AN L^1–ERROR BOUND FOR A SEMIIMPLICIT DIFFERENCE SCHEME APPLIED TO A STIFF SYSTEM OF CONSERVATION LAWS

HANS JOACHIM SCHROLL *, ASLAK TVEITO ** AND RAGNAR WINTHER **

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Abstract. A straightforward semi-implicit finite difference method approximating a system of conservation laws including a stiff relaxation term is analyzed. We show that the error, measured in L^1, is bounded by $O(\sqrt{\Delta t})$ independent of the stiffness, where the time step Δt represents the mesh size. As a simple corollary we obtain that solutions of the stiff system converge towards the solution of an equilibrium model at a rate of $O(\delta^{1/3})$ in L^1 as the relaxation time δ tends to zero.

Key words. Stiff hyperbolic conservation law, error estimate, relaxation term, non-equilibrium

AMS(MOS) subject classifications. 35L65 35L45 65M06 65M15

1. Introduction. The purpose of this paper is to study a finite difference approximation of the following system of conservation laws

$$\begin{aligned} u_t + f(u)_x &= -\frac{1}{\delta}\, g(u,v), \\ v_t &= \frac{1}{\delta}\, g(u,v). \end{aligned} \tag{1.1}$$

The source term is defined by

$$g(u,v) := A(u) - v. \tag{1.2}$$

Here A and f are given increasing functions of u. The variables u and v are the unknowns and $\delta > 0$ is referred to as the relaxation time. Further assumptions on the model will be given in Section 2.

The system (1.1) arises in chromatography and is discussed in the books [13, 14, 19]. In this framework, u denotes the density of some species contained in a fluid flowing through a fixed bed, and v denotes the density of the species adsorbed on the material in the bed. The source term g models the adsorption.

The above system is a stiff system of conservation laws in the sense of Pember [11], [12], that is: The relaxation time δ is small compared to the time scale determined by the characteristic speeds of the system and some appropriate length scale.

Systems consisting of one conservation equation and one equation with a relaxation term have been studied by several authors, cf. [1, 7, 8, 10, 17] and references therein. Existence and uniqueness for systems of the form (1.1) are proved in [7, 16, 17].

* Numerische Mathematik, RWTH–Aachen, Templergraben 55, D-52056 Aachen.
This author has been supported by The Norwegian Research Council (NFR), program no. 100284/431.
** Department of Informatics, P. O. Box 1080 Blindern, University of Oslo, Norway.
This author has been supported by The Norwegian Research Council (NFR), program no. 100284/431 and STP.29643.
e-mail: schroll@igpm.rwth-aachen.de, aslak@ifi.uio.no, ragnar@ifi.uio.no

Computational studies of such systems are presented by e.g. Colella, Majda and Roytburd [2] and by Pember [11, 12].

In [17] it was observed, that one of Pembers' models (cf. "model II" of [11]) is a system of the form (1.1) and (1.2). Thus our results apply to Pembers' model.

The system (1.1) satisfies the so-called subcharacteristic condition, which relates the non-equilibrium system to the corresponding equilibrium system:

$$(w + A(w))_t + f(w)_x = 0. \tag{1.3}$$

Let $\lambda_1 = 0$ and $\lambda_2 = f'$ denote the characteristic speeds of (1.1) and $\lambda^* = f'/(1 + A')$ the characteristic speed of (1.3). Then, since A is increasing we have $\lambda_1 \leq \lambda^* \leq \lambda_2$, which is called the subcharacteristic condition. Liu [8], Chen and Liu [1] and LeVeque and Wang [5] observed that the subcharacteristic condition is necessary for the stability of the non-equilibrium model.

The particular finite difference scheme that will be analyzed in this paper is a straightforward semi-implicit method:

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{f(u_j^n) - f(u_{j-1}^n)}{\Delta x} &= -\frac{1}{\delta}\, g(u_j^{n+1}, v_j^{n+1}), \\ \frac{v_j^{n+1} - v_j^n}{\Delta t} &= \frac{1}{\delta}\, g(u_j^{n+1}, v_j^{n+1}). \end{aligned} \tag{1.4}$$

Exactly the same scheme was used in [17] to analyze the system (1.1). Based on proper TV-bounds on the approximate solutions, independent of δ, convergence of a subsequence of $\{u_j^n, v_j^n\}$ to the unique entropy solution of (1.1) was established by standard compactness arguments. However, the main interest of [17] was to study the relation between the model (1.1) and the corresponding equilibrium model (1.3).

The main result of this paper is to improve the convergence result for the difference scheme (1.4) by establishing an error bound of order $\sqrt{\Delta t}$ in L^1, independent of the relaxation time δ. To the knowledge of the authors, this is the first error bound for a finite difference scheme applied to a stiff system of conservation laws. The result is based on the observation in [17], that solutions which are initially close to equilibrium stay close to equilibrium for all time.

In [17] we proved that the solutions of the non-equilibrium model (1.1) converge towards the solution of the equilibrium model (1.3) at a rate of $O(\delta^{1/3})$ in L^1. This result was proved by introducing a parabolic regularization. In Section 6 of the present paper we adress this issue again and show that the rate 1/3 can also be obtained from the error estimate of order $\sqrt{\Delta t}$ of this paper. Thus we give an alternative proof of convergence towards equilibrium that does not rely on a parabolic regularization.

The results of this paper rely on the monotonicity of the right hand side g, which is expressed in (2.2) below. The present source term allows only one unique equilibrium state $v = A(u)$. It is well known that more general stiff relaxation terms can cause problems in computing the entropy solution cf. [6, 15, 11, 12].

The paper is organized as follows: In Section 2 we state the assumptions on the model (1.1), introduce some notation and recall properties of the solutions of the PDE and the scheme. Furthermore, the main result is stated. Section 3 prepares the proof of the main result. A comparison function is introduced and an entropy formulation of the scheme is developed. In Section 4 we prove the main result and Section 5 contains the proof of an auxiliary lemma. Finally, Section 6 is devoted to the convergence to equilibrium as $\delta \to 0$.

2. Preliminaries and statement of the main result. Well-posedness of the Cauchy problem for (1.1) and (1.2) was proved in [17] under the following conditions:

$$\begin{aligned} &A \text{ and } f \text{ are smooth functions,} \\ &f(0) = 0, \quad f'(u) \geq 0, \\ &A(0) = 0, \quad A(1) = 1, \quad A'(u) \geq 0, \text{ for } u \in [0,1]. \end{aligned} \tag{2.1}$$

In [17] an additional condition on A'' was used to estimate the rate of convergence to equilibrium. This condition is not needed here. In Section 6 we prove the result in [17] under the milder conditions (2.1).

