A SYSTEM OF CONSERVATION LAWS WITH A RELAXATION TERM

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ABSTRACT. We consider a 2×2 system of conservation laws including a stiff relaxation term. Well-posedness of the system, the rate of convergence to equilibrium, and the rate of convergence for a finite difference scheme is discussed. Also a numerical example is presented.

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1. INTRODUCTION

The purpose of this paper is to review some recent results concerning the following system of conservation laws

$$\begin{aligned} (u+v)_t + f(u)_x &= 0 \\ \delta v_t &= A(u) - v. \end{aligned} \tag{1}$$

This system has been studied in the papers [4, 5], and proofs of the results presented here can be found in those papers. In (1), u and v are the unknowns, $\delta > 0$ is referred to as the relaxation time, and $f = f(u)$ and $A = A(u)$ are given functions. This system arises in chromatography, cf. [6].

For the system (1), we prove the existence of an entropy solution of bounded variation for any $\delta > 0$, and it is interesting to note that the variation bound is independent of δ. Thus, we may pass to the limit of zero-relaxation time, and prove that the limit is an entropy solution of the following scalar conservation law,

$$(w + A(w))_t + f(w)_x = 0. \tag{2}$$

Moreover, we show the rate of convergence towards equilibrium, measured in L^1, is 1/3. We also study a finite difference scheme applied to the system (1), and prove that the numerical solutions converge towards the entropy solution with the rate 1/2, again measured in L^1, as the mesh size goes to zero. Since the order 1/2 is known to be optimal for scalar conservation laws, it is also optimal for this system.

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Systems of conservation with relaxation terms appear in a series of important applications. A nice introduction is provided in Whithams book [6], where such models arising in e.g. chromatography, traffic modelling, water waves and gas dynamics are discussed. For more recent results we refer to Chen, Levermore and Liu [1] and references given therein.

The rest of this paper is organized as follows; in the next section we review the results concerning the well-posedness of the system (1), in Section 3 the rate of convergence for the finite difference scheme is discussed, in Section 4 the rate of convergence towards the solution of the scalar equation is analyzed, and finally in Section 5 we present some numerical computations.

2. Well-posedness

In this section we will review the existence and uniqueness theory of the system (1) developed in [5], and we start by giving the precise assumptions on the model.

We assume that $f = f(u)$ is a smooth function satisfying $f(0) = 0$, and

$$f'(u) \geq 0 \quad \text{for } u \in [0,1].$$

The function $A = A(u)$ will be assumed to satisfy:

$$\begin{aligned} &A(0) = 0, A(1) = 1, \\ &A'(u) \geq 0, \\ &|A''(u)| \leq \alpha < 1, \end{aligned} \tag{3}$$

for all relevant values of u. We shall consider solutions of (1) in the state-space

$$\mathcal{S} = [0,1] \times [0,1]$$

and solutions of (2) in $[0,1]$.

The initial conditions (u^0, v^0) for the system (1) are supposed to satisfy:

$$\begin{aligned} &\text{i) } (u^0(x), v^0(x)) \in \mathcal{S} \quad \forall x \in \mathcal{R}, \\ &\text{ii) } TV(u^0) + TV(v^0) \leq M, \\ &\text{iii) } \|p^0\|_1 = \|A(u^0) - v^0\|_1 \leq M\delta, \\ &\text{iv) } u^0(\pm\infty), v^0(\pm\infty) = 0. \end{aligned} \tag{4}$$

Here M denotes a finite constant not depending on δ. Note that the variable $p = A(u) - v$ measures the deviation from equilibrium, and that we assume the initial conditions are almost in equilibrium.

For the precise definition of the various norms etc. used in the present paper, we refer to [5]. The entropy solution of the system is defined as follows.

Definition 2.1. *Let (u^0, v^0), satisfying (4), be the given initial data. Then (u, v) is called a entropy solution of the system (1) if the following requirements are satisfied:*

1. $(u, v) \in \mathcal{S} \quad \forall (x,t) \in \mathcal{R} \times \mathcal{R}_0^+$,
2. $TV(u(\cdot,t)), TV(v(\cdot,t)) \leq M \quad \forall t \in \mathcal{R}_0^+$,
3. $\|u(\cdot,t) - u(\cdot,\tau)\|_1 + \|v(\cdot,t) - v(\cdot,\tau)\|_1 \leq M|t - \tau| \quad \forall t, \tau \in \mathcal{R}_0^+$,

