Discontinuous Solutions of the Navier-Stokes Equations for Multidimensional Heat-Conducting Flow

by David Hoff*

Abstract: We prove the global existence of weak solutions of the Navier-Stokes equations for compressible, heat–conducting flow in two and three space dimensions when the initial density is close to a constant in $L^2 \cap L^\infty$, the initial temperature is close to a constant in L^2, and the initial velocity is small in $H^s \cap L^4$, where $s = 0$ when $n = 2$ and $s > 1/3$ when $n = 3$. (The L^p norms must be weighted slightly when $n = 2$.) In particular, the initial data may be discontinuous across a hypersurface of $\mathbb{R}^n$. A great deal of qualitative information about the solution is obtained. For example, we show that the velocity, vorticity, and temperature are relatively smooth in positive time, as is the "effective viscous flux" F, which is the divergence of the velocity minus a certain multiple of the pressure. We find that F plays a central role in the entire analysis, particularly in closing the required energy estimates and in understanding rates of regularization near the initial layer. Moreover, F is precisely the quantity through which the hyperbolicity of the corresponding inviscid equations shows itself, an effect which is crucial for obtaining time-independent pointwise bounds for the density.

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§1 Introduction

We prove the global existence of weak solutions of the Navier-Stokes equations

$$\rho_t + \operatorname{div}(\rho u) = 0 \tag{1.1}$$

$$(\rho u^j)_t + \operatorname{div}(\rho u^j u) + P(\rho,e)_{x_j} = \varepsilon\Delta u^j + \lambda \operatorname{div} u_{x_j}, \;\; j=1,\dots,n \tag{1.2}$$

$$\begin{aligned}(\rho E)_t + \operatorname{div}(\rho E u + Pu) &= \Delta\left(T(e) + \tfrac{1}{2}\varepsilon|u|^2\right)\\ &+ \;\varepsilon \operatorname{div}\big((\nabla u)u\big) + (\lambda-\varepsilon)\operatorname{div}\big((\operatorname{div} u)u\big)\end{aligned} \tag{1.3}$$

in two and three space dimensions when the initial data (ρ_0, u_0, E_0) is small but possibly discontinuous. Here $t \geq 0$ is time, $x \in \mathbb{R}^n$, $n = 2,3$, is the spatial coordinate, and $\rho(x,t)$, $u(x,t) = \big(u^1(x,t),\dots,u^n(x,t)\big)^{tr}$, $e(x,t)$, $P(\rho,e)$, and $T(e)$ represent respectively the fluid density, velocity, specific internal energy, pressure, and normalized temperature, and $E = e+\frac{1}{2}|u|^2$ is the specific total energy. ε and λ are positive viscosity coefficients, div and Δ are the usual spatial divergence and Laplace operators, and ∇u is the $n\times n$ Jacobian matrix $\left[\dfrac{\partial u^j}{\partial x_k}\right]$. The equations (1.1) - (1.3) then express respectively the conservation of mass, the balance of momentum, and the balance of energy, under internal pressure and viscosity forces and the conduction of thermal energy.

We fix positive constants $\tilde\rho$ and $\tilde e$, and assume that $\rho_0 - \tilde\rho$ is small in $L^2\cap L^\infty$, that $e_0-\tilde e$ is small in L^2 with e_0 bounded below away from zero, and that u_0 is small in $H^s\cap L^4$, where $s=0$ when $n=2$ and $s\in\left(\frac{1}{3},\frac{1}{2}\right)$ when $n=3$. (The L^2 and L^4 norms must be weighted slightly when $n=2$.) In particular, the initial data may be discontinuous across a hypersurface of $\mathbb{R}^n$. We then obtain the global existence of weak solutions, together with a number of important results concerning their partial regularity.

The questions addressed here are motivated by the fact that discontinuous solutions are fundamental both in the physical theory of nonequilibrium thermodynamics as well as in the mathematical theory of inviscid models for compressible flow. It is natural, therefore, to seek a rigorous theory which accommodates discontinuities in solutions of the Navier-Stokes system. This leads to a number of significant mathematical issues, owing particularly to the fact that the viscosity matrix in (1.1) - (1.3) is not positive definite. Foremost among these is the problem of achieving pointwise control of the density ρ. We remark in this regard that the Navier-Stokes system is a model for *nondilute* fluids, that is, fluids in which the density is bounded below away from zero. A different difficulty arises from the observation that, in the natural splitting of the velocity gradient ∇u, the antisymmetric part, which is the vorticity, is relatively smooth, but the symmetric part is not. This prevents our obtaining by the Sobolev imbeddings the L^4 bounds for ∇u required to control the convective terms in (1.1) - (1.3). On the other hand, we find also that the trace of the symmetric part of ∇u, which is div u, plus a certain multiple of the pressure, is in fact in H^1 at positive times. We call this combination the "effective viscous flux." The effective viscous flux F and the vorticity ω are complementary variables in a sense, and play a central role in the entire analysis. In particular, the various regularity estimates for F and ω may be applied, via the Marcinkiewicz theorem, to derive the required L^4 bounds for ∇u. Moreover, F is precisely the variable through which an important underlying hyperbolic effect is revealed, an effect which is crucial in obtaining time-independent pointwise bounds for the density ρ.

This hyperbolic effect depends also upon the positivity of the energy e; the derivation of the required pointwise bounds for e therefore comprises another important and interesting part of the analysis. First, because the $O(|\nabla u|)^2$ mechanical work terms in the energy equation (1.3) are highly singular in the initial layer near $t = 0$, it is necessary to derive adequate smoothing rates into L^2 for $\nabla u, \nabla e$, and the convective derivative $u_t + u \cdot \nabla u$. These rates are coupled in a complicated way, and depend ultimately upon a sort of quasilinear Riesz-Thorin type interpolation, starting from the hypothesis that $u_0 \in H^s$. Second, we find that, even away from $t = 0$, there is insufficient regularity to bound e via the Sobolev imbeddings. Instead, we detect a parabolic smoothing from L^2 into L^∞ in the evolution of $e(\cdot, t) - \tilde{e}$ by estimating the corresponding adjoint operator backwards in time. The adjoint equation has nonconstant coefficients, so that the required estimates, based on Fourier transform techniques, are similar to the first steps in the Nash theory. Neither of these aspects of the analysis, relating to the derivation of pointwise bounds for e, occurs in the simpler isentropic or isothermal cases treated in [9] and [10], in which $P(\rho) = K\rho^\gamma, \gamma \geq 1$, and the energy equation (1.3) is omitted.

