcal{S}$ *for all* $(j, n) \in \mathcal{Z} \times \mathcal{Z}^+$

II. $TV(u^n) + TV(v^n) \leq TV(u^0) + TV(v^0)$

III. $\|p^n\|_1 \leq M\delta$

IV. $\|u^n - u^m\|_1 + \|v^n - v^m\|_1 \leq M|n-m|\Delta t$

V. $\|u^n - \bar{u}^n\|_1 + \|v^n - \bar{v}^n\|_1 \leq \|u^0 - \bar{u}^0\|_1 + \|v^0 - \bar{v}^0\|_1,$

where $(\bar{u}^n, \bar{v}^n)$ is a discrete solution based on initial data $(\bar{u}^0, \bar{v}^0)$ satisfying (2.9). □

Note that since the entropy solution will inherit the properties of the approximate solutions, it will satisfy a maximum principle (I), be TV-stable (II), be close to equilibrium (III) be L^1-continuous in time (IV), and be stable in L^1 (V).

The next subsections are devoted to the proof of the lemma above.

3.1. Proof of I; (Maximum principle). For a fixed pair of (j, n), we let $(u, v) = (u_j^{n+1}, v_j^{n+1})$, $(\bar{u}, \bar{v}) = (u_j^n, v_j^n)$ and $(u^L, v^L) = (u_{j-1}^n, v_{j-1}^n)$. Then, by (2.6), we get the equations

$$
\begin{aligned}
u + v &= \bar{u} + \bar{v} - \frac{\Delta t}{\Delta x}(f(\bar{u}) - f(u^L)) \\
(\delta + \Delta t)v - \Delta t A(u) &= \delta \bar{v}.
\end{aligned}
\tag{3.1}
$$

Let us first verify that (3.1) has a unique solution (u, v) for any given $(\bar{u}, \bar{v}, u^L, v^L)$. Denote by r_1 and r_2 the two right hand sides, such that

$$v = r_1 - u,$$

and thus the single equation

$$K(u) \equiv \Delta t A(u) + (\delta + \Delta t)u + r_2 - r_1(\delta + \Delta t) = 0$$

determines u. Since K is monotone, and $K(\pm\infty) = \pm\infty$, it is clear that (u, v) is uniquely determined by (3.1).

Next, we assume that $0 \leq \bar{u}, \bar{v}, u^L, v^L \leq 1$, and we want to prove that this implies

$$0 \leq u, v \leq 1, \tag{3.2}$$

which, in fact, proves part I of the lemma. In order to prove (3.2), we consider u and v as functions of u^L,$\bar{u}$,$\bar{v}$, i.e.

$$u = u(u^L, \bar{u}, \bar{v}) \text{ and } v = u(u^L, \bar{u}, \bar{v}).$$

Now, we want to show that u and v are monotone in all arguments, and we start by considering $\frac{\partial u}{\partial \bar{v}}$ and $\frac{\partial v}{\partial \bar{v}}$. From the first equation of (3.1), we get

$$\frac{\partial u}{\partial \bar{v}} + \frac{\partial v}{\partial \bar{v}} = 1, \tag{3.3}$$

and the second equation gives

$$(\delta + \Delta t)\frac{\partial v}{\partial \bar{v}} - \Delta t A'(u)\frac{\partial u}{\partial \bar{v}} = \delta,$$

hence

$$\frac{\partial v}{\partial \bar{v}} = \frac{\delta + \Delta t A'(u)}{\delta + \Delta t + \Delta t A'(u)} \in (0, 1)$$

and then, by (3.3) $\frac{\partial u}{\partial \bar{v}} \in (0, 1)$ and we conclude that

$$\frac{\partial u}{\partial \bar{v}}, \frac{\partial v}{\partial \bar{v}} > 0. \tag{3.4}$$

Similarly, by differentiating (3.1) with resect to $\bar{u}$, we get

$$\frac{\partial u}{\partial \bar{u}} + \frac{\partial v}{\partial \bar{u}} = 1 - \mu f'(\bar{u}), \tag{3.5}$$

and

$$(\delta + \Delta t)\frac{\partial v}{\partial \bar{u}} - \Delta t A'(u)\frac{\partial u}{\partial \bar{u}} = 0,$$

hence

$$\frac{\partial v}{\partial \bar{u}} = \frac{\Delta t A'(u)}{\delta + \Delta t + \Delta t A'(u)}(1 - \mu f'(\bar{u})) \equiv \omega(1 - \mu f'(\bar{u})) \tag{3.6}$$

where $\omega \in [0, 1)$. Now (3.5) gives

$$\frac{\partial u}{\partial \bar{u}} = (1 - \mu f'(\bar{u}))(1 - \omega),$$

and then, by the CFL-condition (2.8), we can conclude that

$$\frac{\partial u}{\partial \bar{u}}, \frac{\partial v}{\partial \bar{u}} \geq 0 \tag{3.7}$$

Finally, by differentiating (3.1) with respect to u^L, we get

$$\frac{\partial u}{\partial u^L} + \frac{\partial v}{\partial u^L} = \mu f'(u^L)$$

and

$$(\delta + \Delta t)\frac{\partial v}{\partial u^L} - \Delta t A'(u)\frac{\partial u}{\partial u^L} = 0,$$

hence

$$\frac{\partial v}{\partial u^L} = \frac{\Delta t A'(u)}{\delta + \Delta t + \Delta t A'(u)}\mu f'(u^L) \equiv \bar{\omega}\mu f'(u^L), \tag{3.8}$$

where $\bar{\omega} \in [0, 1)$, and

$$\frac{\partial u}{\partial u^L} = \mu f'(u^L)(1 - \bar{\omega}) \tag{3.9}$$

Now (3.8) and (3.9) give

$$\frac{\partial u}{\partial u^L}, \frac{\partial v}{\partial u^L} \geq 0 \tag{3.10}$$

By using the monotonicity properties (3.4), (3.7) and (3.10) of $u = u(u^L, \bar{u}, \bar{v})$ and $v = v(u^L, \bar{u}, \bar{v})$, we have

$$u(0, 0, 0) \leq u(u^L, \bar{u}, \bar{v}) \leq u(1, 1, 1)$$

and

$$v(0, 0, 0) \leq v(u^L, \bar{u}, \bar{v}) \leq v(1, 1, 1).$$

It is easily seen that $u(0, 0, 0) = v(0, 0, 0) = 0$ and $u(1, 1, 1) = v(1, 1, 1) = 1$, and then the maximum principle I of the lemma follows by induction.

3.2. Proof of II; the total variation estimate. Recall that the total variation of $\{u_j^n\}$ at time $t = n\Delta t$ is defined by

$$TV(u^n) = \sum_j |u_{j+1}^n - u_j^n|.$$

In order to prove the TV-bound, it is convenient to introduce the following notation:

$$U_j^n = u_{j+1}^n - u_j^n$$

and

$$V_j^n = v_{j+1}^n - v_j^n.$$

Then by rewriting the scheme (2.6) in the form

$$\begin{aligned} u_j^{n+1} &= u_j^n - \mu(f(u_j^n) - f(u_{j-1}^n)) - \frac{\Delta t}{\delta}(A(u_j^{n+1}) - v_j^{n+1}) \\ v_j^{n+1} &= v_j^n + \frac{\Delta t}{\delta}(A(u_j^{n+1}) - v_j^{n+1}), \end{aligned} \tag{3.11}$$

we get

$$U_j^{n+1} = U_j^n - \mu f'(\tilde{u}_{j+1/2}^n)U_j^n + \mu f'(\tilde{u}_{j-1/2}^n)U_{j-1}^n - \frac{\Delta t}{\delta}A'(\hat{u}_{j+1/2}^{n+1})U_j^{n+1} + \frac{\Delta t}{\delta}V_j^{n+1} \tag{3.12}$$

$$V_j^{n+1} = V_j^n + \frac{\Delta t}{\delta}A'(\hat{u}_{j+1/2}^{n+1})U_j^{n+1} - \frac{\Delta t}{\delta}V_j^{n+1}. \tag{3.13}$$