4. *For any* $(k, q) \in \mathcal{S}$ *and any* $\varphi, \psi \in \mathcal{D}_+(T)$*, where* $T > 0$ *is arbitrary,*

$$\begin{aligned}
&\int_0^T \int_{\mathcal{R}} [|u - k|\varphi_t + |f(u) - f(k)|\varphi_x + |v - q|\psi_t] dx\, dt \\
&+ \int_{\mathcal{R}} [|u^0 - k|\varphi(x, 0) + |v^0 - q|\psi(x, 0)] dx \\
&- \int_{\mathcal{R}} [|u(x, T) - k|\varphi(x, T) + |v(x, T) - q|\psi(x, T)] dx \\
&\geq \frac{1}{\delta} \int_0^T \int_{\mathcal{R}} [\sigma(u - k)\varphi - \sigma(v - q)\psi](A(u) - v) dx\, dt,
\end{aligned} \tag{5}$$

where σ *denotes the sign function.*

In this definition, M denotes a finite constant not depending on δ, and $\mathcal{D}_+(T)$ denotes the set of all nonnegative C^∞-functions with compact support in $\mathcal{R} \times [0, T]$.

The following result concerning well-posedness of (1) is proved in [5].

Theorem 2.1. *Let* (u^0, v^0) *be initial data satisfying (4). Then there is a unique solution* (u, v) *of (1) satisfying the requirements of Definition 2.1. The unique solution also satisfies the following bound,*

$$\|p(\cdot, t)\|_1 = \|A(u) - v\|_1 \leq M\delta \quad \text{for} \quad t \geq 0,$$

where M *is a finite constant not depending on* δ*.*

Furthermore, if $(\bar{u}^0, \bar{v}^0)$ *is another pair of initial data satisfying (4), there is a unique entropy solution* $(\bar{u}, \bar{v})$ *such that*

$$\|u(\cdot, t) - \bar{u}(\cdot, t)\|_1 + \|v(\cdot, t) - \bar{v}(\cdot, t)\|_1 \leq \|u^0 - \bar{u}^0\|_1 + \|v^0 - \bar{v}^0\|_1 \quad \text{for} \quad t \geq 0. \qquad \square$$

This strong regularity result enables us both to prove an error estimate for a finite difference scheme and to derive a rate of convergence towards equilibrium.

3. A finite difference scheme

One way to prove the existence part of Theorem 2.1 is to study the following semi-implicit finite difference scheme

$$\begin{aligned}
&\frac{(u_j^{n+1} + v_j^{n+1}) - (u_j^n + v_j^n)}{\Delta t} + \frac{f(u_j^n) - f(u_{j-1}^n)}{\Delta x} = 0, \\
&\frac{\delta(v_j^{n+1} - v_j^n)}{\Delta t} = A(u_j^{n+1}) - v_j^{n+1}.
\end{aligned} \tag{6}$$

The discrete values (u_j^n, v_j^n) computed from (6) are considered as approximations of (u, v) in the cell $I_j := [(j - 1/2)\Delta x, (j + 1/2)\Delta x)$, at time level $t_n = n\Delta t$. The initial data for the scheme is taken as cell averages of u^0 and v^0

$$u_j^0 = \frac{1}{\Delta x}\int_{I_j} u^0(x)\, dx \quad \text{and} \quad v_j^0 = \frac{1}{\Delta x}\int_{I_j} v^0(x)\, dx. \tag{7}$$

Due to the explicit treatment of the convective term one has to impose a CFL-condition

$$\mu f'(u) \leq 1 \text{ for all } u \in [0,1], \text{ where } \mu = \Delta t/\Delta x. \tag{8}$$

It should be noted that the step size is not related to the relaxation time δ.

In [5] the following a-priori bounds for solutions of the scheme are established.

Lemma 3.1. *Suppose that the initial data (u^0, v^0) satisfy (4), and that the grid-parameters Δt and Δx satisfy the CFL-condition (8). Then, for any $\delta > 0$, there is a finite constant M independent of δ, Δt and Δx such that:*

$$\begin{aligned}
&I. \quad (u_j^n, v_j^n) \in \mathcal{S} \quad \text{for all } (j,n) \in \mathcal{Z} \times \mathcal{Z}^+\\
&II. \quad TV(u^n) + TV(v^n) \leq TV(u^0) + TV(v^0)\\
&III. \quad \|p^n\|_1 \leq M\delta\\
&IV. \quad \|u^n - u^m\|_1 + \|v^n - v^m\|_1 \leq M|n-m|\Delta t\\
&V. \quad \|u^n - \bar{u}^n\|_1 + \|v^n - \bar{v}^n\|_1 \leq \|u^0 - \bar{u}^0\|_1 + \|v^0 - \bar{v}^0\|_1,
\end{aligned}$$

where $(\bar{u}^n, \bar{v}^n)$ is a discrete solution based on initial data $(\bar{u}^0, \bar{v}^0)$ satisfying (4). □

Here the total variation and norm for grid functions is defined by

$$TV(u) = \sum_{j \in \mathcal{Z}} |u_j - u_{j-1}|, \text{ and } \|u\|_1 = \Delta x \sum_{j \in \mathcal{Z}} |u_j|.$$

Furthermore, $p_j^n = A(u_j^n) - v_j^n$ is the discrete deviation from equilibrium.