The question of global existence of discontinuous solutions of the Navier-Stokes equations is largely settled for the case of one space dimension; see for example Hoff [5] - [7], Serre [15], and Shelukhin [18]. A complete theory for spherically symmetric flows, even with large, discontinuous initial data, but for isentropic or isothermal equations of state, is given in Hoff [8]. As for flows which are truly multidimensional, the most important work until recently had been that of Matsumura and Nishida [13], who proved the global existence of solutions to the full Navier-Stokes system when the initial data is close to a constant in $H^3(\mathbb{R}^n)$, hence in $C^1(\mathbb{R}^n)$. In more recent work [9] and [10], we proved the global existence of solutions for the simpler isothermal and isentropic cases described above. Just as here, the initial data may be discontinuous, but must also be small in a suitable sense. Finally, Lions [12] has obtained global solutions in the isentropic case for *large*, discontinuous initial data, in any number of space dimensions, assuming again that $P(\rho) = K\rho^\gamma$, where now $\gamma \geq \gamma_0(n) > 1$. The present paper therefore contains the first results for discontinuous solutions of the full Navier-Stokes system (1.1) - (1.3), and the problem of extending these results to the case of large data, discontinuous or not, remains open.

We now give a more precise formulation of our results. First, we say that (ρ,u,E) is a weak solution of (1.1) - (1.3) with initial data (ρ_0,u_0,E_0) provided that, for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^n \times (-\infty,\infty))$ and all times $t_2 > t_1 \geq 0$,

$$\int_{\mathbb{R}^n} (\rho\varphi)(x,\cdot)\, dx\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\mathbb{R}^n} (\rho\varphi_t + \rho u \cdot \nabla\varphi)\, dxdt = 0, \tag{1.4}$$

$$\begin{aligned}\int_{\mathbb{R}^n} (\rho u^j\varphi)(x,\cdot)\, dx\Big|_{t_1}^{t_2} &- \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left[\rho u^j\varphi_t + \rho u^j u \cdot \nabla\varphi + P(\rho,e)\varphi_{x_j}\right]\, dxdt \\ &= -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left[\varepsilon\nabla u^j \cdot \nabla\varphi + \lambda(\operatorname{div} u)\varphi_{x_j}\right]\, dxdt,\ j = 1,\dots,n,\end{aligned} \tag{1.5}$$

$$\begin{aligned}\int_{\mathbb{R}^n} (\rho E\varphi)(x,\cdot)\, dx\Big|_{t_1}^{t_2} &- \int_{t_1}^{t_2}\int_{\mathbb{R}^n} [\rho E\varphi_t + \rho E u \cdot \nabla\varphi + P(\rho,e)u \cdot \nabla\varphi]\, dxdt \\ &= -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left[\nabla T + \tfrac{1}{2}\nabla(\varepsilon|u|^2) + \varepsilon(\nabla u)u + (\lambda-\varepsilon)(\operatorname{div} u)u\right] \cdot \nabla\varphi\, dxdt.\end{aligned} \tag{1.6}$$

It is understood that, when $t_1 = 0$, $\rho(\cdot,0)$, $u(\cdot,0)$, $E(\cdot,0)$ are to be replaced by ρ_0, u_0, E_0. The regularity and integrability properties of the solutions (ρ, u, E) that we construct here will be more than adequate for the above integrals to be defined, and the requirement that $\varphi \in \mathcal{D}$ can be relaxed considerably.

We shall assume that the fluid under consideration is ideal, so that

$$P(\rho,e) = (\gamma-1)\rho e, \tag{1.7}$$

where $\gamma > 1$ is the adiabatic constant, and T is a smooth function of $e \in (0,\infty)$, satisfying

$$C^{-1} \leq T'(e),\ T(e)/e \leq C \tag{1.8}$$

for a positive constant C. Recall that an ideal fluid is said to be polytropic if $T = Ce$. Observe also that a more general heat-conduction term $\operatorname{div}(k(e)\nabla T(e))$ can be reduced to the simpler term ΔT appearing in (1.3) by a redefinition of the temperature T.

Concerning the viscosity parameters ε and λ, we must assume that

$$\varepsilon, \lambda > 0 \quad \text{and} \quad \lambda < \left(1+\sqrt{2}\right)\varepsilon \tag{1.9}$$

when $n = 2$, and

$$\varepsilon > 0 \quad \text{and} \quad \tfrac{1}{3}\varepsilon < \lambda < \left(1+\tfrac{1}{3}\sqrt{15}\,\right)\varepsilon \tag{1.10}$$

when $n = 3$. Recall that $(\lambda - \frac{1}{3}\varepsilon)\mathrm{div}\, u$ is the difference between the thermodynamical, or equilibrium pressure, which we have denoted by P, and the mechanical pressure, which is minus one-third the trace of the stress tensor. (The stress tensor is defined below in (1.15).) Our assumptions (1.9) - (1.10) therefore include physically realistic flow regimes. See Batchelor [1] for a complete discussion of the background physics.

We shall use subscripts to denote differentiation with respect to x; thus $u_j^k = \dfrac{\partial u^k}{\partial x_j}$, etc. The material derivative $\dfrac{d}{dt}$ is defined, for functions w of x and t, by

$$\frac{dw}{dt} = \dot{w} = \frac{\partial w}{\partial t} + u \cdot \nabla w = w_t + u^j w_j$$

(summation over repeated indices is understood). Next, because of its frequent appearance, $\mathrm{div}\, u$ will be denoted simply by D:

$$D = \mathrm{div}\, u = u_j^j.$$

The vorticity matrix ω is defined by

$$\omega^{j,k} = u_k^j - u_j^k,$$

and the effective viscous flux F by

$$F = (\varepsilon + \lambda)D - P(\rho,e) + P(\tilde{\rho},\tilde{e}). \tag{1.11}$$

The important roles played by ω and F will be discussed below, following the statements of Theorems 1.1 and 1.2.