Here $\tilde{u}_{j+1/2}^n$ satisfies

$$f(u_{j+1}^n) - f(u_j^n) = f'(\tilde{u}_{j+1/2}^n)U_j^n, \tag{3.14}$$

and $\hat{u}_{j+1/2}^n$ satisfies

$$A(u_{j+1}^n) - A(u_j^n) = A'(\hat{u}_{j+1/2}^n)U_j^n. \tag{3.15}$$

We multiply (3.12) by $\sigma(U_j^{n+1})$ and (3.13) by $\sigma(V_j^{n+1})$, (recall that σ denotes the sign function), and get

$$\begin{aligned} |U_j^{n+1}| &\le (1 - \mu f'(\tilde{u}_{j+1/2}^n)|U_j^n| + \mu f'(\tilde{u}_{j-1/2}^n)|U_{j-1}^n| - \frac{\Delta t}{\delta}A'(\hat{u}_{j+1/2}^{n+1})|U_j^{n+1}| + \frac{\Delta t}{\delta}|V_j^{n+1}| \\ |V_j^{n+1}| &\le |V_j^n| + \frac{\Delta t}{\delta}A'(\hat{u}_{j+1/2}^{n+1})|U_j^{n+1}| - \frac{\Delta t}{\delta}|V_j^{n+1}|. \end{aligned}$$

By adding these inequalities, we get

$$\sum_j |U_j^{n+1}| + \sum_j |V_j^{n+1}| \leq \sum_j |U_j^n| + \sum_j |V_j^n|,$$

thus

$$TV(u^n) + TV(v^n) \leq TV(u^0) + TV(v^0)$$

by induction.

3.3. Proof of III; deviation from equilibrium. Note that $p = A(u) - v$ measures the deviation from equilibrium in the non-equilibrium model. Part III of the lemma states that if we give initial data close to equilibrium for the non-equilibrium model, the solution will remain close to equilibrium for all time. To prove this, we start by noting that

$$p_j^{n+1} - p_j^n = A'(\hat{u}_j^{n+1})(u_j^{n+1} - u_j^n) - (v_j^{n+1} - v_j^n)$$

where $\hat{u}_j^{n+1}$ satisfies

$$A(u_j^{n+1}) - A(u_j^n) = A'(\hat{u}_j^{n+1})(u_j^{n+1} - u_j^n).$$

Then, using the scheme (3.11), we get

$$p_j^{n+1} = p_j^n - \mu A'(\hat{u}_j^{n+1}) f'(\tilde{u}_{j-1/2}^n)(u_j^n - u_{j-1}^n) - \frac{\Delta t}{\delta}(1 + A'(\hat{u}_j^{n+1}))p_j^{n+1}, \tag{3.16}$$

where $\tilde{u}_{j-1/2}^n$ satisfies (3.14). Multiplying this equation by $\sigma(p_j^{n+1})$ we obtain

$$|p_j^{n+1}| \leq |p_j^n| + \mu M |u_j^n - u_{j-1}^n| - \frac{\Delta t}{\delta}|p_j^{n+1}|,$$

where M is a finite constant independent of Δx,Δt and δ, and where we have used the fact that

$$A'(u) \geq 0$$

for all $u \in [0, 1]$, cf (2.3). Since the total variation of u^n is bounded, we have

$$\|p^{n+1}\|_1 \leq \|p^n\|_1 + \tilde{M}\Delta t - \frac{\Delta t}{\delta}\|p^{n+1}\|_1 \tag{3.17}$$

where again $\tilde{M}$ is another finite constant independent of Δx,Δt and δ. From (3.17), we get

$$\|p^n\|_1 \le \tilde{M}\delta, \tag{3.18}$$

provided that the estimate holds for $n = 0$. This concludes the proof of III in the lemma.

Remark:

As mentioned above, our interest in this paper is solutions which are close to equilibrium. For this purpose, the estimate (3.18) is exactly what we need. It should, however, be mentioned that some further insight could be gained from (3.17). In fact, by assuming that Δt is sufficiently small, i.e. $\Delta t \le$ const.δ, the inequality (3.17) implies that

$$\|p^n\|_1 \le M_1\delta + e^{-M_2 t_n/\delta}\|p^0\|_1 \tag{3.19}$$

where M_1 and M_2 are finite constants independent of δ, Δt, Δx. Hence, for any initial data of bounded variation lying in the state spaces, the non-equilibrium discrete solution approaches a state close to equilibrium at exponential rate. $\square$

3.4. Proof of IV; L^1-continuity in time. By using the property of p_j^n derived above, the L^1 continuity in time is easily verified. Consider the scheme (3.14), (3.15) using the definition of p_j^n:

$$\begin{aligned} u_j^{n+1} &= u_j^n - \mu(f(u_j^n) - f(u_{j-1}^n)) - \frac{\Delta t}{\delta} p_j^{n+1} \\ v_j^{n+1} &= v_j^n + \frac{\Delta t}{\delta} p_j^{n+1}. \end{aligned} \tag{3.20}$$

Now,

$$\|u^{n+1} - u^n\|_1 + \|v^{n+1} - v^n\|_1 \le M\Delta t \sum_j |u_j^n - u_{j-1}^n| + \frac{\Delta t}{\delta}\|p^{n+1}\|_1,$$

where M is a finite constant independent of Δx,Δt and δ. Thus, using part I,II and III of the lemma, we get

$$\|u^{n+1} - u^n\|_1 + \|v^{n+1} - v^n\|_1 = O(\Delta t),$$

and then IV follows using the triangle inequality.

3.5. Proof of V; stability in L^1. The proof of the L^1 stability is quite similar to the proof of the TV-bound, and it is therefore omitted.

3.6. The discrete entropy inequalities. Finally we would like to show that the solutions of the difference scheme (2.6) satisfy an entropy inequality.

LEMMA 3.2. *Suppose that (u_j^0, v_j^0) satisfies (2.9), and let $\phi, \psi \in \mathcal{D}_+(T)$ for some $T > 0$. Furthermore, let N be a positive integer such that $t_N \leq T$, and let $E : [0,1] \to \mathcal{R}$ be a convex C^∞ entropy function with an associated entropy flux F, i.e. $F : [0,1] \to \mathcal{R}$ satisfies $F' = E'f'$. Then the following inequalities hold:*

$$\begin{aligned}
&\Delta t \sum_{n=0}^{N-1} \Delta x \sum_{j\in\mathcal{Z}} \left(\frac{\phi(x_j, t_{n+1}) - \phi(x_j, t_n)}{\Delta t} E(u_j^{n+1}) + \frac{\phi(x_{j+1}, t_n) - \phi(x_j, t_n)}{\Delta x} F(u_j^n) \right) \\
&\qquad + \Delta x \sum_{j\in\mathcal{Z}} \Big(E(u_j^0)\phi(x_j, 0) - E(u_j^N)\phi(x_j, t_N) \Big) \\
&\geq (\Delta t/\delta) \sum_{n=0}^{N-1} (\Delta x) \sum_{j\in\mathcal{Z}} E'(u_j^{n+1})\phi(x_j, t_n)(A(u_j^{n+1}) - v_j^{n+1})
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\Delta t \sum_{n=0}^{N-1} \Delta x \sum_{j\in\mathcal{Z}} \left(\frac{\psi(x_j, t_{n+1}) - \psi(x_j, t_n)}{\Delta t} E(v_j^{n+1}) \right) \\
&+ \Delta x \sum_{j\in\mathcal{Z}} \Big(E(v_j^0)\psi(x_j, 0) - E(v_j^N)\psi(x_j, t_N) \Big) \\
&\geq -(\Delta t/\delta) \sum_{n=0}^{N-1} (\Delta x) \sum_{j\in\mathcal{Z}} E'(v_j^{n+1})\psi(x_j, t_n)(A(u_j^{n+1}) - v_j^{n+1}).
\end{aligned} \tag{3.22}$$