The following property of the source term is a consequence of the monotonicity of A and is fundamental in the proof below. For all (u,v), $(\bar{u},\bar{v}) \in \mathcal{S}$ we have:

$$[\sigma(u-\bar{u}) - \sigma(v-\bar{v})][g(u,v) - g(\bar{u},\bar{v})] \geq 0. \tag{2.2}$$

Here σ denotes the sign function.

The initial conditions (u^0, v^0) are supposed to satisfy:

$$\begin{aligned} \text{i)} &\quad (u^0(x), v^0(x)) \in \mathcal{S} \quad \forall x \in \mathcal{R}, \\ \text{ii)} &\quad TV(u^0) + TV(v^0) \leq M, \\ \text{iii)} &\quad \|g(u^0,v^0)\|_1 \leq M\delta, \\ \text{iv)} &\quad u^0(\pm\infty) = v^0(\pm\infty) = 0. \end{aligned} \tag{2.3}$$

Here, and in the rest of this paper, $\mathcal{S} := [0,1]^2$ is the state space and M denotes a generic finite constant independent of δ. The L^1-norm is denoted by $\|\cdot\|_1$ and $TV(\cdot)$ denotes the total-variation, defined by

$$TV(z) := \sup_{h \neq 0} \int_{\mathcal{R}} \frac{|z(x+h) - z(x)|}{|h|} dx.$$

The subset of L^1_{loc} consisting of functions with bounded total variation will be denoted by BV.

We recall that the above definition of the total variation is equivalent to the more classical definition used by Kuznetsov [4]. In fact we have for all $z \in BV$

$$TV(z) = \operatorname{ess\,sup} \sum_j |z(x_j) - z(x_{j-1})|,$$

where the supremum to the set of all finite increasing sequences $\{x_j\}$. Furthermore we define the total variation on a interval $J \subset \mathcal{R}$, $TV(z)|_J$, by restricting the supremum to all finite increasing sequences $\{x_j\}$ in J.

Based on the assumptions (2.1) and (2.3), we know ([17], Theorem 2.1) that there exists a unique entropy solution of (1.1) and (1.2) with initial data (u^0, v^0). This solution is defined by the following weak entropy formulation (cf. [17] and [3]):

$$
\begin{aligned}
&\int_0^T \int_{\mathcal{R}} [|u-k|\varphi_t + |f(u)-f(k)|\varphi_x + |v-q|\psi_t] dx\, dt \\
&+\int_{\mathcal{R}} [|u^0-k|\varphi(x,0) + |v^0-q|\psi(x,0)] dx \\
&-\int_{\mathcal{R}} [|u(x,T)-k|\varphi(x,T) + |v(x,T)-q|\psi(x,T)] dx \\
&\geq \frac{1}{\delta}\int_0^T \int_{\mathcal{R}} [\sigma(u-k)\varphi - \sigma(v-q)\psi] g(u,v) dx\, dt,
\end{aligned} \tag{2.4}
$$

for any $T > 0, (k,q) \in \mathcal{S}$ and any $\varphi, \psi \in \mathcal{D}_+(T)$.

Here $\mathcal{D}_+(T)$ is the set of all nonnegative C^∞-functions with compact support in $\mathcal{R} \times [0,T]$. Furthermore, the solution has the properties (cf. [17]):

$$
\begin{array}{rl}
\text{i)} & (u,v) \in \mathcal{S} \quad \forall (x,t) \in \mathcal{R} \times \mathcal{R}_0^+, \\
\text{ii)} & TV(u(\cdot,t)), TV(v(\cdot,t)) \leq M \quad \forall t \in \mathcal{R}_0^+, \\
\text{iii)} & \|g(u(\cdot,t), v(\cdot,t))\|_1 \leq M\delta \quad \forall t \in \mathcal{R}_0^+, \\
\text{iv)} & \|u(\cdot,t) - u(\cdot,\tau)\|_1 + \|v(\cdot,t) - v(\cdot,\tau)\|_1 \leq M|t-\tau| \quad \forall t,\tau \in \mathcal{R}_0^+.
\end{array} \tag{2.5}
$$

These are derived from the corresponding estimates for solutions of the scheme.

Using grid sizes Δt and Δx in t and x direction we define cell boundaries $x_{j+1/2} := (j+1/2)\Delta x$ and computational cells $I_j := [x_{j-1/2}, x_{j+1/2})$. The discrete values (u_j^n, v_j^n) computed from (1.4) are considered as approximations of (u,v) in the cell I_j, at time level $t_n = n\Delta t$. The initial data for the scheme is taken as cell averages of u^0 and v^0

$$
u_j^0 = \frac{1}{\Delta x}\int_{I_j} u^0(x)\, dx \quad \text{and} \quad v_j^0 = \frac{1}{\Delta x}\int_{I_j} v^0(x)\, dx. \tag{2.6}
$$

Assuming a CFL-condition

$$
\mu f'(u) \leq 1 \text{ for } u \in [0,1], \tag{2.7}
$$

where $\mu = \Delta t/\Delta x$, the implicit system (1.4) has a unique solution satisfying similar bounds as (2.5) (cf. [17], Lemma 3.1):

$$
\begin{array}{rl}
\text{i)} & (u_j^n, v_j^n) \in \mathcal{S} \quad \forall (j,n) \in \mathcal{Z} \times \mathcal{Z}_0^+, \\
\text{ii)} & TV(u^n), TV(v^n) \leq M \quad \forall n \in \mathcal{Z}_0^+, \\
\text{iii)} & \|g(u^n, v^n)\|_1 \leq M\delta \quad \forall n \in \mathcal{Z}_0^+, \\
\text{iv)} & \|u^n - u^m\|_1 + \|v^n - v^m\|_1 \leq M\Delta t|n-m| \quad \forall n,m \in \mathcal{Z}_0^+.
\end{array} \tag{2.8}
$$

For grid functions the total variation is defined by

$$
TV(u) = \sum_{j\in\mathcal{Z}} |u_j - u_{j-1}|
$$

and $\|\cdot\|_1$ denotes the discrete L^1 norm

$$\|u\|_1 = \Delta x \sum_{j\in\mathcal{Z}} |u_j|.$$

Now, we are in the position to state the main result:

THEOREM 2.1. *Let (u^0, v^0) be initial data satisfying (2.3). Define (u_j^0, v_j^0) as cell averages by (2.6) and consider grid parameters satisfying the CFL-condition (2.7). Let (u, v) be the entropy solution of (1.1) and (u^N, v^N) a piecewise constant representation of the data $\{u_j^N, v_j^N\}$ generated by the scheme (1.4).*

Then for any fixed $T = N\Delta t > 0$ there is a finite constant M independent of Δt, Δx and δ such that

$$\|u(\cdot, T) - u^N\|_1 + \|v(\cdot, T) - v^N\|_1 \le M\sqrt{\Delta t}. \tag{2.9}$$

¿From the computational point of view it is important to note, that due to the implicit treatment of the stiff source term the stability condition (2.7) does not depend on the relaxation time δ. Hence, the system can be solved numerically as efficient as a nonstiff system of hyperbolic conservation laws. The authors find this feature quite remarkable, although it mimics similar well known properties from stiff ordinary differential equations.