We now derive from (1.1) - (1.3) the equations for the kinetic and internal energy densities. Multiplying (1.2) by u^j, applying (1.1), and summing over j, we obtain the kinetic energy equation

$$\begin{aligned} &\left(\tfrac{1}{2}\rho|u|^2\right)_t + \mathrm{div}\,\left(\tfrac{1}{2}\rho|u|^2 u\right) + \nabla P \cdot u \\ &\qquad = \varepsilon\left[\Delta\left(\tfrac{1}{2}|u|^2\right) - |\nabla u|^2\right] + \lambda \nabla D \cdot u\,. \end{aligned} \tag{1.12}$$

We now subtract (1.12) from (1.3) and rearrange to obtain the equation for e:

$$
\begin{aligned}
(\rho e)_t + \operatorname{div}(\rho e u) + PD& \\
&= \Delta T + \varepsilon|\nabla u|^2 + \lambda D^2 + \varepsilon \operatorname{div}\big((\nabla u)u - Du\big) \\
&= \Delta T + \varepsilon|\nabla u|^2 + \lambda D^2 + \varepsilon(u_j^k u_k^j - D^2).
\end{aligned}
\tag{1.13}
$$

We record here the weak form of (1.13), which will be of importance later on: for suitable test functions φ,

$$
\begin{aligned}
\int_{\mathbb{R}^n} (\rho e\varphi)(x,\cdot)dx\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\mathbb{R}^n} &\left[\rho e(\varphi_t + \nabla\varphi\cdot u) - PD\varphi\right] dxdt \\
= -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} &\left[\nabla T + \varepsilon\big((\nabla u)u - Du\big)\right]\cdot\nabla\varphi\, dxdt \\
+ \int_{t_1}^{t_2}\int_{\mathbb{R}^n} &\left[\varepsilon|\nabla u|^2 + \lambda D^2\right]\varphi\, dxdt.
\end{aligned}
\tag{1.14}
$$

We can rewrite the equations (1.1), (1.2), (1.13) in a fairly compact form by introducing the stress tensor σ, given by

$$
\sigma^{j,k} = \varepsilon(u_j^k + u_k^j) + [(\lambda-\varepsilon)D - P]\,\delta_{jk},
\tag{1.15}
$$

where δ_{jk} is the Kronecker symbol. (1.1), (1.2), (1.13) then become

$$
\begin{cases}
\dot\rho = -\rho D \\
\rho\dot u^j = \sigma^{j,k}_k = -P_j + \varepsilon\Delta u^j + \lambda D_j, \\
\rho\dot e = \Delta T + \sigma^{j,k}u_k^j \\
\qquad = \Delta T - PD + \varepsilon\left(|\nabla u|^2 + u_k^j u_j^k\right) + (\lambda-\varepsilon)D^2.
\end{cases}
\tag{1.16}
$$

We shall measure the initial data (ρ_0, u_0, e_0) in the norm given by

$$
\begin{aligned}
C_0 = \|\rho_0 - \tilde\rho\|_{L^\infty}^2 + \|u_0\|_{H^s}^2& \\
+ \int_{\mathbb{R}^n} &\left[(\rho_0(x)-\tilde\rho)^2 + |u_0(x)|^2 + (e_0(x)-\tilde e)^2\right](1+|x|^2)^\ell dx \\
+ &\left[\int_{\mathbb{R}^n} |u_0(x)|^4(1+|x|^2)^\ell dx\right]^{1/2},
\end{aligned}
\tag{1.17}
$$

where $\tilde\rho$ and $\tilde e$ are fixed positive constants, $s = 0$ when $n = 2$ and $s \in \left(\frac{1}{3}, \frac{1}{2}\right)$ when $n = 3$, and ℓ is positive but arbitrarily small when $n = 2$, and $\ell = 0$ when $n = 3$. Recall that the H^s norm of a function $w \in L^2(\mathbb{R}^n)$ is given by

$$
\|w\|_{H^s}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat w(\xi)|^2 d\xi.
$$

One easily computes, for example, that the characteristic function of a cube in $\mathbb{R}^n$ is in H^s for all $s < \frac{1}{2}$. Our hypothesis that $u_0 \in H^s$ for $s \in \left(\frac{1}{3},\frac{1}{2}\right)$ therefore allows initial velocities which are discontinuous across hypersurfaces of $\mathbb{R}^n$. This assumption is required to insure sufficiently rapid smoothing of the variables u and e in the initial layer near $t = 0$. The weighting of the L^2 and L^4 norms, on the other hand, serves to compensate for the growth of the fundamental solution of the Laplacian at infinity in two space dimensions.

Solutions of (1.1) - (1.3) will be measured in norms depending on s, as well as on two parameters r and β, which are chosen as follows: when $n = 2$, $r = 2$ and $\beta = 0$; when $n = 3$ we first choose $r \in (3 - 3s,2)$, then $\beta \in \left(0,\frac{1}{84}\right)$ so that $3 - 3s + 2\beta < r < 2$. Thus

$$\begin{cases} n = 2: \ s = 0, \ r = 2, \ \beta = 0; \\ n = 3: \ s \in \left(\frac{1}{3},\frac{1}{2}\right), \ \beta \in \left(0,\frac{1}{84}\right), \ 3 - 3s + 2\beta < r < 2. \end{cases} \tag{1.18}$$