Proof. It will be sufficient to establish the inequality (3.21) since (3.22) will follow by completely similar arguments. From the difference scheme (2.6) we observe that

$$\begin{aligned}
u_j^{n+1} - k &= u_j^n - k - \mu(f(u_j^n) - f(k)) + \mu(f(u_{j-1}^n) - f(k)) \\
&\quad -(\Delta t/\delta)(A(u_j^{n+1}) - v_j^{n+1}),
\end{aligned} \tag{3.23}$$

for any $k \in [0,1]$. By multiplying this equality by $\sigma(u_j^{n+1} - k)$, and by observing that the CFL-condition (2.8) implies that

$$|u_j^n - k - \mu(f(u_j^n) - f(k))| = |u_j^n - k| - \mu|f(u_j^n) - f(k)|,$$

we obtain

$$\begin{aligned}|u_j^{n+1}-k| \quad &\leq \quad |u_j^n-k|-\mu|f(u_j^n)-f(k)|+\mu|f(u_{j-1}^n)-f(k)| \\ &\qquad -(\Delta t/\delta)\sigma(u_j^{n+1}-k)(A(u_j^{n+1})-v_j^{n+1}).\end{aligned}$$

If we multiply this final inequality by $\phi(x_j,t_n)$, sum over $0 \leq n \leq N-1$ and $j \in \mathcal{Z}$, and apply summation by parts with respect to time and space the following inequality appears

$$\begin{aligned}&\Delta t \sum_{n=0}^{N-1} \Delta x \sum_{j\in\mathcal{Z}} \left(\frac{\phi(x_j,t_{n+1})-\phi(x_j,t_n)}{\Delta t}|u_j^{n+1}-k| + \frac{\phi(x_{j+1},t_n)-\phi(x_j,t_n)}{\Delta x}|f(u_j^n)-f(k)|\right) \\ &\qquad +\Delta x \sum_{j\in\mathcal{Z}} \left(|u_j^0-k|\phi(x_j,0)-|u_j^N-k|\phi(x_j,t_N)\right) \\ &\geq (\Delta t/\delta)\sum_{n=0}^{N-1}(\Delta x)\sum_{j\in\mathcal{Z}} \sigma(u_j^{n+1}-k)\phi(x_j,t_n)(A(u_j^{n+1})-v_j^{n+1}).\end{aligned} \tag{3.24}$$

Let now $E_m : I \to \mathcal{R}$ be a convex piecewise linear function of the form

$$E_m(u) = \beta_0(u-k_0) + \sum_{i=1}^{m} \beta_i|u-k_i| \tag{3.25}$$

where $\beta_i \geq 0$ for $i = 1, 2, \ldots, m$ and let F_m be the corresponding flux given by

$$F_m(u) = \beta_0(f(u)-f(k_0)) + \sum_{i=1}^{m} \beta_i|f(u)-f(k_i)|.$$

Observe that $F_m'(u) = E_m'(u)f'(u)$ and that the equality (3.23) and the inequality (3.24) imply that the desired inequality (3.21) holds if the smooth functions E and F are replaced by the polygonal functions E_m and F_m. However, for any smooth convex function E we can choose E_m of the form (3.25) such that E_m and E_m' converge uniformly on $[0,1]$, as $m \to \infty$, to E and E', respectively, and hence we obtain (3.21). □

4. Properties of the entropy solutions. In this section we shall conclude the proof of Theorem 2.1. The properties of the entropy solutions of the system (2.2) will be derived from the corresponding properties derived for the finite difference scheme (2.6) above. First, we will establish existence of entropy solutions for (2.2) by a limit argument. Thereafter, uniqueness and continuous dependence with respect to the initial data in L^1 is proved. From a proper application of Helly's theorem the following lemma can be derived, by standard arguments, from I-IV of Lemma 3.1 (cf. [16] or [23], chapter 16).

LEMMA 4.1. *Suppose that* (u^0, v^0) *satisfies the requirements (2.4) above. Then, as the mesh parameters* Δx *and* Δt *tend to zero, there is a subsequence* $\{(u_\Delta, v_\Delta)\}$ *of the family of approximate solutions generated by (2.6), converging in* $(L^1_{loc}(\mathcal{R} \times \mathcal{R}_0^+))^2$ *to a pair of functions* (u, v). *Furthermore,* $u(\cdot, t), v(\cdot, t) \in BV$, *for all* $t \geq 0, (u(x,t), v(x,t)) \in \mathcal{S}$ *for* $(x,t) \in \mathcal{R} \times \mathcal{R}_0^+$ *and the estimates*

$$TV(u(\cdot,t)) + TV(v(\cdot,t)) \leq TV(u^0) + TV(v^0), \tag{4.1}$$

$$\|p(\cdot,t)\|_1 \leq M\delta, \tag{4.2}$$

$$\|u(\cdot,t) - u(\cdot,\tau)\|_1 + \|v(\cdot,t) - v(\cdot,\tau)\|_1 \leq M|t - \tau| \tag{4.3}$$

hold, where $p(\cdot,t) = A(u(\cdot,t)) - v(\cdot,t)$ *and* M *is independent of* t *and* δ. □

Since the discrete solutions satisfy the entropy inequalities (3.21), (3.22) we also easily derive that the limit (u, v) is an entropy solution of (2.2).

LEMMA 4.2. *Assume that* (u^0, v^0) *satisfies the requirements (2.4), and let* (u, v) *be the pair of functions constructed in Lemma 4.1. Then* (u, v) *is an entropy solution of (2.2).*

Proof. We have to show that (u, v) satisfy the variational inequality (2.5) for any $\varphi, \psi \in \mathcal{D}_+(T)$ and $(k, q) \in \mathcal{S}$. However, by letting $\Delta t, \Delta x \to 0$ in (3.21) it follows that for any smooth entropy/entropy flux pair (E, F):

$$\begin{aligned} &\int_0^T \int_{\mathcal{R}} [E(u)\varphi_t + F(u)\varphi_x] dx\, dt \\ &+ \int_{\mathcal{R}} [E(u^0)\varphi(x,0) - E(u(x,T))\varphi(x,T)] dx \\ &\geq \frac{1}{\delta} \int_0^T \int_{\mathcal{R}} E'(u)\varphi(A(u) - v) dx\, dt. \end{aligned} \tag{4.4}$$

Hence, by choosing a sequence of smooth entropy/entropy flux pairs (E_θ, F_θ), such that, as $\theta \to 0$,

$$E_\theta(u) \to |u - k| \quad \text{and} \quad E'_\theta \to \sigma(u - k)$$

pointwise, the dominated convergence theorem implies that

$$\begin{aligned}&\int_0^T\int_{\mathcal{R}}[|u-k|\varphi_t+|f(u)-f(k)|\varphi_x]dx\,dt\\&+\int_{\mathcal{R}}[|u^0-k|\varphi(x,0)-|u(x,T)-k|\varphi(x,T)]dx\\&\geq\frac{1}{\delta}\int_0^T\int_{\mathcal{R}}\sigma(u-k)\varphi(A(u)-v)dx\,dt\end{aligned}$$

for all $k\in[0,1]$ and $\varphi\in\mathcal{D}_+(T)$. The rest of the desired inequality (2.5) can be derived from (3.22) by similar arguments. □

In order to show that the entropy solutions are unique we will use arguments inspired by Kruzkov [9], Kuznetsov [10] and Lucier [14]. For any $\theta\in(0,1]$ we introduce the mollifier function ω_θ on $\mathcal{R}$ given by

$$\omega_\theta(x)=\frac{1}{\theta}\Omega(x/\theta)$$

where $\Omega:\mathcal{R}\to\mathcal{R}$ is a nonnegative, symmetric C^∞-function with support in $[-1,1]$ and satisfying

$$\int_{\mathcal{R}}\Omega(x)dx=1.$$

Hence

$$\int_{\mathcal{R}}\omega_\theta(x)dx=1$$

and $\mathrm{supp}(\omega_\theta)\subset[-\theta,\theta]$. Furthermore, we define a smooth approximation σ_θ of the sign function σ by

$$\sigma_\theta(x)=-1+2\int_{-\infty}^x\omega_\theta(y)dy$$

and a smooth approximation μ_θ of the absolute value function by

$$\mu_\theta(x)=\theta+\int_{-\theta}^x\sigma_\theta(y)dy.$$

We observe that ${\mu_\theta}'=\sigma_\theta$ and ${\sigma_\theta}'=2\omega_\theta$. Also,

$$\sigma_\theta(x)=\sigma(x)\quad\text{for }|x|\geq\theta,$$

$$\mu_\theta(x)=|x|\quad\text{for }|x|\geq\theta$$

and

$$|\mu_\theta(x) - |x|| \leq \theta \quad \text{for all } x \in \mathcal{R}.$$

The next result shows that the entropy solutions of (2.2) are unique, and depend continuously, independently of δ, on the initial data in L^1.