3. Entropy formulation of the scheme. In order to compare the approximation generated by the scheme to the entropy solution of (1.1), we define a comparison function $(\bar{u}_\Delta, \bar{v}_\Delta)$ on $\mathcal{R}\times\mathcal{R}_0^+$ which is discontinuous at the discrete time levels t_n and interpolates the sequence $\{u_j^n, v_j^n\}$ in the sense that

$$(\bar{u}_\Delta, \bar{v}_\Delta)(y, t_n^+) = \lim_{\substack{\tau\to t_n \\ \tau > t_n}} (\bar{u}_\Delta, \bar{v}_\Delta)(y, \tau) = (u_j^n, v_j^n) \text{ for } y \in I_j. \tag{3.1}$$

Obviously, $(\bar{u}_\Delta, \bar{v}_\Delta)$ depends on the step sizes Δt and Δx. Nevertheless, the subscript Δ will usually be suppressed.

Furthermore, an entropy inequality for the comparison function will be derived which can be seen as an entropy formulation of the scheme. Using this formulation and the inequality (2.4) we will be able to compare $(\bar{u}, \bar{v})$ and (u, v).

3.1. The comparison function. Let us introduce the averaging operator P which computes a piecewise constant approximation of a L^1_{loc} function by

$$P(u)(x) := \frac{1}{\Delta x}\int_{I_j} u(z)\, dz \text{ for } x \in I_j.$$

The following bound on the error of this representation is well known:

$$\|(P - I)u\|_1 \le \Delta x\, TV(u), \quad u \in BV. \tag{3.2}$$

Here I denotes the identity operator. In fact, below we will need a strengtened version of (3.2) of the form:

$$\int_{\mathcal{R}} \sup_{k\in\mathcal{R}} |(P-I)(|u-k|)(x)|\,dx \le \Delta x\, TV(u), \quad u \in BV. \tag{3.3}$$

The proof of this estimate is straightforward.

The comparison function is defined iteratively. In the context below it will be convenient to use the arguments y and τ instead of x and t. First, we initialize

$$\begin{aligned} \bar u(y,0) &= P(u^0)(y), \\ \bar v(y,0) &= P(v^0)(y). \end{aligned}$$

Then, we iterate the following three steps for $n \in \mathcal{Z}_0^+$.

1) In (t_n, t_{n+1}), $(\bar u, \bar v)$ is the solution of

$$\begin{aligned} \bar u_\tau + f(\bar u)_y &= 0, \\ \bar v_\tau &= 0 \end{aligned} \tag{3.4}$$

with initial data $(\bar u, \bar v)(y, t_n^+)$.

2) At t_{n+1} we take cell averages

$$\begin{aligned} \bar u(y,t_{n+1}) &= P(\bar u(\cdot, t_{n+1}^-))(y), \\ \bar v(y,t_{n+1}) &= \bar v(y, t_{n+1}^-), \end{aligned} \tag{3.5}$$

where $\bar u(\cdot, t_n^-) := \lim_{\substack{\tau\to t_n \\ \tau<t_n}} \bar u(\cdot,\tau)$.

3) The initial data for the next iteration is defined by solving a backward Euler step

$$\begin{aligned} \bar u(y,t_{n+1}^+) &= \bar u(y,t_{n+1}) - \frac{\Delta t}{\delta} g(\bar u(y,t_{n+1}^+), \bar v(y,t_{n+1}^+)), \\ \bar v(y,t_{n+1}^+) &= \bar v(y,t_{n+1}) + \frac{\Delta t}{\delta} g(\bar u(y,t_{n+1}^+), \bar v(y,t_{n+1}^+)). \end{aligned} \tag{3.6}$$

From the monotonicity property of g it follows that $\bar u(y, t_{n+1}^+)$ and $\bar v(y, t_{n+1}^+)$ are well-defined by (3.6).

The interpolation property (3.1) follows by induction. Assuming $(\bar u, \bar v)(y, t_n^+) = (u_j^n, v_j^n)$ for $y \in I_j$, we conclude from the integral form of (3.4) on the grid block $I_j \times [t_n, t_{n+1})$:

$$\begin{aligned} \bar u(y,t_{n+1}) &= u_j^n - \mu(f(u_j^n) - f(u_{j-1}^n)), \\ \bar v(y,t_{n+1}) &= v_j^n, \quad y \in I_j. \end{aligned}$$

Now $(\bar u, \bar v)(y, t_{n+1}^+) = (u_j^{n+1}, v_j^{n+1})$ for $y \in I_j$ is an obvious consequence.

For later reference, we summarize some properties of $(\bar u, \bar v)$ which basically follow from the interpolation (3.1) and the discrete bounds (2.8).

LEMMA 3.1. *For all $t, \tau \in \mathcal{R}_0^+$, $y \in \mathcal{R}$ and $n, m \in \mathcal{Z}_0^+$ we have:*

$$
\begin{aligned}
&i) \quad (\bar{u}, \bar{v})(y, \tau) \in \mathcal{S},\\
&ii) \quad TV(\bar{u}(\cdot, \tau)), TV(\bar{v}(\cdot, \tau)) \leq M,\\
&iii) \quad \|g(\bar{u}(\cdot, t_n^+), \bar{v}(\cdot, t_n^+))\|_1 \leq M\delta,\\
&iv) \quad \|\bar{u}(\cdot, t_n^+) - \bar{u}(\cdot, t_m^+)\|_1 + \|\bar{v}(\cdot, t_n^+) - \bar{v}(\cdot, t_m^+)\|_1 \leq M|t_n - t_m|,\\
&v) \quad \|\bar{u}(\cdot, t) - \bar{u}(\cdot, \tau)\|_1 + \|\bar{v}(\cdot, t) - \bar{v}(\cdot, \tau)\|_1 \leq M(|t - \tau| + \Delta t).
\end{aligned}
$$

Note that g is given by (1.2) and A satisfies (2.1).