Then for a given solution (ρ,u,e) we define

(1.19)

$$\begin{aligned} A(t) = &\sup_{0 \le t' \le 1\wedge t} \int \left[\left(\rho(x,t') - \tilde{\rho}\right)^2 + |u(x,t')|^2 + |u(x,t')|^4 + \left(e(x,t') - \tilde{e}\right)^2 \right] (1 + |x|^2)^{\ell} dx \\ &+ \int_0^{1\wedge t} \int \left[(1 + |u|^2)|\nabla u|^2 + |\nabla e|^2\right] (1 + |x|^2)^{\ell} dx dt' \\ &+ \sup_{1\wedge t < t' \le t} \int \left[\left(\rho(x,t') - \tilde{\rho}\right)^2 + |u(x,t')|^2 + \left(e(x,t') - \tilde{e}\right)^2 \right] dx \\ &+ \int_{1\wedge t}^{t} \int \left(|\nabla u|^2 + |\nabla e|^2\right) dx dt', \end{aligned}$$

and

$$\begin{aligned} B(t) = \sup_{0 < t' \le t} \int \Big[&(1 \wedge t')^{1-s} |\nabla u(x,t')|^2 \\ &+ (1 \wedge t')^r \left(|\dot{u}(x,t')|^2 + |\nabla \omega(x,t')|^2 + |\nabla e(x,t')|^2\right) \Big] dx \\ &+ \int_0^t \int \left[(1 \wedge t')^{1-s+\beta} (|\dot{u}|^2 + |\nabla \omega|^2) + (1 \wedge t')^r (|\nabla \dot{u}|^2 + \dot{e}^2) \right] dx dt', \end{aligned} \tag{1.20}$$

where $1 \wedge t = \min\{1,t\}$.

Finally, we employ the usual notation for Hölder norms,

$$\langle w \rangle^{\alpha} = \sup_{\substack{x,y \in \mathbb{R}^n \\ x \ne y}} \frac{|w(x) - w(y)|}{|x - y|^{\alpha}}$$

for functions $w : \mathbb{R}^n \to \mathbb{R}^m$; and for $w : Q \subseteq \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^m$,

$$\langle w \rangle_Q^{\alpha_1,\alpha_2} = \sup_{\substack{(x,t),(y,s) \in Q \\ (x,t) \ne (y,s)}} \frac{|w(x,t) - w(y,s)|}{|x - y|^{\alpha_1} + |t - s|^{\alpha_2}}.$$

The following theorem then contains the principle results of this paper:

Theorem 1.1. *Let $n = 2$ or 3, and fix the system parameters ε, λ, P, and T as above in (1.7) - (1.10); choose ℓ as in (1.17) and s, r, and β as in (1.18), and let positive constants $\overline{\rho} > \tilde{\rho} > \underline{\rho} > 0$ and $\tilde{e} > e_1 > \underline{e} > 0$ be given. Then there are positive constants a and C depending on these quantities such that: given initial data (ρ_0, u_0, e_0) for which*

$$C_0 \le a \quad \text{and} \quad \operatorname{ess\,inf} e_0 \ge e_1, \tag{1.21}$$

there is a global weak solution $\left(\rho, u, E = \frac{1}{2}|u|^2 + e\right)$ of (1.1) - (1.3) with initial data $\left(\rho_0, u_0, E_0 = \frac{1}{2}|u_0|^2 + e_0\right)$, in the sense of (1.4) - (1.6), satisfying

$$\begin{cases} \rho - \tilde{\rho} \in C\left([0,\infty);\ H^{-1}(\mathbb{R}^n)\right),\ \rho(\cdot,t) - \tilde{\rho} \in (L^2 \cap L^\infty)(\mathbb{R}^n),\ t \ge 0, \\ u \in C\left((0,\infty);\ L^2(\mathbb{R}^n)\right), \\ e - \tilde{e} \in C\left((0,\infty);\ L^2(\mathbb{R}^n)\right); \end{cases} \tag{1.22}$$

$$\begin{cases} \rho \in [\underline{\rho}, \overline{\rho}] \text{ a.e.}, \\ e(\cdot,t) \ge \underline{e} \text{ a.e.}; \end{cases} \tag{1.23}$$

$$u(\cdot,t),\ \omega(\cdot,t),\ F(\cdot,t),\ e(\cdot,t) - \tilde{e} \in H^1(\mathbb{R}^n),\ t > 0; \tag{1.24}$$

$$u_t(\cdot,t),\ \dot{u}(\cdot,t) \in L^2(\mathbb{R}^n),\ t > 0; \tag{1.25}$$

$$A(t),\ B(t) \le CC_0,\ t > 0; \tag{1.26}$$

$$\lim_{t\to\infty} \left\|\left(\rho(\cdot,t) - \tilde{\rho},\ u(\cdot,t),\ e(\cdot,t) - \tilde{e}\right)\right\|_{L^p(\mathbb{R}^n)} = 0 \tag{1.27}$$

for all $p \in (2,\infty]$. In addition, for global positive constants θ and q, which depend only on n,

$$\begin{aligned} &\|\rho(\cdot,t) - \tilde{\rho}\|_{L^\infty(\mathbb{R}^n)} + (1 \wedge t)^q \left(\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} + \|e(\cdot,t) - \tilde{e}\|_{L^\infty(\mathbb{R}^n)}\right) \\ &\qquad \le CC_0^\theta,\ t > 0; \end{aligned} \tag{1.28}$$

$$\langle u \rangle^{\alpha,\alpha/(n+2\alpha)}_{\mathbb{R}^n \times [t,\infty)} \le CC_0^\theta (1 \wedge t)^{-q},\ t > 0, \tag{1.29}$$

where $\alpha = \frac{1}{2}$ *when* $n = 3$ *and* $\alpha \in (0,1)$ *when* $n = 2$ *(*C *may depend additionally on* α *in the* $n = 2$ *case). The following additional regularity results hold in the polytropic case that* $T = Ce$*:*

$$\sup_{t>0}(1 \wedge t)^q \int \dot{e}(x,t)^2 dx + \int_0^\infty \int (1 \wedge t)^q |\nabla \dot{e}(x,t)|^2 dx dt \le C C_0^\theta \tag{1.30}$$

and

$$\langle e \rangle^{\alpha,\alpha/(n+2\alpha)}_{\mathbb{R}^n \times [t,\infty)} \le C C_0^\theta (1 \wedge t)^{-q}, \ t > 0, \tag{1.31}$$

where α *and* C *are just as in* (1.29)*, and* q *and* θ *depend only on* n*.*

Finally, whether the fluid is polytropic or not, the weak form (1.14) *of the equation* (1.13) *for* e *holds for times* $t_2 > t_1 > 0$ *when* $n = 2$ *and times* $t_2 > t_1 \ge 0$ *when* $n = 3$*; and* (1.14) *holds for* $t_1 = 0$ *when* $n = 2$ *provided that we make the additional assumption that* $u_0 \in H^{s_1}$ *for some* $s_1 > 0$*.*