By a proper application of Helly's theorem it follows from I-IV of Lemma 3.1 that a subsequence of the family of approximate solutions generated by (6) converges to a pair of functions $(u,v) \in (L^1_{loc}(\mathcal{R} \times \mathcal{R}_0^+))^2$ as the mesh size tends to zero. Using a discrete entropy inequality (see [5]), it can be seen that the limit (u,v) is a entropy solution of (1). Having established the existence of the entropy solution, we are able to derive an error estimate for solutions of the scheme:

Theorem 3.1. *Let (u^0, v^0) be initial data satisfying (4). Define (u_j^0, v_j^0) as cell averages by (7) and consider grid parameters satisfying the CFL-condition (8). Let (u,v) be the entropy solution of (1) and (u^N, v^N) a piecewise constant representation of the data $\{u_j^N, v_j^N\}$ generated by the scheme (6).*

Then for any fixed $T = N\Delta t > 0$ there is a finite constant M independent of Δt, Δx and δ such that

$$\|u(\cdot,T) - u^N\|_1 + \|v(\cdot,T) - v^N\|_1 \leq M\sqrt{\Delta t}. \quad \square \tag{9}$$

A proof of this theorem can be found in [4]. Here we will only sketch the main steps of the proof.

Sketch of proof: In order to compare the discrete data $\{u_j^n, v_j^n\}$ generated by the scheme to the entropy solution (u,v) of (1) we define a comparison function $(\bar{u}_\Delta, \bar{v}_\Delta)$ on $\mathcal{R} \times \mathcal{R}_0^+$ which extends the data in the sense that

$$(\bar{u}_\Delta, \bar{v}_\Delta)(x, t_n^+) = \lim_{\substack{t \to t_n \\ t > t_n}} (\bar{u}_\Delta, \bar{v}_\Delta)(x,t) = (u_j^n, v_j^n) \text{ for } x \in I_j.$$

This comparison function is defined iteratively. First, we initialize

$$\begin{aligned} \bar{u}_\Delta(x,0) &= \frac{1}{\Delta x}\int_{I_j} u^0(z)\,dz, \\ \bar{v}_\Delta(x,0) &= \frac{1}{\Delta x}\int_{I_j} v^0(z)\,dz \text{ for } x \in I_j. \end{aligned}$$

Then, we iterate the following three steps for $n \in \mathcal{Z}_0^+$.

1) In (t_n, t_{n+1}), $(\bar{u}_\Delta, \bar{v}_\Delta)$ is the solution of

$$\begin{aligned} (\bar{u}_\Delta)_t + f(\bar{u}_\Delta)_x &= 0, \\ (\bar{v}_\Delta)_t &= 0 \end{aligned} \tag{10}$$

with initial data $(\bar{u}_\Delta, \bar{v}_\Delta)(x, t_n^+)$.

2) At t_{n+1} we take cell averages

$$\begin{aligned} &\bar{u}_\Delta(x, t_{n+1}) = \frac{1}{\Delta x}\int_{I_j} \bar{u}_\Delta(z, t_{n+1}^-)dz, \text{ for } x \in I_j \\ &\bar{v}_\Delta(x, t_{n+1}) = \bar{v}_\Delta(x, t_{n+1}^-), \end{aligned} \tag{11}$$

where $\bar{u}_\Delta(\cdot, t_n^-) := \lim\limits_{\substack{t \to t_n \\ t < t_n}} \bar{u}_\Delta(\cdot, t)$.

3) The initial data for the next iteration is defined by solving a backward Euler step

$$\begin{aligned} \bar{u}_\Delta(x, t_{n+1}^+) &= \bar{u}_\Delta(x, t_{n+1}) - \frac{\Delta t}{\delta} p(\bar{u}_\Delta(x, t_{n+1}^+), \bar{v}_\Delta(x, t_{n+1}^+)), \\ \bar{v}_\Delta(x, t_{n+1}^+) &= \bar{v}_\Delta(x, t_{n+1}) + \frac{\Delta t}{\delta} p(\bar{u}_\Delta(x, t_{n+1}^+), \bar{v}_\Delta(x, t_{n+1}^+)). \end{aligned} \tag{12}$$

From the monotonicity property of p it follows that $(\bar{u}_\Delta(x, t_{n+1}^+), \bar{v}_\Delta(x, t_{n+1}^+))$ are well-defined by (12).