Proof. The interpolation condition (3.1) implies, that the a-priori bounds for the scheme (2.8) carry over to the comparison function evaluated at t_n^+ and hence iii) and iv) follow. Furthermore, i) and ii) follow from the corresponding estimates of (2.8) and basic properties of the operator P and the homogeneous conservation law (3.4).

In order to prove v) we assume $t \geq \tau$ and pick indices n and m such that $t \in [t_n, t_{n+1})$ and $\tau \in [t_{m-1}, t_m)$. Then

$$
\begin{aligned}
\|\bar{u}(\cdot, t) - \bar{u}(\cdot, \tau)\|_1 \leq \; & \|\bar{u}(\cdot, t) - \bar{u}(\cdot, t_n^+)\|_1 + \|\bar{u}(\cdot, t_n^+) - \bar{u}(\cdot, t_m^+)\|_1 \\
+ \; & \|\bar{u}(\cdot, t_m^+) - \bar{u}(\cdot, t_m)\|_1 + \|\bar{u}(\cdot, t_m) - \bar{u}(\cdot, t_m^-)\|_1 \\
+ \; & \|\bar{u}(\cdot, t_m^-) - \bar{u}(\cdot, \tau)\|_1.
\end{aligned}
$$

Because of (3.4), $\bar{u}$ is time-L^1-Lipschitz continuous in (t_n, t_{n+1}) and the Lipschitz constant is independent of δ:

$$
\begin{aligned}
\|\bar{u}(\cdot, t) - \bar{u}(\cdot, t_n^+)\|_1 &\leq \Delta t TV(\bar{u}(\cdot, t_n^+)) \leq \Delta t M,\\
\|\bar{u}(\cdot, t_m^-) - \bar{u}(\cdot, \tau)\|_1 &\leq \Delta t TV(\bar{u}(\cdot, t_{m-1}^+)) \leq \Delta t M.
\end{aligned}
$$

Furthermore, $\|\bar{u}(\cdot, t_n^+) - \bar{u}(\cdot, t_m^+)\|_1$ is estimated by iv) and

$$
\|\bar{u}(\cdot, t_m^+) - \bar{u}(\cdot, t_m)\|_1 \leq 2\frac{\Delta t}{\delta}\|g(\bar{u}(\cdot, t_m^+), \bar{v}(\cdot, t_m^+))\|_1 \leq \Delta t M
$$

is a consequence from iii). Finally, from (3.2) and the CFL-condition (2.7) it follows

$$
\|\bar{u}(\cdot, t_m) - \bar{u}(\cdot, t_m^-)\|_1 \leq \Delta x TV(\bar{u}(\cdot, t_m^-)) \leq \Delta t M.
$$

A similar argument for $\bar{v}$ concludes the proof. □

3.2. The entropy inequality. Deriving the entropy inequality for the comparison function, we consider first the time intervals (t_n, t_{n+1}). Since $(\bar{u}, \bar{v})$ as a function of y and τ is an entropy solution of the hyperbolic system (3.4) the usual Kruzkov entropy formulation is valid (cf. [3]):

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}} \int_{\mathcal{R}} [|\bar{u} - k|\varphi_\tau + |f(\bar{u}) - f(k)|\varphi_y + |\bar{v} - q|\psi_\tau] \, dy \, d\tau \\
&+\int_{\mathcal{R}} [|\bar{u}(y, t_n^+) - k|\varphi(y, t_n) + |\bar{v}(y, t_n^+) - q|\psi(y, t_n)] \, dy \\
&-\int_{\mathcal{R}} [|\bar{u}(y, t_{n+1}^-) - k|\varphi(y, t_{n+1}) + |\bar{v}(y, t_{n+1}^-) - q|\psi(y, t_{n+1})] \, dy \geq 0
\end{aligned}
$$

for all $(k, q) \in \mathcal{S}$ and any $\varphi, \psi \in \mathcal{D}_+(T)$.

Summing from $n=0$ to $N-1$ and using $(\bar{u},\bar{v})(y,t_0^-)=(\bar{u},\bar{v})(y,t_0^+)=(\bar{u}^0,\bar{v}^0)$, we find

$$
\begin{aligned}
&\int_0^T\int_{\mathcal{R}}[|\bar{u}-k|\varphi_\tau+|f(\bar{u})-f(k)|\varphi_y+|\bar{v}-q|\psi_\tau]\;dy\;d\tau\\
&+\sum_{n=1}^{N}\int_{\mathcal{R}}[|\bar{u}(y,t_n^+)-k|-|\bar{u}(y,t_n^-)-k|]\varphi(y,t_n)\;dy\\
&+\sum_{n=1}^{N}\int_{\mathcal{R}}[|\bar{v}(y,t_n^+)-q|-|\bar{v}(y,t_n^-)-q|]\psi(y,t_n)\;dy\\
&+\int_{\mathcal{R}}[|\bar{u}^0-k|\varphi(y,0)+|\bar{v}^0-q|\psi(y,0)]\;dy\\
&-\int_{\mathcal{R}}[|\bar{u}(y,t_N^+)-k|\varphi(y,T)+|\bar{v}(y,t_N^+)-q|\psi(y,T)]\;dy\geq 0.
\end{aligned}
\tag{3.7}
$$

¿From the Euler step (3.6) we observe that for any $(k,q)\in\mathcal{S}$

$$
\begin{aligned}
\bar{u}(y,t_n^+)-k &= \bar{u}(y,t_n)-k-\frac{\Delta t}{\delta}g(\bar{u}(y,t_n^+),\bar{v}(y,t_n^+)),\\
\bar{v}(y,t_n^+)-q &= \bar{v}(y,t_n)-q+\frac{\Delta t}{\delta}g(\bar{u}(y,t_n^+),\bar{v}(y,t_n^+)).
\end{aligned}
$$

Multiplying by $\sigma(\bar{u}(y,t_n^+)-k)$ or $\sigma(\bar{v}(y,t_n^+)-q)$ respectively, we find

$$
\begin{aligned}
|\bar{u}(y,t_n^+)-k| &\leq |\bar{u}(y,t_n)-k|-\frac{\Delta t}{\delta}\;\sigma(\bar{u}(y,t_n^+)-k)\;g(\bar{u}(y,t_n^+),\bar{v}(y,t_n^+)),\\
|\bar{v}(y,t_n^+)-q| &\leq |\bar{v}(y,t_n)-q|+\frac{\Delta t}{\delta}\;\sigma(\bar{v}(y,t_n^+)-q)\;g(\bar{u}(y,t_n^+),\bar{v}(y,t_n^+)).
\end{aligned}
$$