We record here a version of Theorem 1.1 for the isentropic and isothermal systems corresponding to (1.1) - (1.3). In these systems P is assumed to be a function of ρ alone,

$$P(\rho) = K\rho^\gamma, \ \gamma \ge 1, \ K = \text{const.} \tag{1.32}$$

and the energy equation is omitted. The system (1.1) - (1.3) then takes the simpler form

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0 \\ (\rho u^j)_t + \operatorname{div}(\rho u^j u) + P(\rho)_{x_j} = \varepsilon \Delta u^j + \lambda \operatorname{div} u_{x_j} \end{cases} \tag{1.33}$$

for the unknown functions ρ and u, with given initial data (ρ_0,u_0). In place of (1.9) - (1.10) we assume that

$$0 \le \frac{\lambda}{\varepsilon} < \begin{cases} 1 + \sqrt{2}, & n = 2, \\ 1 + \frac{1}{3}\sqrt{15}, & n = 3, \end{cases} \tag{1.34}$$

and we define C_0, A, and B just as in (1.17), (1.19), (1.20), but with all the terms involving e omitted. We then have the following theorem, which is a slight strengthening of the results of [9] and [10]:

Theorem 1.2. *Fix the system parameters* $P, \varepsilon,$ *and* λ *as above in* (1.32) *and* (1.34)*, choose* ℓ *as in* (1.17) *and* $s, r,$ *and* β *as in* (1.18)*, and let* $\overline{\rho} > \tilde{\rho} > \underline{\rho} > 0$ *be given. Then there are positive constants* a *and* C *such that, given initial data* (ρ_0,u_0) *with* $C_0 \le a$*, there is* *a global weak solution of the system* (1.33) *in the sense of* (1.4) - (1.5), *satisfying all the conclusions* (1.22) - (1.29) *of Theorem 1.1 which pertain to* ρ *and* u.

We now give a brief summary of the heuristic observations made in Hoff [9] concerning the propagation of singularities in solutions of (1.1) - (1.3). A similar discussion at the formal level is given in Serre [16]; see also the remarks of Serrin [17], pg. 220, who refers to the much earlier article of Duhem [4]. First, the equations (1.2) and (1.13) for u and e are parabolic, so that some smoothing for u and e is to be expected. The same is true for the vorticity ω, which satisfies an equation similar to (1.2), but with the pressure and divergence terms absent. There is no viscosity in the mass equation (1.1), however, so that singularities in ρ can be expected to persist for all time. A straightforward application of the Rankine-Hugoniot condition to (1.1) and (1.2) shows first that these singularities convect along particle trajectories, and second that the quantity $F = (\varepsilon + \lambda)D - P + \tilde{P}$ is free of jump discontinuities. This is the first indication of the important role played by the effective viscous flux F. To summarize, we anticipate some parabolic smoothing for u, e, ω, and F, but not for ρ, and consequently not for P. Therefore, if $F(\cdot,t) \in H^1$ for $t > 0$, but $P(\cdot,t) - \tilde{P} \notin H^1$ (we abbreviate $P(\rho(x,t), e(x,t))$ by $P(x,t)$), then $D(\cdot,t) \notin H^1$, so that $u(\cdot,t) \in H^1 - H^2$. In particular, $\rho(\cdot,t), P(\cdot,t)$, and $\nabla u(\cdot,t)$ may be discontinuous across hypersurfaces of $\mathbb{R}^n$.

We shall make frequent use of the pde for F, which is derived as follows. First write

$$\begin{aligned} \Delta u^j &= u^j_{kk} = u^k_{kj} + (u^j_k - u^k_j)_k \\ &= D_j + \omega^{j,k}_k \\ &= (\varepsilon + \lambda)^{-1}(F_j + P_j) + \omega^{j,k}_k. \end{aligned} \tag{1.35}$$

As we shall see shortly, it is by way of this decomposition that estimates for ∇u can be derived from properties of the smoother variables F and ω. Substituting (1.35) into (1.2), we obtain the following alternative form of the momentum equation:

$$\rho \dot{u}^j - F_j = \varepsilon \omega^{j,k}_k;$$

and taking $\dfrac{\partial}{\partial x_k}$ and summing over j, we then arrive at the fundamental relation

$$\Delta F = \operatorname{div} \rho \dot{u}. \tag{1.36}$$

Now, the solution (ρ, u, e) of Theorem 1.1 will be obtained as the limit of approximate, smooth solutions which result by solving the system (1.1) - (1.2), (1.13) with mollified initial data. The bulk of the analysis therefore consists in obtaining *a priori* bounds for these approximate solutions, bounds which depend only on the initial norm given in (1.17). The logical connections among these bounds may be pictured as follows:

I. Energy Estimates
$A, B \leq CC_0$.

III. Pointwise Bounds
a. $e \geq \underline{e} > 0$.
b. $\|e(\cdot,t) - \tilde{e}\|_{L^\infty} \leq CC_0^\theta (1 \wedge t)^{-q}$.
c. $\|\rho - \tilde{\rho}\|_{L^\infty} \leq CC_0^\theta$.

II. Auxiliary Bounds
a. $\langle u(\cdot,t)\rangle^\alpha \leq CC_0^\theta (1 \wedge t)^{-q}$.
b. $\int_1^t \|F(\cdot,t')\|_{L^\infty} dt' \leq C(t) C_0^\theta$.
c. $\int_0^1 \|e(\cdot,t) - \tilde{e}\|_{L^\infty} dt \leq CC_0^\theta$.