LEMMA 4.3. *Let (u, v) and $(\bar{u}, \bar{v})$ be two entropy solutions of (2.2), with initial data (u^0, v^0) and $(\bar{u}^0, \bar{v}^0)$, respectively, satisfying the requirements (2.4). Then*

$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_1 + \|v(\cdot,t) - \bar{v}(\cdot,t)\|_1 \leq \|u^0 - \bar{u}^0\|_1 + \|v^0 - \bar{v}^0\|_1 \quad \textit{for all } t \geq 0.$$

Proof. The result follows by generalizing Kuznetsov's argument in [10] for scalar conservation laws. Let $T > 0$ be given and choose $(k, q) = (\bar{u}(y,\tau), \bar{v}(y,\tau))$ and $\varphi(x,t) = \psi(x,t) = \omega_\theta(x-y)\omega_\theta(t-\tau)$ in (2.5) for the solution (u, v). Integrating the result over $\mathcal{R} \times [0, T]$ with respect to y and τ we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathcal{R}} \int_0^T \int_{\mathcal{R}} (|u-\bar{u}| + |v-\bar{v}|)\omega_\theta{}'(t-\tau)\omega_\theta(x-y)dx\,dt\,dy\,d\tau \\
&+\int_0^T \int_{\mathcal{R}} \int_0^T \int_{\mathcal{R}} |f(u) - f(\bar{u})|\omega_\theta(t-\tau)\omega_\theta{}'(x-y)dx\,dt\,dy\,d\tau \\
&+\int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} [|u^0(x) - \bar{u}| + |v^0(x,t) - \bar{v}|]\omega_\theta(\tau)\omega_\theta(x-y)dx\,dy\,d\tau \\
&-\int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} [|u(x,T) - \bar{u}| + |v(x,T) - \bar{v}|]\omega_\theta(T-\tau)\omega_\theta(x-y)dx\,dy\,d\tau \\
&\geq \frac{1}{\delta}\int_0^T \int_{\mathcal{R}} \int_0^T \int_{\mathcal{R}} (\sigma(u-\bar{u}) - \sigma(v-\bar{v}))(A(u) - v)\omega_\theta(x-y)\omega_\theta(t-\tau)dx\,dt\,dy\,d\tau,
\end{aligned} \tag{4.5}$$

where $u = u(x,t)$, $v = v(x,t)$, $\bar{u} = \bar{u}(y,\tau)$ and $\bar{v} = \bar{v}(y,\tau)$. By performing a similar operation on the inequality (2.5) for the solution $(\bar{u}, \bar{v})$, but where we reverse the role of the variables (x,t) and (y,τ), and by adding the obtained inequality to (4.5) we observe that the terms which contain derivatives will cancel. Hence we obtain an inequality of the form

$$\begin{aligned}
&R(\theta) - L(\theta) \\
&\geq \frac{1}{\delta}\int_0^T \int_{\mathcal{R}} \int_0^T \int_{\mathcal{R}} (\sigma(u-\bar{u}) - \sigma(v-\bar{v}))(A(u) - A(\bar{u}) - (v-\bar{v}))\omega_\theta(x-y)\omega_\theta(t-\tau)dx\,dt\,dy\,d\tau
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned} R(\theta) &= \int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} (|u^0(x) - \bar{u}| + |v^0(x) - \bar{v}|)\omega_\theta(x-y)\omega_\theta(\tau) dx\, dy\, d\tau \\ &+ \int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} (|u - \bar{u}^0(y)| + |v - \bar{v}^0(y)|)\omega_\theta(x-y)\omega_\theta(t) dx\, dy\, dt \end{aligned}$$

and

$$\begin{aligned} L(\theta) &= \int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} (|u(x,T) - \bar{u}| + |v(x,T) - \bar{v}|)\omega_\theta(T-\tau)\omega_\theta(x-y) dx\, dy\, d\tau \\ &+ \int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} (|u - \bar{u}(y,T)| + |v - \bar{v}(y,T)|)\omega_\theta(T-t)\omega_\theta(x-y) dx\, dy\, dt. \end{aligned}$$

Due to the monotonicity of $A = A(u)$, cf. (2.3), the integrand on the right hand side of (4.6) is non-negative, and thus we get

$$L(\theta) \le R(\theta). \tag{4.7}$$

By using the properties (4.1) and (4.3) for both pair of solutions it follows easily (cf. e.g. [26]) that

$$|R(\theta) - (\|u^0 - \bar{u}^0\|_1 + \|v^0 - \bar{v}^0\|_1)| \le M\theta$$

and

$$|L(\theta) - (\|u(\cdot,T) - \bar{u}(\cdot,T)\|_1 + \|v(\cdot,T) - \bar{v}(\cdot,T)\|_1)| \le M\theta,$$

where M is independent of θ. Thus, the desired result follows from (4.7) by letting $\theta \to 0$. □

Remarks:

1) Note that Lemma 4.1, 4.2 and 4.3 prove Theorem 2.1. In particular, the uniqueness result of Lemma 4.3 implies that the complete sequence $\{(u_\Delta, v_\Delta)\}$ converges towards (u, v).

2) In the proof of Theorem 2.1 we have not used the assumption on A'' given in (2.3).

3) The stability result of Lemma 4.3 could also be proved by appealing to the similar property of the discrete scheme, cf. V of Lemma 3.1. But in order to prove both stability *and uniqueness* of the entropy solution, the argument presented above is needed. □

5. A regularized model. The purpose of the two final sections of this paper is to analyze the convergence, as δ tends to zero, of the solutions of the model (2.2) to the corresponding solutions of the equilibrium model given by (2.1). The main purpose of the discussion is to establish that the difference between these solutions, measured in L^1-norm, is bounded by $O(\delta^{1/3})$.

In order to study this convergence, a regularized model is introduced. For any $\epsilon, \delta \in (0,1]$ consider the pure initial value problem

$$\begin{aligned} &u_t^\epsilon + f(u^\epsilon)_x = -\frac{1}{\delta}(A(u^\epsilon) - v^\epsilon) + \epsilon u_{xx}^\epsilon, \\ &v_t^\epsilon = \frac{1}{\delta}(A(u^\epsilon) - v^\epsilon) + \epsilon v_{xx}^\epsilon, \\ &u^\epsilon(x,0) = u^{\epsilon,0}(x), \quad v^\epsilon(x,0) = v^{\epsilon,0}(x), \end{aligned} \tag{5.1}$$

where f and A are the given functions of u satisfying (2.3).

Since the system (5.1) contains proper diffusion terms, the solutions will in general be smooth functions of x and t. In order to state a suitable regularity result for the system we let $H^m(\mathcal{R})$, $m \ge 0$, denote the L^2-based Sobolev spaces of order m on $\mathcal{R}$ and we let $H^\infty(\mathcal{R}) = \bigcap_{m\ge 0} H^m(\mathcal{R})$.

Hence, $H^\infty(\mathcal{R})$ consists of all C^∞-functions with the property that any derivative is in $L^2(\mathcal{R})$. In particular, if $u \in H^\infty(\mathcal{R})$ then

$$\frac{\partial^j}{\partial x^j} u(x) \to 0 \quad \text{as } x \to \pm\infty \tag{5.2}$$

for any $j \ge 0$.

In order to avoid technical difficulties with respect to smoothness we will simply assume that the initial functions $u^{\epsilon,0}, v^{\epsilon,0}$ of (5.1) are in $H^\infty(\mathcal{R})$. The following regularity result can then be established.