Finally, we insert the last two estimates into (3.7) and make use of $\bar{v}(y,t_n)=\bar{v}(y,t_n^-)$ (cf. (3.5)) to obtain the entropy inequality

$$
\begin{aligned}
&\int_0^T\int_{\mathcal{R}}[|\bar{u}-k|\varphi_\tau+|f(\bar{u})-f(k)|\varphi_y+|\bar{v}-q|\psi_\tau]\;dy\;d\tau\\
&+\sum_{n=1}^{N}\int_{\mathcal{R}}[|\bar{u}(y,t_n)-k|-|\bar{u}(y,t_n^-)-k|]\varphi(y,t_n)\;dy\\
&+\int_{\mathcal{R}}[|\bar{u}^0-k|\varphi(y,0)+|\bar{v}^0-q|\psi(y,0)]\;dy\\
&-\int_{\mathcal{R}}[|\bar{u}(y,t_N^+)-k|\varphi(y,T)+|\bar{v}(y,t_N^+)-q|\psi(y,T)]\;dy\\
&\geq\frac{\Delta t}{\delta}\sum_{n=1}^{N}\int_{\mathcal{R}}[\sigma(\bar{u}(y,t_n^+)-k)\varphi(y,t_n)-\sigma(\bar{v}(y,t_n^+)-q)\psi(y,t_n)]\\
&\qquad\qquad\qquad\times g(\bar{u}(y,t_n^+),\bar{v}(y,t_n^+))\;dy
\end{aligned}
\tag{3.8}
$$

for all $(k,q)\in\mathcal{S}$ and any $\varphi,\psi\in\mathcal{D}_+(T)$.

4. Proof of the main result. Our argument to prove Theorem 2.1 is inspired by the work of Kruzkov [3], Kuznetsov [4] and Lucier [9].

For any $\epsilon > 0$ we define the mollifier function

$$\omega_\epsilon(x) := \frac{1}{\epsilon}\Omega(x/\epsilon),$$

where $\Omega : \mathcal{R} \to \mathcal{R}$ is a nonnegative, symmetric C^∞-function with support in $[-1, 1]$ satisfying $\int_{\mathcal{R}} \Omega(x)\, dx = 1$. This mollifier has the following properties

$$\int_{\mathcal{R}} \omega_\epsilon(x)\, dx = 1, \quad \int_{\mathcal{R}} |x|\omega_\epsilon(x)\, dx \le \epsilon, \quad \int_{\mathcal{R}} |\omega'_\epsilon(x)|\, dx \le \frac{1}{\epsilon} \quad \text{and } \operatorname{supp}(\omega_\epsilon) \subset [-\epsilon, \epsilon].$$

For arbitrary $T > 0$ we proceed by selecting the constants (k, q) and the smooth functions (φ, ψ).

In (3.8) set $(k, q) = (u, v)(x, t)$, $\varphi(y, \tau) = \psi(y, \tau) := \omega_\epsilon(x - y)\omega_\epsilon(t - \tau)$ and integrate in x and t:

$$\begin{aligned}
&-\int_0^T \int_{\mathcal{R}} \int_0^T \int_{\mathcal{R}} \quad [|\bar{u} - u| + |\bar{v} - v|]\, \omega_\epsilon(x - y)\omega'_\epsilon(t - \tau) \\
&\qquad\qquad + \quad |f(\bar{u}) - f(u)|\, \omega'_\epsilon(x - y)\omega_\epsilon(t - \tau)\, dy\, d\tau\, dx\, dt \\
&+\int_0^T \int_{\mathcal{R}} \sum_{n=1}^N \int_{\mathcal{R}} [|\bar{u}(y, t_n) - u| - |\bar{u}(y, t_n^-) - u|]\omega_\epsilon(x - y)\omega_\epsilon(t - t_n)\, dy\, dx\, dt \\
&+\int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} [|\bar{u}^0 - u| + |\bar{v}^0 - v|]\omega_\epsilon(x - y)\omega_\epsilon(t)\, dy\, dx\, dt \\
&-\int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} [|\bar{u}(y, t_N^+) - u| + |\bar{v}(y, t_N^+) - v|]\omega_\epsilon(x - y)\omega_\epsilon(t - T)\, dy\, dx\, dt \\
&\ge \frac{1}{\delta}\int_0^T \int_{\mathcal{R}} \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}} [\sigma(\bar{u}(y, t_n^+) - u) - \sigma(\bar{v}(y, t_n^+) - v)]\bar{g}(y, t_n^+) \\
&\qquad\qquad\qquad\qquad \times\omega_\epsilon(x - y)\omega_\epsilon(t - t_n)\, dy\, d\tau\, dx\, dt.
\end{aligned} \tag{4.1}$$

Here $u = u(x, t)$, $v = v(x, t)$, $\bar{u} = \bar{u}(y, \tau)$, $\bar{v} = \bar{v}(y, \tau)$ and $\bar{g}(y, \tau) = g(\bar{u}(y, \tau), \bar{v}(y, \tau))$. Furthermore, we will also use $g(x, t) = g(u(x, t), v(x, t))$.

In the entropy inequality (2.4) we choose $(k, q) = (\bar{u}, \bar{v})(y, t_n^+)$ and $\varphi(x, t) = \psi(x, t) := \omega_\epsilon(x - y)\omega_\epsilon(t - \tau)$. Furthermore, we integrate over $\mathcal{R} \times [t_{n-1}, t_n]$ with respect to y and τ and sum in n. Adding the result to (4.1), we obtain the following inequality:

$$L(\epsilon) \le R(\epsilon) + F(\epsilon) + G(\epsilon) + H(\epsilon). \tag{4.2}$$