The energy estimates I are derived under the assumption that ρ and e satisfy the pointwise bounds in III. These pointwise bounds are required for the application of the standard entropy estimate, Lemma 2.5, which gives time-independent bounds in L^2 for $\rho(\cdot,t) - \tilde{\rho}$, $u(\cdot,t)$, and $e(\cdot,t) - \tilde{e}$. Time-independent estimates for the other terms in A and B are then derived in a fairly lengthy sequence of energy estimates in sections 2 and 3. We remark here that there seems to be no mechanism which enforces an upper bound on $e(x,t)$ for t near 0; the entropy estimate of Lemma 2.5 is therefore applied only away from $t = 0$, and a different derivation of L^2 bounds is given in Theorem 2.2 for small times. The lower bound IIIa, on the other hand, plays a much more important role, and is required even for t near 0. In this regard we remark that the corresponding inviscid system, consisting of the compressible Euler equations, is hyperbolic only when $P_\rho > 0$, that is, when $e > 0$. As we shall see, this hyperbolicity plays an important role in the present analysis as well.

There is a deeper issue which arises in the derivation of the bounds I for A and B: due to the quadratic convection terms $u \cdot \nabla u^j$ in (1.2), there occur in these energy estimates higher order terms, such as $\displaystyle\int\!\!\int (1 \wedge t)^q |\nabla u|^4 dx dt$. Now, a bound for Δu in L^2 would suffice to control such terms, since $W^{2,2}$ is continuously imbedded in $W^{1,4}$ in two and three space dimensions. However, as we have observed above, ∇u can be discontinuous across hypersurfaces of $\mathbb{R}^n$, so that $D_x^2 u$ need not be locally integrable. We are therefore prevented from applying the Sobolev imbedding $W^{2,2} \to W^{1,4}$ in a straightforward way. On the other hand, the decomposition (1.35) shows that Δu^j is a linear combination of ∇F, ∇P, and $\nabla \omega$. Integrating once, we find that u_k^j is a linear combination of convolutions of appropriate Riesz kernels with $F, P - \tilde{P}$, and components of ω. The imbedding $W^{1,2} \to L^4$ may then be applied to the F and ω terms, and the inclusion $L^2 \cap L^\infty \subset L^4$ to the $P - \tilde{P}$ term.

Applying the Marcinkiewicz theorem, we can then obtain a bound for the higher order integral $\iint (1\wedge t)^q |\nabla u|^4 dx dt$ in terms of quantities appearing in the definitions of A and B; the overall bounds for A and B are then closed by a smallness argument.

The estimates in II follow in a straightforward way from those in I by way of the Sobolev imbeddings, and are given in Lemmas 4.3 and 4.4. We note here that it is estimate IIc which determines the values of the parameters r, s, and β in the definition of B: the integral in IIc is finite precisely because of the t^r smoothing rate near $t = 0$ implied in the definition of B for $\int |\nabla e|^2 dx$. Working through the somewhat complicated coupling of the higher-order norms for u and e, we find that this t^r smoothing rate depends in turn on the t^s smoothing rate for $\int |\nabla u|^2 dx$; it is for this reason that we assume that $u_0 \in H^s$.

This leads us back to the pointwise bounds III for ρ and e. The lower bound IIIa for e is obtained in Theorem 2.1 by a straightforward maximum principle argument, which is based on the special form of the viscosity terms in (1.13) and on our assumptions (1.9) and (1.10) concerning ε and λ. To prove the bound IIIb, we observe a parabolic smoothing from L^2 to L^∞ in the mapping $e_0 - \tilde{e} \to e(\cdot,t) - \tilde{e}$ by showing that the adjoint mapping backwards in time is bounded from L^1 to L^2. The adjoint operator solves a parabolic equation with nonconstant coefficients; the required estimates, which are given in Lemma 4.2, are reminiscent of the first steps in the Nash theory. Observe that IIIb does not follow from the bounds in I, which give only that $e(\cdot,t) - \tilde{e} \in H^1$.

The pointwise bound in IIIc for ρ is more subtle, and, in fact, is one of the focal points of the entire analysis, since it brings together nearly all the other estimates. We shall comment in some detail on the proof of IIIc, which is given in Lemma 4.5. First, writing the mass equation (1.1) in the form $\dot{\rho} = -\rho D$ and adding and subtracting terms, we obtain

$$(\varepsilon+\lambda)\frac{d}{dt}[\log\rho - \log\tilde{\rho}] = -(\varepsilon+\lambda)D = -F - P + \tilde{P},$$

so that, by (1.7),

$$\begin{aligned}(\varepsilon+\lambda)\frac{d}{dt}[\log\rho - \log\tilde{\rho}] &+ (\gamma-1)\tilde{e}(\rho-\tilde{\rho}) \\ &= -F - (\gamma-1)\rho(e-\tilde{e}).\end{aligned} \tag{1.37}$$

There are two important points to be made concerning this ode. The first is that, since $\log\rho$ is an increasing function of ρ, the second term on the left here is dissipative, enforcing an exponential decay in t as $t\to\infty$. We can therefore obtain the time-independent bound IIIc for $\rho - \tilde{\rho}$, at least away from $t = 0$, by applying this exponential decay together with the estimates IIb and IIIb for the driving terms on the right side of (1.37). Observe that this device for obtaining a large-time estimate for ρ succeeds precisely because we substituted for D in terms of F in the mass equation, thereby introducing $P - \tilde{P}$ with the "right sign." The fact that the pressure enters here with the right sign is no accident, but rather is a direct consequence of the hyperbolicity of the corresponding inviscid equations. There is a second important issue arising in the proof of IIIc, namely, the singularity of the driving terms in (1.37) at $t = 0$. Integrating (1.37) along a particle path $x(t)$, we obtain an estimate for $\log\rho - \log\tilde{\rho}$ which involves

$$\int_0^t \left[F\left(x(s),s\right) + (\gamma - 1)\rho\left(x(s),s\right)\left(e\left(x(s),s\right) - \tilde{e}\right)\right] ds.$$