For this pair of functions $(\bar{u}_\Delta, \bar{v}_\Delta)$ we derive an Kruzkov type inequality which can be compared to inequality (5). Similar as in Lucier's paper [3], the error estimate is a result of this comparison. For details see [4]. □

4. Convergence towards equilibrium

This section is devoted to the question how well the solution of the scalar equilibrium equation (2) approximates the δ–dependent solution of the relaxed system (1). The result is:

Theorem 4.1. *Let (u^0, v^0) be a pair of initial data satisfying (4), and let $w^0 = u^0$. Then, for any $T > 0$, there is a finite constant M such that*

$$\|u(\cdot,t) - w(\cdot,t)\|_1 \le M\delta^{1/3} \quad \textit{for all } \; 0 \le t \le T.$$

Here (u,v) and w are solutions of (1) and (2) respectively. □

There are two independent arguments to obtain this result. The first one was established in [5] and is based on a parabolic regularization of the system (1):

$$u_t^\epsilon + f(u^\epsilon)_x = \frac{1}{\delta}(v^\epsilon - A(u^\epsilon)) + \epsilon u_{xx}^\epsilon,$$
$$v_t^\epsilon = \frac{1}{\delta}(A(u^\epsilon) - v^\epsilon) + \epsilon v_{xx}^\epsilon.$$

In [5], Lemma 6.1 it is proved that

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\|_1 + \|v^\epsilon(\cdot,t) - v(\cdot,t)\|_1 \le M\epsilon^{1/2},$$

where (u^ϵ, v^ϵ) solves the regularized problem and (u,v) solves (1). Then we introduce a regularized equilibrium model

$$(w^\epsilon + A(w^\epsilon))_t + f(w^\epsilon)_x = \epsilon(w^\epsilon + A(w^\epsilon))_{xx}. \tag{13}$$

For this scalar equation, it is well known, cf. Kuznetsov [2], that

$$\|w - w^\epsilon\|_1 \le M\epsilon^{1/2},$$

where w solves (2) and w^ϵ solves (13). Furthermore, Lemma 6.2 of [5] states that

$$\|u^\epsilon - w^\epsilon\|_1 \le M\delta/\epsilon.$$

Finally, choosing $\epsilon = \delta^{2/3}$, the result follows from

$$\|u - w\|_1 \le \|u - u^\epsilon\|_1 + \|u^\epsilon - w^\epsilon\|_1 + \|w^\epsilon - w\|_1 \le M(\epsilon^{1/2} + \delta/\epsilon + \epsilon^{1/2}).$$

The second alternative argument is stated in [4] and uses the error estimate for the finite difference scheme above. We introduce the discretized equilibrium equation

$$\frac{w_j^{n+1} + A(w_j^{n+1}) - (w_j^n + A(w_j^n))}{\Delta t} + \frac{f(w_j^n) - f(w_{j-1}^n)}{\Delta x} = 0 \tag{14}$$

and set $w(x,0) = u(x,0)$. Using the error estimate for the scheme (6) and the scalar Godunov scheme (14), we get

$$\begin{aligned} \|u(\cdot,T) - w(\cdot,T)\|_1 &\le \|u(\cdot,T) - u^N\|_1 + \|u^N - w^N\|_1 + \|w^N - w(\cdot,T)\|_1 \\ &= O(\Delta t^{1/2}) + \|u^N - w^N\|_1. \end{aligned} \tag{15}$$

The discrete deviation $\|u^N - w^N\|_1$ can be estimated by comparing the two schemes. From (6) we get

$$u_j^{n+1} + A(u_j^{n+1}) = u_j^n + A(u_j^n) - \mu(f(u_j^n) - f(u_{j-1}^n)) + p(u_j^{n+1}, v_j^{n+1}) - p(u_j^n, v_j^n)$$

and similarly, from (14), we have

$$w_j^{n+1} + A(w_j^{n+1}) = w_j^n + A(w_j^n) - \mu(f(w_j^n) - f(w_{j-1}^n)).$$

Hence, by using the CFL-condition (8), we obtain

$$\begin{aligned} & \Delta x \sum_j |u_j^{n+1} + A(u_j^{n+1}) - (w_j^{n+1} + A(w_j^{n+1}))| \\ \leq \; & \Delta x \sum_j |u_j^n + A(u_j^n) - (w_j^n + A(w_j^n))| \\ + \; & \Delta x \sum_j (|p(u_j^{n+1}, v_j^{n+1})| + |p(u_j^n, v_j^n)|). \end{aligned}$$