LEMMA 5.1. *Assume that $(u^{\epsilon,0}(x), v^{\epsilon,0}(x)) \in \mathcal{S}$ for all $x \in \mathcal{R}$ and that $u^{\epsilon,0}, v^{\epsilon,0} \in H^\infty(\mathcal{R})$. Then there exists a unique classical solution (u^ϵ, v^ϵ) of (5.1) such that*

$$\frac{\partial^j u^\epsilon}{\partial t^j}(\cdot,t), \ \ \frac{\partial^j v^\epsilon}{\partial t^j}(\cdot,t) \in H^\infty(\mathcal{R})$$

for all $t \ge 0$ and integers $j \ge 0$. Furthermore,

$$(u^\epsilon(x,t), v^\epsilon(x,t)) \in \mathcal{S}$$

for all $x \in \mathcal{R}$, $t \geq 0$. $\square$

The regularity part of this lemma is rather standard and a proof can for example be found in [8] (cf. chapter 5 of [8]), while the invariant region part follows from the assumptions on A and the result of Chueh, Conley, Smoller [4], (cf. Theorem 4.4. of their paper).

In Section 3 we derived bounds, independently of the relaxation parameter δ and the discretization parameters Δx and Δt, for the total variation of the solution of the discrete scheme (2.6). We will need similar bounds for the solution of the regularized model (5.1), i.e. total variation bounds independent of the parameters ϵ and δ. We observe that if $u \in H^\infty(\mathcal{R})$ then $u \in BV$ if and only if $u_x \in L^1(\mathcal{R})$. Note that for these functions, $TV(u) = \|u_x\|_1$.

LEMMA 5.2. *Assume that* $(u^{\epsilon,0}(x), v^{\epsilon,0}(x)) \in \mathcal{S}$ *for all* $x \in \mathcal{R}$ *and that* $u^{\epsilon,0}, v^{\epsilon,0} \in H^\infty(\mathcal{R}) \bigcap BV$. *If* (u^ϵ, v^ϵ) *is the solution of (5.1) then* $u^\epsilon(\cdot,t), v^\epsilon(\cdot,t) \in H^\infty(\mathcal{R}) \bigcap BV$ *for all* $t \geq 0$ *and*

$$\|u_x^\epsilon(\cdot,t)\|_1 + \|v_x^\epsilon(\cdot,t)\|_1 \leq \|u_x^{\epsilon,0}\|_1 + \|v_x^{\epsilon,0}\|_1. \tag{5.3}$$

Furthermore, if the initial data is chosen such that $u_t^\epsilon(\cdot,0), v_t^\epsilon(\cdot,0) \in L^1(\mathcal{R})$ *then* $u_t^\epsilon(\cdot,t), v_t^\epsilon(\cdot,t) \in L^1(\mathcal{R})$ *for all* $t \geq 0$ *and*

$$\|u_t^\epsilon(\cdot,t)\|_1 + \|v_t^\epsilon(\cdot,t)\|_1 \leq \|u_t^\epsilon(\cdot,0)\|_1 + \|v_t^\epsilon(\cdot,0)\|_1. \tag{5.4}$$

Proof. For notational convenience we will throughout this proof denote the solution of (5.1) by (u, v), i.e. the explicit dependence on the regularization parameter ϵ is suppressed. Let $\mu_\theta : \mathcal{R} \to \mathcal{R}$, $\theta \in (0, 1]$, be the smooth approximation of the absolute value function introduced in Section 4. Hence, ${\mu_\theta}' = \sigma_\theta$, where σ_θ is the smooth approximation of the sign function, while

$${\mu_\theta}'' = {\sigma_\theta}' = 2\omega_\theta.$$

Consider now the system (5.1) and differentiate both equations with respect to x. We then obtain the system

$$(u_x)_t + (f'(u)u_x)_x = -\frac{1}{\delta}(A'(u)u_x - v_x) + \epsilon(u_x)_{xx},$$
$$(v_x)_t = \frac{1}{\delta}(A'(u)u_x - v_x) + \epsilon(v_x)_{xx}. \tag{5.5}$$

Furthermore, multiply the first equation of (5.5) by $\sigma_\theta(u_x)$, the second equation by $\sigma_\theta(v_x)$ and integrate over $(-M,M)\times(0,T)$ for $M,T>0$. If we add the results from the two equations we obtain

$$\begin{aligned}&\int_{-M}^{M}(\mu_\theta(u_x(x,T)) + \mu_\theta(v_x(x,T)))dx + \int_0^T\int_{-M}^{M}(f'(u)u_x)_x\sigma_\theta(u_x)dx\ dt = \\ &\int_{-M}^{M}(\mu_\theta(u_x^0(x)) + \mu_\theta(v_x^0(x)))dx - \frac{1}{\delta}\int_0^T\int_{-M}^{M}(A'(u)u_x - v_x)(\sigma_\theta(u_x) - \sigma_\theta(v_x))dx\ dt + \\ &\epsilon\int_0^T\int_{-M}^{M}(u_{xxx}\sigma_\theta(u_x) + v_{xxx}\sigma_\theta(v_x))dx\ dt.\end{aligned} \tag{5.6}$$

We observe that since $|\sigma_\theta(r)|\le 1$ and since $\sigma_\theta(r)\to\sigma(r)$ for all $r\in\mathcal{R}$ it follows from the dominated convergence theorem that

$$\begin{aligned}\lim_{\theta\to 0}\quad & -\int_0^T\int_{-M}^{M}(A'(u)u_x - v_x)(\sigma_\theta(u_x) - \sigma_\theta(v_x))dx\ dt \\ &= -\int_0^T\int_{-M}^{M}(A'(u)u_x - v_x)(\sigma(u_x) - \sigma(v_x))dx\ dt \le 0.\end{aligned} \tag{5.7}$$

By using integration by parts with respect to x we derive

$$\int_0^T\int_{-M}^{M}(f'(u)u_x)_x\sigma_\theta(u_x)dx\ dt = \int_0^T[f'(u)u_x\sigma_\theta(u_x)]\,|_{x=-M}^{M}dt - 2\int_0^T\int_{-M}^{M}f'(u)u_xu_{xx}\omega_\theta(u_x)dx\ dt$$

Since $r\omega_\theta(r)$ is uniformly bounded and since for any $r\in\mathcal{R}$, $\lim_{\theta\to 0} r\omega_\theta(r) = 0$, it follows again from the dominated convergence theorem that

$$\lim_{\theta\to 0}\int_0^T\int_{-M}^{M}f'(u)u_xu_{xx}\omega_\theta(u_x)dx\ dt = 0.$$

Hence, (5.2) implies that

$$\lim_{M\to\infty}\lim_{\theta\to 0}\int_0^T\int_{-M}^{M}(f'(u)u_x)_x\sigma_\theta(u_x)dx\ dt = 0. \tag{5.8}$$

Finally we observe that integration by parts with respect to x implies that

$$\int_0^T\int_{-M}^{M}(u_x)_{xx}\sigma_\theta(u_x)dx\ dt = \int_0^T u_{xx}\sigma_\theta(u_x)\,|_{x=-M}^{M}dt - 2\int_0^T\int_{-M}^{M}u_{xx}^2\omega_\theta(u_x)dx\ dt$$

Since $\omega_\theta(r) \geq 0$ and by treating the v-term in the same way we therefore obtain

$$\lim_{M\to\infty} \limsup_{\theta\to 0} \epsilon \int_0^T \int_{-M}^{M} ((u_x)_{xx}\sigma_\theta(u_x) + (v_x)_{xx}\sigma_\theta(v_x))dx\, dt \leq 0.$$

Together with (5.6), (5.7) and (5.8) this implies that

$$\lim_{M\to\infty} \int_{-M}^{M} (|u_x(x,T)| + |v_x(x,T)|)dx \leq \|u_x^0\|_1 + \|v_x^0\|_1$$

and hence $u(\cdot,T), v(\cdot,T) \in BV$ and (5.3) holds. The inequality (5.4) can be derived by a completely analogous argument by differentiating the system (5.1) with respect to t instead of x. The details of this derivation are therefore omitted. □

As above, we need to measure the deviation from equilibrium,

$$p^\epsilon = A(u^\epsilon) - v^\epsilon.$$

LEMMA 5.3. *Assume that the initial data $u^{\epsilon,0}, v^{\epsilon,0}$ of (5.1) satisfies all the assumptions given in Lemma 5.2. Furthermore, assume that $\|p^\epsilon(\cdot,0)\|_1 \leq M_0\delta$ and $\|p_x^\epsilon(\cdot,0)\|_1 \leq M_0\delta/\epsilon$, where M_0 is independent of ϵ and δ. Then there is a constant M, independent of ϵ and δ, such that*

$$\|p^\epsilon(\cdot,t)\|_1 \leq M\delta \tag{5.9}$$

and

$$\|p_x^\epsilon(\cdot,t)\|_1 \leq M\delta/\epsilon \tag{5.10}$$

for all $t \geq 0$.