Now, the $e - \tilde{e}$ term here is controlled by the estimate in IIc. We do not have sufficient information about F, however, to conclude that $\int_0^t \|F(\cdot,s)\|_{L^\infty} ds$ is finite; indeed, elementary considerations suggest that it is not. We therefore instead apply (1.36) to write

$$\begin{aligned} \int_0^t F\left(x(s),s\right) ds &= \int_0^t \Delta^{-1}\mathrm{div}\ \left[(\rho\dot{u})\left(x(s),s\right)\right]\ ds \\ &= \int_0^t \left(\Gamma_j * (\rho\dot{u}^j)(\cdot,s)\right)\left(x(s),s\right) ds, \end{aligned} \tag{1.38}$$

where Γ is the fundamental solution of the Laplace operator. This integral is taken over the particle path $x(t)$, as is the material derivative $\dot{u}^j$. There is therefore a crucial cancellation (integration by parts) which results in an integral and boundary terms which involve lower-order regularity of u. Explicit computation of the smoothing rate t^{-q} in the Hölder continuity estimate IIa then suffices to bound the resulting integral.

The effective viscous flux F thus plays a crucial role at three different points of the analysis, and in three distinct ways. First, it is the *regularity* of F (and of ω) which enables us to estimate higher L^p norms of ∇u by way of the decomposition (1.35). Second, by replacing D by F in the large-time estimate for $\rho - \tilde{\rho}$, we find that the pressure enters with the right sign, thereby revealing an important *hyperbolic* effect. Third, we are able to complete the small-time estimate for $\rho - \tilde{\rho}$ only by substituting for F in (1.38) in terms of the *convective* derivative $\dot{u}$, thereby obtaining the required cancellation.

The energy estimates I for the quantities A and B are given in sections 2 and 3; the auxiliary bounds II, as well as the pointwise bounds in III, are then derived in section 4. In section 5 we apply these *a priori* bounds to the smooth solutions corresponding to mollifed initial data, thereby obtaining the compactness required to extract a limiting solution (ρ,u,e), as well as the regularity properties (1.22) - (1.26) and (1.28) - (1.31). The large-time result (1.27) is proved in section 6.

We shall make repeated use of various standard Sobolev inequalities and imbedding theorems. The most basic of these is the bound

$$\int |w|^{n/(n-1)} dx \le C(n) \left(\int |\nabla w| dx \right)^{n/(n-1)} \tag{1.39}$$

for $w \in W^{1,1}(\mathbb{R}^n)$ (see [20], Theorem 2.4.1). Applying Hölder's inequality in an elementary way, we then easily derive from (1.39) the estimates

$$\int |w|^p dx \le C(p) \left(\int w^2 dx \right) \left(\int |\nabla w|^2 dx \right)^{(p-2)/2} \tag{1.40}$$

for $p \ge 2$ when $n = 2$, and

$$\int |w|^p dx \le C(p) \left(\int w^2 dx \right)^{(6-p)/4} \left(\int |\nabla w|^2 dx \right)^{(3p-6)/4} \tag{1.41}$$

for $p \in [2,6]$ when $n = 3$. Next, for $p > n$, we have that

$$\|w\|_{L^\infty(\mathbb{R}^n)} \le C(n,p) \|w\|_{W^{1,p}(\mathbb{R}^n)} \tag{1.42}$$

and

$$\langle w \rangle^{\alpha}_{\mathbb{R}^n} \le C(n,p) \|\nabla w\|_{L^p(\mathbb{R}^n)}, \tag{1.43}$$

where $\alpha = 1 - n/p$ (see [20], Remark 2.4.3 and Theorem 2.4.4). Finally, we note the following elementary fact about Riesz potentials: If Γ is the fundamental solution of the Laplace operator on $\mathbb{R}^n$, then given $p_1 \in [1,n)$ and $p_2 \in (n,\infty]$, there is a constant $C = C(n,p_1,p_2)$ such that

$$\|\Gamma_{x_j} * w\|_{L^\infty(\mathbb{R}^n)} \le C \left[\|w\|_{L^{p_1}(\mathbb{R}^n)} + \|w\|_{L^{p_2}(\mathbb{R}^n)} \right] \tag{1.44}$$

for $w \in L^{p_1} \cap L^{p_2}$. The proof, which is quite easy, is given at the end of section 1 of [9].

§2. L^2 Bounds

In this section we derive *a priori* bounds in L^2 for smooth solutions (ρ,u,e) of (1.1) - (1.2), (1.13). Specifically, we show in Theorem 2.2 below that the standard parabolic norm A defined in (1.19) is bounded in terms of the initial data, independently of time. First, however, we need to obtain an estimate from below for the specific internal energy e:

Theorem 2.1. *There is a positive time τ, depending on the same parameters as a and C in Theorem 1.1, such that, if (ρ,u,e) is any smooth solution of (1.1) - (1.2), (1.13) on $\mathbb{R}^n \times [0,\bar{t}]$ with $\rho(x,t) \in [\underline{\rho}\,,\bar{\rho}]$, whose initial data (ρ_0,u_0,e_0) satisfies the hypotheses (1.21) of Theorem 1.1, then*

$$e(x,t) \geq \underline{e} \tag{2.1}$$

for $x \in \mathbb{R}^n$ and $0 \leq t \leq \tau \wedge \bar{t}$.