Since $\|p(u^n, v^n)\|_1 \leq M\delta$ for all $n \in \mathcal{Z}_0^+$, cf. III of Lemma 3.1, we get

$$\Delta x \sum_j |u_j^N + A(u_j^N) - (w_j^N + A(w_j^N))| \leq \tilde{M} N \delta = \tilde{M} T \delta / \Delta t,$$

where the constant $\tilde{M}$ is independent of Δt and δ. Using the monotonicity properties of A, we obtain

$$\|u^N - w^N\|_1 \leq \tilde{M} T \delta / \Delta t. \tag{16}$$

Now, the desired result follows from (15) and (16) choosing $\Delta t = \delta^{2/3}$.

5. A numerical example

In this section will present a numerical example of convergence towards equilibrium. Further experiments are discussed in [4] where it is observed, numerically, that the rate of convergence towards equilibrium is about 1/2 for one particular example. This shows that the convergence towards equilibrium is slower than a straightforward analysis of smooth solutions would indicate, but we have not been able to find numerical evidence for the rate of 1/3. We therefore consider the problem of determining the optimal rate of convergence to be open.

In the present example, we consider the functions

$$A(u) = \frac{2u}{u+1}, \quad \text{and} \quad f(u) = \frac{1}{2}u^2,$$

and the initial data,

$$u(x,0) = w(x,0) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0, \end{cases}$$

and $v(x,0) = A(u(x,0))$.

Figure 1 shows the exact solution of the equilibrium model (2) and the numerical solution of the non-equilibrium model (1) generated by the finite difference scheme (6) for some values of the relaxation time δ. In the present computations, we used the parameters

$$\Delta x = 0.0005 \quad \Delta t = 0.00025 \quad \text{and } T = 1.$$

Note that the non-equilibrium solutions converge towards the single shock of the scalar equation. For this example, the rate of convergence is estimated to be about 1, cf. the numerical section of [4].

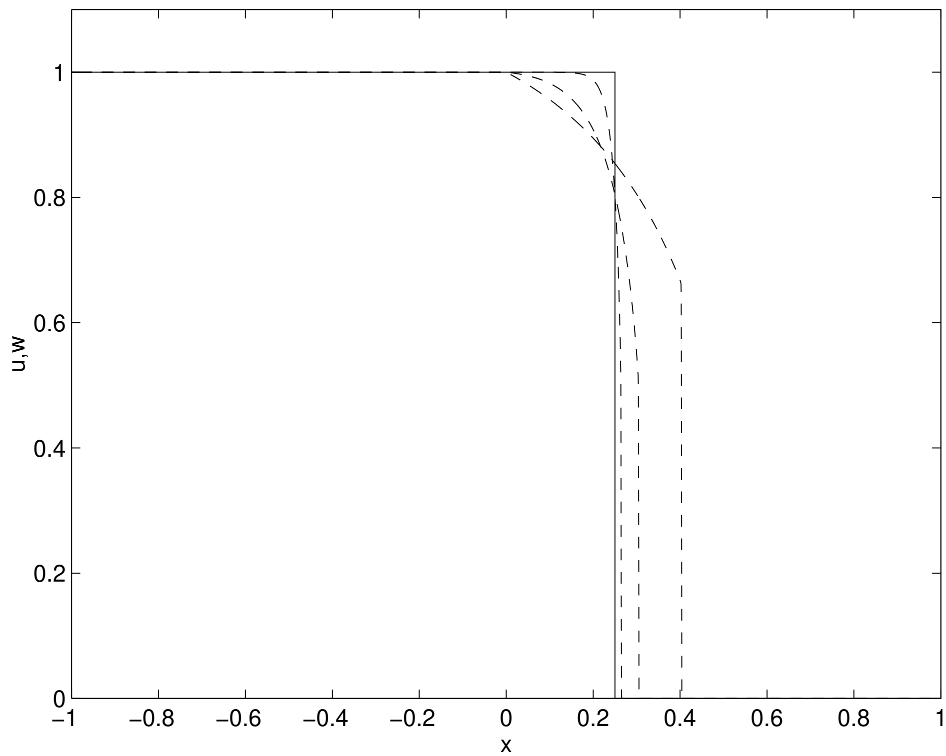

FIGURE 1. *The figure shows the solution of the equilibrium model (solid lines) and numerical solutions of the non-equilibrium model (dotted lines) for $\delta = 1, 1/4, 1/16$, at the time $t = 1$.*

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