Proof. As above we suppress the explicit dependence of ϵ throughout the proof. From the definition of the function p and from (5.1) it follows that

$$p_t = A'(u)u_t - v_t = A'(u)(\epsilon u_{xx} - \frac{1}{\delta}p - f(u)_x) - \epsilon v_{xx} - \frac{1}{\delta}p.$$

Since

$$p_{xx} = A'(u)u_{xx} + A''(u){u_x}^2 - v_{xx}$$

the expression above can be written

$$p_t = \epsilon p_{xx} - \frac{1}{\delta}(1 + A'(u))p - \epsilon A''(u){u_x}^2 - A'(u)f(u)_x. \tag{5.11}$$

Observe that if we multiply the first equation of (5.1) by $u(x,t)$ and integrate with respect to x we obtain

$$\epsilon\int_{\mathcal{R}} {u_x}^2(x,t)dx \leq 1/\delta\int_{\mathcal{R}} |p(x,t)||u(x,t)|dx + \int_{\mathcal{R}} |u_t(x,t)||u(x,t)|dx.$$

Hence the results of lemma 5.1 and 5.2 imply that there is a constant M, independent of ϵ, δ and t such that

$$\epsilon\int_{\mathcal{R}} {u_x}^2(x,t)dx \leq \frac{\|p(\cdot,t)\|_1}{\delta} + M.$$

If we multiply (5.11) by $\sigma_\theta(p)$, integrate with respect to x and let θ tend to zero we obtain

$$\frac{d}{dt}\|p(\cdot,t)\|_1 \leq M - \frac{1-\alpha}{\delta}\|p(\cdot,t)\|_1$$

where $0 \leq \alpha < 1$ (cf. (2.3)). Gronwall's lemma therefore implies that

$$\|p(\cdot,t)\|_1 \leq \exp(-(1-\alpha)t/\delta)\|p(\cdot,0)\|_1 + (M\delta/(1-\alpha))(1-\exp(-(1-\alpha)t/\delta) \leq \|p(\cdot,0)\|_1 + \frac{M}{1-\alpha}\delta.$$

We have therefore established (5.9).

In order to derive (5.10) we differentiate (5.11) with respect to x and obtain the equation

$$(p_x)_t = \epsilon(p_x)_{xx} - \frac{1}{\delta}(1 + A'(u))p_x + R \tag{5.12}$$

where $R = R(x,t)$ is given by

$$R = -\epsilon(A'''(u){u_x}^3 + 2A''(u)u_x u_{xx}) - \frac{1}{\delta}A''(u)u_x p - A''(u)f'(u){u_x}^2 - A'(u)(f'(u)u_{xx} - f''(u){u_x}^2).$$

Observe now that $\|u_t(\cdot,t)\|_1$, $\|u_x(\cdot,t)\|_1$ and $\frac{1}{\delta}\|p(\cdot,t)\|_1$ is bounded independently of ϵ, δ and t. Hence, the first equation of (5.1) implies that $\epsilon\|u_{xx}\|_1$ admits a similar bound and since

$$\lim_{x\to\pm\infty} u_x(x,t) = 0$$

we obtain

$$\|u_x(\cdot,t)\|_\infty \le M/\epsilon, \tag{5.13}$$

where M is independent of ϵ, δ and t. However, from (5.13) and the earlier bounds (5.3) and (5.9), we derive that

$$\|R(\cdot,t)\|_1 \le M/\epsilon \tag{5.14}$$

for all $t \ge 0$. Arguing as we did above we therefore obtain from (5.12) that

$$\frac{d}{dt}\|p_x(\cdot,t)\|_1 \le M/\epsilon - 1/\delta\|p_x(\cdot,t)\|_1$$

and hence (5.10) follows from Gronwall's Lemma. $\square$

6. The rate of convergence. The purpose of this final section of the paper is to complete the proof of Theorem 2.2. Let (u,v) be the entropy solution of (2.2) with initial data (u^0,v^0). As in Theorem 2.1 we assume that the initial data satisfies the requirements (2.4), i.e. $(u^0(x),v^0(x)) \in \mathcal{S}$ for $x \in \mathcal{R}$, $u^0, v^0 \in BV$,

$$\lim_{x\to\pm\infty} u^0(x) = \lim_{x\to\pm\infty} v^0(x) = 0$$

and that

$$\|p^0\|_1 = \|A(u^0) - v^0\|_1 \le M\delta,$$

where M is independent of δ. Furthermore, let w be the entropy solution of the equilibrium model (2.1) with $w(x,0) = u^0(x)$.

Our goal is to show that

$$\|u(\cdot,t) - w(\cdot,t)\|_1 \le M\delta^{1/3}, \quad \text{for } 0 \le t \le T, \tag{6.1}$$

where M is independent of δ.

In order to establish the desired estimate (6.1) we shall rely on properties of the regularized model (5.1). Throughout this section we assume that $\delta \le \epsilon$. In fact, at the end of this section we will choose $\epsilon = \delta^{2/3}$.

Let $\{u^{\epsilon,0}\}$ be a sequence of functions in $H^\infty(\mathcal{R})$ such that

$$\begin{aligned} &\|u^{\epsilon,0} - u^0\|_1 \le M\epsilon, \\ &TV(u^{\epsilon,0}) \le TV(u^0), \qquad (6.2) \\ &\|u^{\epsilon,0}_{xx}\|_1 \le M/\epsilon, \end{aligned}$$

where the constant M is independent of ϵ and δ. Such a sequence can be constructed by a standard averaging procedure using the mollifier function ω_ϵ. Furthermore, let

$$v^{\epsilon,0} = A(u^{\epsilon,0}), \qquad (6.3)$$

then $p^{\epsilon,0} \equiv 0$ and

$$\begin{aligned} &(u^{\epsilon,0}(x), v^{\epsilon,0}(x)) \in \mathcal{S} \quad \text{for all } x \in \mathcal{R}, \\ &\|v^{\epsilon,0} - v^0\|_1 \le M\epsilon, \qquad (6.4) \\ &TV(v^{\epsilon,0}) \le M, \\ &\|v^{\epsilon,0}_{xx}\|_1 \le M/\epsilon. \end{aligned}$$

Let (u^ϵ, v^ϵ) denote the solution of the system (5.1) corresponding to the initial data $(u^{\epsilon,0}, v^{\epsilon,0})$. We observe that (6.2) and (6.4) imply that $\|u^\epsilon_t(\cdot,0)\|_1$ and $\|u^\epsilon_t(\cdot,0)\|_1$ is bounded independently of ϵ and δ. Hence, the initial data $(u^{\epsilon,0}, v^{\epsilon,0})$ satisfies all the assumption of the lemma 5.1-5.3.