The individual expressions are:

$$\begin{aligned}
L(\epsilon) := &\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}} \int_{\mathcal{R}} [|u(x, T) - \bar{u}(y, t_n^+)| + |v(x, T) - \bar{v}(y, t_n^+)|] \\
&\qquad\qquad\qquad \times\omega_\epsilon(x - y)\omega_\epsilon(T - \tau)\, dy\, dx\, d\tau \\
&+\int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} [|\bar{u}(y, t_N^+) - u| + |\bar{v}(y, t_N^+) - v|]\, \omega_\epsilon(x - y)\omega_\epsilon(t - T)\, dy\, dx\, dt,
\end{aligned}$$

$$R(\epsilon) := \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}} \int_{\mathcal{R}} [|u^0 - \bar{u}(y, t_n^+)| + |v^0 - \bar{v}(y, t_n^+)|] \, \omega_\epsilon(x-y)\omega_\epsilon(\tau) \, dy \, dx \, d\tau$$
$$+ \int_0^T \int_{\mathcal{R}} \int_{\mathcal{R}} [|\bar{u}^0 - u| + |\bar{v}^0 - v|] \, \omega_\epsilon(x-y)\omega_\epsilon(t) \, dy \, dx \, dt,$$

$$\begin{aligned} F(\epsilon) := \int_0^T \int_{\mathcal{R}} \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}} & [|u - \bar{u}(y, t_n^+)| - |\bar{u} - u|] \, \omega_\epsilon(x-y)\omega_\epsilon'(t-\tau) \\ & + [|f(u) - f(\bar{u}(y, t_n^+))| - |f(\bar{u}) - f(u)|] \, \omega_\epsilon'(x-y)\omega_\epsilon(t-\tau) \\ & + [|v - \bar{v}(y, t_n^+)| - |\bar{v} - v|] \, \omega_\epsilon(x-y)\omega_\epsilon'(t-\tau) \, dy \, d\tau \, dx \, dt, \end{aligned}$$

$$\begin{aligned} G(\epsilon) := -\frac{1}{\delta} \int_0^T \int_{\mathcal{R}} \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}} & [\sigma(u - \bar{u}(y, t_n^+)) - \sigma(v - \bar{v}(y, t_n^+))] \\ & \times [g(x,t) \, \omega_\epsilon(t-\tau) - \bar{g}(y, t_n^+) \, \omega_\epsilon(t - t_n)] \, \omega_\epsilon(x-y) \, dy \, d\tau \, dx \, dt, \end{aligned}$$

and finally,

$$H(\epsilon) := \int_0^T \int_{\mathcal{R}} \sum_{n=1}^{N} \int_{\mathcal{R}} [|\bar{u}(y, t_n) - u| - |\bar{u}(y, t_n^-) - u|] \, \omega_\epsilon(x-y)\omega_\epsilon(t - t_n) \, dy \, dx \, dt.$$

For these expressions we have the following bounds:

Lemma 4.1. *For any $T > 0$ there is a constant M, independent of step sizes, relaxation time and ϵ such that:*

i) $|L(\epsilon) - \|u(\cdot, T) - \bar{u}(\cdot, t_N^+)\|_1 - \|v(\cdot, T) - \bar{v}(\cdot, t_N^+)\|_1| \leq M\epsilon,$

ii) $|R(\epsilon) - \|u^0 - \bar{u}^0\|_1 - \|v^0 - \bar{v}^0\|_1| \leq M(\epsilon + \Delta t),$

iii) $|F(\epsilon)| \leq M\dfrac{\Delta t}{\epsilon},$

iv) $G(\epsilon) \leq M\dfrac{\Delta t}{\epsilon},$

v) $|H(\epsilon)| \leq M\dfrac{\Delta t}{\epsilon}.$

This lemma will be proved in the next section and we conclude this section by using the lemma to prove Theorem 2.1. From (4.2) and Lemma 4.1 we find

$$\|u(\cdot, T) - \bar{u}(\cdot, t_N^+)\|_1 + \|v(\cdot, T) - \bar{v}(\cdot, t_N^+)\|_1 \leq \|u^0 - \bar{u}^0\|_1 + \|v^0 - \bar{v}^0\|_1 + M\Big(\epsilon + \Delta t + \frac{\Delta t}{\epsilon}\Big).$$

The desired result (2.9) then follows from (3.1), (3.2) and (2.7) by choosing $\epsilon = \sqrt{\Delta t}$. ∎

5. Proof of Lemma 4.1. Using total variation bounds [(2.5) *ii*) and Lemma 3.1 *ii*)] and time-L^1-Lipschitz continuity [(2.5) *iv*) and Lemma 3.1 *iv*)], it is easy to show *i*) (cf. [17] and [18]).

Concerning *ii*) there is an additional $O(\Delta t)$-term caused by:

$$\begin{aligned}
&\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}}\int_{\mathcal{R}}|u^0-\bar{u}(y,t_n^+)|\omega_\epsilon(x-y)\omega_\epsilon(\tau)\,dy\,dx\,d\tau \\
&\le \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}}\int_{\mathcal{R}}[|u^0-\bar{u}^0(x)|+|\bar{u}^0(x)-\bar{u}^0(y)|+|\bar{u}^0(y)-\bar{u}(y,t_n^+)|] \\
&\qquad\qquad\qquad\qquad\times\omega_\epsilon(x-y)\omega_\epsilon(\tau)\,dy\,dx\,d\tau \\
&\le \frac{1}{2}\|u^0-\bar{u}^0\|_1+TV(\bar{u}^0)\int_{\mathcal{R}}|z|\omega_\epsilon(z)\,dz+\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|\bar{u}^0-\bar{u}(\cdot,t_n^+)\|_1\omega_\epsilon(\tau)\,d\tau \\
&\le \frac{1}{2}\|u^0-\bar{u}^0\|_1+M\epsilon+M\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}t_n\,\omega_\epsilon(\tau)\,d\tau.
\end{aligned}$$

Here, we estimate $t_n\le\tau+\Delta t$ and find

$$\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}t_n\,\omega_\epsilon(\tau)\,d\tau\le\int_0^T\tau\omega_\epsilon(\tau)\,d\tau+\Delta t\int_0^T\omega_\epsilon(\tau)\,d\tau\le\epsilon+\Delta t.$$

That is the reason for the extra $M\Delta t$ in *ii*). The rest of this estimate is straightforward.

In order to estimate $F=F_1+F_2+F_3$, we consider

$$\begin{aligned}
|F_1(\epsilon)| &= \left|\int_0^T\int_{\mathcal{R}}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}}[|u-\bar{u}(y,t_n^+)|-|\bar{u}-u|]\,\omega_\epsilon(x-y)\omega'_\epsilon(t-\tau)\,dy\,d\tau\,dx\,dt\right| \\
&\le \int_0^T\int_{\mathcal{R}}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}}|\bar{u}(y,t_n^+)-\bar{u}|\,\omega_\epsilon(x-y)\,|\omega'_\epsilon(t-\tau)|\,dy\,d\tau\,dx\,dt \\
&= \int_0^T\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|\bar{u}(\cdot,t_n^+)-\bar{u}(\cdot,\tau)\|_1|\omega'_\epsilon(t-\tau)|\,d\tau\,dt.
\end{aligned}$$

Using Lemma 3.1 *v*) we continue

$$|F_1(\epsilon)|\le M\Delta t\int_0^T\int_0^T|\omega'_\epsilon(t-\tau)|\,d\tau\,dt\le M\frac{\Delta t}{\epsilon}.$$

Concerning F_2 we have

$$\begin{aligned}
|F_2(\epsilon)| &\le M\int_0^T\int_{\mathcal{R}}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\mathcal{R}}|\bar{u}(y,t_n^+)-\bar{u}|\,|\omega'_\epsilon(x-y)|\,\omega_\epsilon(t-\tau)\,dy\,d\tau\,dx\,dt \\
&\le \frac{M}{\epsilon}\int_0^T\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|\bar{u}(\cdot,t_n^+)-\bar{u}(\cdot,\tau)\|_1\omega_\epsilon(t-\tau)\,d\tau\,dt \\
&\le M\frac{\Delta t}{\epsilon}\int_0^T\int_0^T\omega_\epsilon(t-\tau)\,d\tau\,dt\le M\frac{\Delta t}{\epsilon}.
\end{aligned}$$

The estimate for F_3 involving v and $\bar{v}$ is analog to F_1 and $iii)$ is proved.

The 'source error' G can be written as

$$G(\epsilon) := -\frac{1}{\delta}\int_0^T \int_{\mathcal{R}} \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}} [\sigma(u-\bar{u}(y,t_n^+)) - \sigma(v-\bar{v}(y,t_n^+))]$$
$$\times[g(x,t)-\bar{g}(y,t_n^+)]\omega_\epsilon(x-y)\omega_\epsilon(t-\tau)\ dy\ d\tau\ dx\ dt$$
$$-\frac{1}{\delta}\int_0^T \int_{\mathcal{R}} \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_{\mathcal{R}} [\sigma(u-\bar{u}(y,t_n^+)) - \sigma(v-\bar{v}(y,t_n^+))]$$
$$\times\bar{g}(y,t_n^+)\ [\omega_\epsilon(t-\tau)-\omega_\epsilon(t-t_n)]\omega_\epsilon(x-y)\ dy\ d\tau\ dx\ dt.$$

The first term in this sum is non-positive (cf. (2.2)) and therefore

$$G(\epsilon) \le \frac{2}{\delta}\int_0^T \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\bar{g}(\cdot,t_n^+)\|_1\ |\omega_\epsilon(t-t_n)-\omega_\epsilon(t-\tau)|\ d\tau\ dt.$$

Using $\omega_\epsilon(t-t_n)-\omega_\epsilon(t-\tau) = \int_{\tau-t}^{t_n-t} \omega_\epsilon'(s)\ ds$ and Lemma 3.1 $iii)$ we find

$$G(\epsilon) \le M\Delta t\int_0^T\int_0^T |\omega_\epsilon'(s-t)|\ ds\ dt \le M\frac{\Delta t}{\epsilon}.$$

Finally, we have to estimate H, which is a 'Godunov-type' error introduced by the averaging (3.5) in step 2). The argument to estimate this term is due to Lucier [9]. To simplify the notation we write $U^n(y) := \bar{u}(y,t_n^-)$. By (3.5) we have $\bar{u}(y,t_n) = P(U)(y)$. Furthermore,

$$H(\epsilon) = \int_0^T \sum_{n=1}^{N} h^n(t,\epsilon)\omega_\epsilon(t-t_n)\ dt, \tag{5.1}$$

where

$$h^n(t,\epsilon) := \int_{\mathcal{R}}\int_{\mathcal{R}} [|P(U^n)-u| - |U^n-u|]\ \omega_\epsilon(x-y)\ dy\ dx.$$

Since $u = u(x,t)$ is constant with respect to y, $P(U^n)-u = P(U^n-u)$. Hence, since $|P(v)| \le P(|v|)$, it follows that

$$h^n(t,\epsilon) \le \int_{\mathcal{R}}\int_{\mathcal{R}} (P-I)(|U^n-u|)\ \omega_\epsilon(x-y)\ dy\ dx,$$

and, since P is the L^2-projection onto the space of piecewise constants, we therefore have

$$h^n(t,\epsilon) \le \int_{\mathcal{R}}\int_{\mathcal{R}} (P-I)(|U^n-u|)\ (I-P)(\omega_\epsilon(x-\cdot))\ dy\ dx.$$

Next, we take absolute values and express the operators acting on the mollifier, which gives:

$$|h^n(t,\epsilon)| \le$$
$$\sum_{j\in\mathcal{Z}} \int_{I_j} \sup_{k\in\mathcal{S}} |(P-I)(|U^n-k|)(y)|\ \frac{1}{\Delta x}\int_{I_j}\int_{\mathcal{R}} |\omega_\epsilon(x-y)-\omega_\epsilon(x-z)|\ dx\ dz\ dy.$$

¿From the fact that

$$\int_{\mathcal{R}} |\omega_\epsilon(x-y) - \omega_\epsilon(x-z)|\, dx \le \frac{M}{\epsilon}|y-z|,$$

we obtain

$$|h^n(t,\epsilon)| \le M\frac{\Delta x}{\epsilon}\int_{\mathcal{R}} \sup_{k\in\mathcal{S}} |(P-I)(|U^n-k|)(y)|\, dy.$$

Hence, by (3.3),

$$|h^n(t,\epsilon)| \le M\frac{\Delta x^2}{\epsilon} TV(U^n).$$

Since $U^n = \bar{u}(\cdot, t_n^-)$ is TV bounded by Lemma 3.1 ii), the estimate of $H(\epsilon)$ follows from (5.1) and Lemma 4.2 is proved. □

6. Convergence to equilibrium. The purpose of this section is to study the convergence of the solution of the system

$$\begin{aligned} u_t + f(u)_x &= -\frac{1}{\delta}(A(u)-v), \\ v_t &= \frac{1}{\delta}(A(u)-v) \end{aligned} \tag{6.1}$$

towards the solution of the scalar equation

$$(w + A(w))_t + f(w)_x = 0 \tag{6.2}$$

as the relaxation parameter δ tends to zero. This problem was studied in [17], where we proved that

$$\|u(\cdot,t) - w(\cdot,t)\|_1 = O(\delta^{1/3}),$$

for proper conditions on the initial data. The proof in [17] is based on a parabolic regularization. In this section, we shall see that, by using the error estimate derived above, we can give an alternative proof of the rate of convergence. We have, however, not been able to prove that the rate, 1/3, is optimal.