LEMMA 6.1. *Let $u^0, v^0, \{u^{\epsilon,0}\}, \{v^{\epsilon,0}\}$ satisfy (2.4),(6.2) and (6.3). Then, for any $T > 0$ there is a constant M, independent of ϵ and δ such that*

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\|_1 + \|v^\epsilon(\cdot,t) - v(\cdot,t)\|_1 \le M\epsilon^{1/2}, \quad \textit{for } 0 \le t \le T.$$

Proof. The proof is based on the characterization (2.5) of the entropy solution of (2.2) and a corresponding variational inequality for the regularized system (5.1). We first observe that for any $(k,q) \in \mathcal{S}$ the following system holds

$$\begin{aligned} &(u^\epsilon - k)_t + (f(u^\epsilon) - f(k))_x = -\frac{1}{\delta}(A(u^\epsilon) - v^\epsilon) + \epsilon u^\epsilon_{xx}, \qquad (6.5) \\ &(v^\epsilon - q)_t = \frac{1}{\delta}(A(u^\epsilon) - v^\epsilon) + \epsilon v^\epsilon_{xx}. \end{aligned}$$

Let $\varphi, \psi \in \mathcal{D}_+(T)$. For any $\theta \in (0,1)$ multiply the first equation of (6.5) by $\sigma_\theta(u^\epsilon - k)\varphi$, the second equation by $\sigma_\theta(v^\epsilon - k)\psi$, add the two equations and integrate over $\mathcal{R}\times(0,T)$. We then obtain

$$
\begin{aligned}
&\int_0^T\int_{\mathcal{R}}[\mu_\theta(u^\epsilon - k)\varphi_t + F_\theta(u^\epsilon,k)\varphi_x + \mu_\theta(v^\epsilon - q)\psi_t]dx\, dt\\
&+\int_{\mathcal{R}}[\mu_\theta(u^{\epsilon,0}(x) - k)\varphi(x,0) + \mu_\theta(v^{\epsilon,0}(x) - q)\psi(x,0)]dx\\
&-\int_{\mathcal{R}}[\mu_\theta(u^\epsilon(x,T) - k)\varphi(x,T) + \mu_\theta(v^\epsilon(x,T) - q)\psi(x,T)]dx\\
&=\frac{1}{\delta}\int_0^T\int_{\mathcal{R}}(\sigma_\theta(u^\epsilon - k)\varphi - \sigma_\theta(v^\epsilon - q)\psi)(A(u^\epsilon) - v^\epsilon)dx\, dt\\
&-\epsilon\int_0^T\int_{\mathcal{R}}[\sigma_\theta(u^\epsilon - k)\varphi u^\epsilon_{xx} + \sigma_\theta(v^\epsilon - q)\psi v^\epsilon_{xx}]dx\, dt,
\end{aligned}
\tag{6.6}
$$

where the flux-function $F_\theta(u,k)$ satisfies

$$\frac{dF_\theta(u,k)}{du} = \sigma_\theta(u-k)f'(u), \quad F_\theta(k,k) = 0.$$

We observe that

$$
\begin{aligned}
&-\epsilon\int_0^T\int_{\mathcal{R}}\sigma_\theta(u^\epsilon - k)u^\epsilon_{xx}\varphi dx\, dt\\
&= -\epsilon\int_0^T\int_{\mathcal{R}}(\sigma_\theta(u^\epsilon - k)u^\epsilon_x)_x\varphi dx\, dt + 2\epsilon\int_0^T\int_{\mathcal{R}}\omega_\theta(u^\epsilon - k)(u^\epsilon_x)^2\varphi dx\, dt\\
&\geq \epsilon\int_0^T\int_{\mathcal{R}}\sigma_\theta(u^\epsilon - k)u^\epsilon_x\varphi_x dx\, dt,
\end{aligned}
$$

and that a similar inequality can be derived for the corresponding v-term. By letting $\theta \to 0$ in (6.6) we therefore obtain that for any $(k,q) \in \mathcal{S}$ and any $\varphi, \psi \in \mathcal{D}_+(T)$ the following inequality holds

$$
\begin{aligned}
&\int_0^T\int_{\mathcal{R}}[|u^\epsilon - k|\varphi_t + |f(u^\epsilon) - f(k)|\varphi_x + |v^\epsilon - q|\psi_t]dx\, dt\\
&+\int_{\mathcal{R}}[|u^{\epsilon,0} - k|\varphi(x,0) + |v^{\epsilon,0} - q|\psi(x,0)]dx\\
&-\int_{\mathcal{R}}[|u^\epsilon(x,T) - k|\varphi(x,T) + |v^\epsilon(x,T) - q|\psi(x,T)]dx\\
&\geq \frac{1}{\delta}\int_0^T\int_{\mathcal{R}}(\sigma(u^\epsilon - k)\varphi - \sigma(v^\epsilon - q)\psi)(A(u^\epsilon) - v^\epsilon)dx\, dt\\
&+\epsilon\int_0^T\int_{\mathcal{R}}[\sigma(u^\epsilon - k)\varphi_x u^\epsilon_x + \sigma(v^\epsilon - q)\psi_x v^\epsilon_x]dx\, dt.
\end{aligned}
\tag{6.7}
$$

This variational inequality should be compared with the characterization (2.5) of the entropy solution of (2.2).

Let now $k = u(y,\tau)$, $q = v(y,\tau)$ and $\varphi(x,t) = \psi(x,t) = \omega_\theta(x-y)\omega_\theta(t-\tau)$ in (6.7), where $(y,\tau) \in \mathcal{R} \times (0,T)$ is fixed, and integrate the result over $\mathcal{R} \times (0,T)$ with respect to y and τ. Consider similarly (2.5), but where we use (y,τ) as integration variables instead of x and t. Then take $k = u^\epsilon(x,t)$, $q = v^\epsilon(y,\tau)$ and

$$\varphi(y,\tau) = \psi(y,\tau) = \omega_\theta(x-y)\omega_\theta(t-\tau).$$

If we add the two inequalities we then obtain

$$L(\epsilon,\theta) \le R(\epsilon,\theta) - \frac{1}{\delta} I_1(\epsilon,\theta) - \epsilon I_2(\epsilon,\theta). \tag{6.8}$$

Here,

$$\begin{aligned}
L(\epsilon,\theta) &= \int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} (|u^\epsilon(x,T) - u| + |v^\epsilon(x,T) - v|)\omega_\theta(x-y)\omega_\theta(T-\tau)dx\,dy\,d\tau \\
&+ \int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} (|u^\epsilon - u(y,T)| + |v^\epsilon - v(y,T)|)\omega_\theta(x-y)\omega_\theta(T-t)dx\,dy\,dt \\
R(\epsilon,\theta) &= \int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} (|u^{\epsilon,0}(x) - u| + |v^{\epsilon,0}(x) - v|)\omega_\theta(x-y)\omega_\theta(T)dx\,dy\,d\tau \\
&+ \int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} (|u^\epsilon - u^0(y)| + |v^\epsilon - v^0(y)|)\omega_\theta(x-y)\omega_\theta(t)dx\,dy\,dt \\
I_1(\epsilon,\theta) &= \int_0^T \int_{\mathcal{R}} \int_0^T \int_{\mathcal{R}} [(\sigma(u^\epsilon - u))(A(u^\epsilon) - A(u)) + |v^\epsilon - v|]dx\,dt\,dy\,d\tau \\
I_2(\epsilon,\theta) &= \int_0^T \int_{\mathcal{R}} \int_0^T \int_{\mathcal{R}} [(\sigma(u^\epsilon - u)u_x^\epsilon + \sigma(v^\epsilon - v)v_x^\epsilon)\omega_\theta{}'(x-y)\omega_\theta(t-\tau)]dx\,dt\,dy\,d\tau,
\end{aligned}$$

where $u^\epsilon = u^\epsilon(x,t)$, $v^\epsilon = v^\epsilon(x,t)$, $u = u(y,\tau)$ and $v = v(y,\tau)$.

From the total variation bounds and the L^1-continuity properties of the solutions (u^ϵ, v^ϵ) and (u,v) we obtain by standard argument (cf. e.g. [26]) that there is a constant M, independent of ϵ, δ and θ, such that

$$\begin{aligned}
&|L(\epsilon,\theta) - \|u^\epsilon(\cdot,T) - u(\cdot,T)\|_1 + \|v^\epsilon(\cdot,T) - v(\cdot,T)\|_1| \le M\theta \\
&|R(\epsilon,\theta) - \|u^{\epsilon,0} - u^0\|_1 + \|v^{\epsilon,0} - v^0\|_1| \le M\theta.
\end{aligned} \tag{6.9}$$

Furthermore, the monotonicity property of A implies that $I_1(\epsilon,\theta) \ge 0$. Finally, from the properties of (u^ϵ, v^ϵ) we obtain that

$$|I_2(\epsilon,\theta)| \le T \sup_{0\le t\le T} (\|u_x^\epsilon(\cdot,t)\|_1 + \|v_x^\epsilon(\cdot,t)\|_1) \int_{\mathcal{R}} |\omega_\theta{}'(y)|dy \le M/\theta.$$

Hence, together with (6.2), (6.4), (6.8) and (6.9) this implies that

$$\|u^\epsilon(\cdot,T) - u(\cdot,T)\|_1 + \|v^\epsilon(\cdot,T) - v(\cdot,T)\|_1 \le M(\epsilon + \theta + \epsilon/\theta).$$

The desired result now follows by choosing $\theta = \epsilon^{1/2}$. □

In addition to the system studied above we shall also need a regularized version of the equilibrium model. Consider the pure initial value problem

$$\begin{aligned}(w^\epsilon + A(w^\epsilon))_t + f(w^\epsilon)_x &= \epsilon(w^\epsilon + A(w^\epsilon))_{xx} \\ w^\epsilon(x,0) &= u^{\epsilon,0}(x),\end{aligned} \tag{6.10}$$

where $\{u^{\epsilon,0}\} \subset H^\infty(\mathcal{R})$ denotes the initial functions introduced at the beginning of this section. From the properties of $\{u^{\epsilon,0}\}$ and u^0 given above it follows from standard theory for scalar equations (cf. e.g. [10]) that for any $T > 0$ there is a constant M, independent of ϵ, such that

$$\|w(\cdot,t) - w^\epsilon(\cdot,t)\|_1 \le M\epsilon^{1/2}, \quad \text{for } 0 \le t \le T. \tag{6.11}$$

Hence, if we write the error $u - w$ in the form

$$u - w = (u - u^\epsilon) + (u^\epsilon - w^\epsilon) + (w^\epsilon - w),$$

it remains to estimate the term $u^\epsilon - w^\epsilon$. In order to estimate this final error term we shall rely on properties of the system (5.1) derived in the previous section and on well known properties of scalar diffusion equations of the form (6.10). Consider a scalar equation of the form

$$\begin{aligned}(z + A(z))_t + f(z)_x &= \epsilon(z + A(z))_{xx} + G \\ z(x,0) &= z^0(x),\end{aligned} \tag{6.12}$$

where $z^0 = z^0(x)$ represents the initial function and $G = G(x,t)$ is a given source term. It is well known that if the data z^0 and G are smooth, then the problem (6.12) has a unique smooth solution. Furthermore, if z and $\hat{z}$ denote two solutions of (6.12), with data (z^0, G) and $(\hat{z}^0, \hat{G})$, respectively, then we have the estimate

$$\|z(\cdot,t) - \hat{z}(\cdot,t)\|_1 \le M_A(\|z^0 - \hat{z}^0\|_1 + \int_0^t \|G - \hat{G}\|_1 ds), \tag{6.13}$$

where the constant M_A only depends on the function A. In fact, this estimate can easily be proved by using arguments similar to those used in the proofs of lemma 5.2-5.3. Using this property of the equation (6.12) and the properties of the system (5.1), derived in section 5, we now establish the following estimate

LEMMA 6.2. *Let* $\{u^{\epsilon,0}\}, \{v^{\epsilon,0}\}$ *satisfy the requirements of Lemma 6.1. Then, for any* $T > 0$ *there is a constant* M*, independent of* ϵ *and* δ*, such that*

$$\|u^\epsilon(\cdot,t) - w^\epsilon(\cdot,t)\|_1 \leq M\delta/\epsilon, \quad \text{for } 0 \leq t \leq T.$$

Proof. Consider first the system (5.1) and as above we let

$$p^\epsilon = A(u^\epsilon) - v^\epsilon.$$

By adding the two equations of (5.1) we obtain

$$(u^\epsilon + v^\epsilon)_t + f(u^\epsilon)_x = \epsilon(u^\epsilon + v^\epsilon)_{xx}$$

or

$$(u^\epsilon + A(u^\epsilon) - p^\epsilon)_t + f(u^\epsilon)_x = \epsilon(u^\epsilon + A(u^\epsilon) - p^\epsilon)_{xx}. \tag{6.14}$$

Introduce a new function $z^\epsilon(x,t)$ defined by

$$z^\epsilon + A(z^\epsilon) = u^\epsilon + A(u^\epsilon) - p^\epsilon. \tag{6.15}$$

Then the differential equation (6.14) can be written in the form (6.12), i.e.

$$(z^\epsilon + A(z^\epsilon))_t + f(z^\epsilon)_x = \epsilon(z^\epsilon + A(z^\epsilon))_{xx} + G, \tag{6.16}$$

where

$$G = f(z^\epsilon)_x - f(u^\epsilon)_x.$$

From the monotonicity property of A it follows that

$$|z^\epsilon(x,t) - u^\epsilon(x,t)| \leq |p^\epsilon(x,t)|.$$

Hence it follows from (6.2) and Lemma 5.3 that

$$\|z^\epsilon(\cdot,t) - u^\epsilon(\cdot,t)\|_1 \le M\delta, \quad 0 \le t \le T. \tag{6.17}$$

By differentiating (6.15) with respect to x we obtain

$$z_x^\epsilon - u_x^\epsilon + A'(z_x^\epsilon)(z_x^\epsilon - u_x^\epsilon) = (A'(u^\epsilon) - A'(z^\epsilon))u_x^\epsilon - p_x^\epsilon$$

and this implies that

$$\|z_x^\epsilon(\cdot,t) - u_x^\epsilon(\cdot,t)\|_1 \le M_A\|u^\epsilon(\cdot,t) - z^\epsilon(\cdot,t)\|_1\|u_x^\epsilon(\cdot,t)\|_\infty + \|p_x^\epsilon(\cdot,t)\|_1.$$

From the estimates (5.10) (5.13), (6.2) and (6.17) we therefore have

$$\|z_x^\epsilon(\cdot,t) - u_x^\epsilon(\cdot,t)\|_1 \le M\delta/\epsilon, \quad \text{for } 0 \le t \le T, \tag{6.18}$$

where M is independent of ϵ and δ. Observe now that the source term G introduced above can be represented in the form

$$G = f'(z^\epsilon)(z_x^\epsilon - u_x^\epsilon) + (f'(z^\epsilon) - f'(u^\epsilon))u_x^\epsilon$$

and since $\|u_x^\epsilon(\cdot,t)\|_1$ is uniformly bounded on $[0,T]$ and since

$$\|z^\epsilon(\cdot,t) - u^\epsilon(\cdot,t)\|_\infty \le \|z_x^\epsilon(\cdot,t) - u_x^\epsilon(\cdot,t)\|_1$$

it follows from (6.18) that

$$\|G(\cdot,t)\|_1 \le M\delta/\epsilon, \quad \text{for } 0 \le t \le T. \tag{6.19}$$

Hence, the stability estimate (6.13) implies that for $0 \le t \le T$

$$\|z^\epsilon(\cdot,t) - w^\epsilon(\cdot,t)\|_1 \le M(\|z^\epsilon(\cdot,0) - u^{\epsilon,0}\|_1 + \delta/\epsilon) \le M\delta/\epsilon. \tag{6.20}$$

The desired estimate now follows from (6.17), (6.20) and the triangle inequality. □

Proof. [of Theorem 2.2] We can now easily complete the proof of Theorem 2.2. By writing

$$u - w = (u - u^\epsilon) + (u^\epsilon - w^\epsilon) + (w^\epsilon - w)$$

we obtain from Lemma 6.1 and 6.2 and the estimate (6.11) that for $0 \leq t \leq T$

$$\|u(\cdot,t) - w(\cdot,t)\|_1 \leq M(\epsilon^{1/2} + \delta/\epsilon).$$

By choosing $\epsilon = \delta^{2/3}$ the desired estimate follows. $\square$

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