Consider the approximation $\{u_j^n, v_j^n\}$ of (6.1) generated by the scheme (1.4) and the approximation $\{w_j^n\}$ of (6.2) generated by the following scheme

$$\frac{w_j^{n+1} + A(w_j^{n+1}) - (w_j^n + A(w_j^n))}{\Delta t} + \frac{f(w_j^n) - f(w_{j-1}^n)}{\Delta x} = 0. \tag{6.3}$$

We assume that the inital conditions satisfy (2.3), and that

$$w(x,0) = u(x,0).$$

Using the rate of convergence for the scheme (1.4) and the scalar Godunov scheme (6.3), we get

$$
\begin{aligned}
& \|u(\cdot,T) - w(\cdot,T)\|_1 \\
\le\ & \|u(\cdot,T) - u^N\|_1 + \|u^N - w^N\|_1 + \|w^N - w(\cdot,T)\|_1 \\
=\ & O(\Delta t^{1/2}) + \|u^N - w^N\|_1.
\end{aligned} \tag{6.4}
$$

Thus, we need to estimate the discrete deviation, i.e. $u^N - w^N$. ¿From the scheme (1.4), we get

$$
u_j^{n+1} + A(u_j^{n+1}) = u_j^n + A(u_j^n) - \mu(f(u_j^n) - f(u_{j-1}^n)) + g(u_j^{n+1}, v_j^{n+1}) - g(u_j^n, v_j^n),
$$

and similarly, from (6.3), we have,

$$
w_j^{n+1} + A(w_j^{n+1}) = w_j^n + A(w_j^n) - \mu(f(w_j^n) - f(w_{j-1}^n)).
$$

Hence, by using the CFL-condition (2.7), we obtain

$$
\begin{aligned}
\Delta x \sum_j |u_j^{n+1} + A(u_j^{n+1}) - (w_j^{n+1} + A(w_j^{n+1}))| \ &\le\ \Delta x \sum_j |u_j^n + A(u_j^n) - (w_j^n + A(w_j^n))| \\
&+\ \Delta x \sum_j |g(u_j^{n+1}, v_j^{n+1})| + |g(u_j^n, v_j^n)|.
\end{aligned}
$$

Since $\|g(u^n, v^n)\|_1 \le M\delta$ for all $n \in \mathcal{Z}_0^+$, cf. iii) of (2.8), we get

$$
\Delta x \sum_j |u_j^N + A(u_j^N) - (w_j^N + A(w_j^N))| \le \tilde{M} N \delta = \tilde{M} T \delta / \Delta t,
$$

where the constant $\tilde{M}$ is independent of Δt and δ. By using the monotonicity properties of A, we obtain

$$
\|u^N - w^N\|_1 \le \tilde{M} T \delta / \Delta t. \tag{6.5}
$$

Now, choosing $\Delta t = \delta^{2/3}$, it follows from (6.4) that,

$$
\|u(\cdot,T) - w(\cdot,T)\|_1 = O(\delta^{1/3}),
$$

for any finite time T. This is exactly the rate of convergence obtained in [17] using a parabolic regularization.

As mentioned above, we have been unable to prove that this bound is optimal. To gain further insight in this problem, we shall present some computational experiments where the rate of convergence is estimated numerically.

6.1. Example 1. In the first example we consider a nonlinear problem with the functions

$$
A(u) = \frac{2u}{u+1}, \quad \text{and} \quad f(u) = \frac{1}{2}u^2.
$$

The initial data are given by

$$u(x,0) = w(x,0) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0, \end{cases}$$

and $v(x,0) = A(u(x,0))$. Approximate solutions of (6.1) and (6.2) are computed using the schemes discussed above. In Table 1, we show the computed difference between the equilibrium and non-equilibrium model measured in the discrete L^1-norm, i.e.

$$\|u^N - w^N\|_1 = \Delta x \sum_j |u_j^N - w_j^N|,$$

where $N\Delta t = T$. In the computations we have used

$$\Delta x = 0.0005 \qquad \Delta t = 0.00025 \qquad \text{and } T = 1.$$

The rate of convergence is computed by comparing the results of two successive values of δ and assume that the deviation has the form $C\delta^\alpha$ where C is a constant and α denotes the rate. The rate of convergence for this problem seems to be very close to one, which agrees with the estimate (6.5).

δ	$\|u^N - w^N\|_1$	Rate
1.000000	0.133239	-
0.500000	0.086482	0.623545
0.250000	0.049396	0.808006
0.125000	0.025866	0.933337
0.062500	0.013007	0.991769
0.031250	0.006464	1.00879
0.015625	0.003192	1.01797
0.007812	0.001560	1.03282
0.003906	0.000751	1.05466

TABLE 1

The table shows the L^1 deviation between the numerical solution of the equilibrium model and the non-equilibrium model for some values of δ.

6.2. Example 2. In our next example we consider a linear problem with the functions

$$A(u) = u, \quad \text{and} \quad f(u) = u.$$

The initial data is as above. In Table 2, the L^1-difference between the numerical solution of the non-equilibrium model and the analytical solution of the equilibrium model is given. The mesh parameters are the same as in the previous example. Thus we study the convergence of a numerical approximation of the non-equilibrium model towards the exact solution of the equilibrium model as δ goes to zero.

For this example, we observe that the rate seems to be close to 1/2. We have run this problem with several grid-sizes and all the computations indicate a rate close to 1/2.

δ	$\|u^N - w(\cdot, T)\|_1$	Rate
1.000000	0.336945	-
0.500000	0.262074	0.362539
0.250000	0.193158	0.440193
0.125000	0.139198	0.472643
0.062500	0.099536	0.483848
0.031250	0.071082	0.485734
0.015625	0.050949	0.48043
0.007812	0.036879	0.466211
0.003906	0.027209	0.438716

TABLE 2

The table shows the L^1 deviation between the numerical solution of the equilibrium model and the non-equilibrium model for some values of δ

.

6.3. Discussion. We have shown, rigorously, that the rate of convergence of the solution of the non-equilibrium model towards the solution of the equilibrium model is at least 1/3. Based on the computations reported here, and some further computations not reported, we are not able to determine whether our estimate is sharp. But the experiments show that the rate cannot, in general, be larger than 1/2.

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