Proof: First, by expanding and collecting terms appropriately, we obtain that

$$\begin{cases} |\nabla u|^2 + u_k^j u_j^k - D^2 \geq 0, & n = 2, \\ |\nabla u|^2 + u_k^j u_j^k - \frac{2}{3} D^2 \geq 0, & n = 3. \end{cases}$$

It therefore follows from (1.13) that

$$\rho \dot{e} + PD - \Delta T \geq \begin{cases} \lambda D^2, & n = 2, \\ \left(\lambda - \frac{1}{3}\varepsilon\right) D^2, & n = 3, \end{cases}$$

so that, by our assumptions (1.9) and (1.10) on ε and λ,

$$\begin{aligned} \rho \dot{e} - \Delta T &\geq C^{-1} D^2 - (\gamma - 1)\rho |eD| \\ &\geq C^{-1} D^2 - C|eD| \geq -Ce^2. \end{aligned}$$

The maximum principle, together with our hypothesis (1.21) that $e_0(x) \geq e_1 > \underline{e}$, then shows that $e(x,t) \geq \underline{e}$ for all x and for $t \leq \tau \wedge \bar{t}$, for some small, positive time τ. //

Theorem 2.2. *There is a number $b > 0$ such that, if (ρ,u,e) is any solution of (1.1) - (1.2), (1.13) on $\mathbb{R}^n \times [0,\bar{t}]$ satisfying the hypotheses of Theorem 2.1, as well as the pointwise bounds*

$$\begin{gathered} \underline{\rho} \leq \rho(x,t) \leq \bar{\rho}, \quad e(x,t) \geq \underline{e}, \quad 0 \leq t \leq \tau \wedge \bar{t} \\ |\big(\rho(x,t), u(x,t), e(x,t)\big) - (\tilde{\rho},0,\tilde{e})| \leq b, \quad \tau \wedge \bar{t} \leq t \leq \bar{t}, \end{gathered}$$

where τ is as in Theorem 2.1, then

$$A(\bar{t}) \leq CC_0,$$

where C is a positive constant as described in Theorem 1.1. In particular, C is independent of $\bar{t}$.

Proof: First, in this section only, we let $\varphi(x) = (1+|x|^2)^\ell$, where ℓ is as in (1.17), and we define auxiliary norms by

$$\begin{aligned} A_1(t) &= \sup_{0\le t'\le t} \int \left[\big(\rho(x,t') - \tilde\rho\big)^2 + |u(x,t')|^2 + \big(e(x,t') - \tilde e\big)^2 \right] \varphi(x)dx \\ &\quad + \int_0^t \int \left(|\nabla u|^2 + |\nabla e|^2\right) \varphi\, dxdt', \\ A_2(t) &= \sup_{0\le t'\le t} \int |u(x,t')|^4 \varphi(x) dx + \int_0^t \int |u|^2 |\nabla u|^2 \varphi\, dxdt', \end{aligned}$$

so that $A(t) = A_1(t) + A_2(t)$ when $t \le 1$. We prove Theorem 2.2 in a sequence of three lemmas: in Lemmas 2.3 and 2.4 we obtain bounds for A_1 and A_2 for $t \le \bar{t} \wedge 1$; and in Lemma 2.4 we apply an entropy argument to estimate $A(\bar{t})$ in terms of $A(1)$ when $\bar{t} > 1$. The hypotheses of Theorem 2.2 will be in force throughout the remainder of this section.

Lemma 2.3. $A_1(t) \le C\left[C_0 + A_2(t) + t^{1/4} A_1(t) A_2(t)^{1/2}\right],\ 0 \le t \le \bar{t} \wedge 1$.

Proof: We estimate only the terms in A_1 involving e; bounds for the ρ and u terms are similar, and, in fact, significantly simpler. Multiplying (1.13) by $e - \tilde e$ and rearranging, we obtain that

$$\begin{aligned} \tfrac12 \rho \frac{d}{dt}(e-\tilde e)^2 + \nabla e \cdot \nabla T &= (e-\tilde e)\big[\nabla P \cdot u + \varepsilon |\nabla u|^2 + \lambda D^2\big] \\ &\quad + \nabla e \cdot [Pu + \varepsilon D u - \varepsilon(\nabla u) u] \\ &\quad + \operatorname{div}\big[O\big(|e-\tilde e|(e|u| + |\nabla e| + |u||\nabla u|)\big)\big]. \end{aligned} \tag{2.2}$$

We substitute from the kinetic energy equation (1.12) to rewrite the first brackets on the right here as

$$(e-\tilde e)\left[-\rho \frac{d}{dt} \tfrac12 |u|^2 + \varepsilon \Delta\left(\tfrac12 |u|^2\right) + \lambda \operatorname{div}(Du)\right],$$

which after some manipulation becomes

$$\begin{aligned} &- \left(\tfrac12 \rho |e-\tilde e|^2\right)_t \\ &\quad + \operatorname{div}\left[O\left(|e-\tilde e|(|u|^3 + |u||\nabla u|) + |u|^2 |\nabla e|\right)\right] \\ &\quad + O(|\nabla e||u||\nabla u| + e|u|^2|\nabla u| + |u|^2|\nabla u|^2). \end{aligned}$$

Making this substitution into (2.2) and applying the mass equation (1.1) to write the term $\frac{1}{2}\rho\dfrac{d}{dt}|e-e|^2$ in divergence form, we then obtain that

$$\begin{aligned}\left(\tfrac{1}{2}\rho|e-\tilde{e}|^2\right)_t + \nabla e\cdot\nabla T &= -\left(\tfrac{1}{2}\rho(e-\tilde{e})|u|^2\right)_t \\ &+ \operatorname{div}\left[O\left((e-\tilde{e})^2|u| + |e-\tilde{e}|(e|u| + |\nabla e| + |u||\nabla u| + |u|^3) + |\nabla e||u|^2\right)\right] \\ &+ O\left[|\nabla e|(|u||\nabla u| + e|u|) + e|u|^2|\nabla u| + |u|^2|\nabla u|^2\right].\end{aligned}$$

We now multiply by the weight $\varphi(x) = (1+|x|^2)^\ell$ and integrate, noting that

$$|\nabla\varphi| \le C\varphi, \tag{2.3}$$

to obtain

$$\begin{aligned}\tfrac{1}{2}\int \rho(e-\tilde{e})^2\varphi\,dx\Big|_0^t &+ \int_0^t\int \nabla e\cdot\nabla T\,\varphi\,dxdt' \\ &= \int_0^t\int O\big(|e-\tilde{e}||u|^3 + |e-\tilde{e}|^2|u| + |e-\tilde{e}||u|^2|\nabla u| + e|\nabla e|u\big)\varphi\,dxdt' \\ &\quad + \dots,\end{aligned}$$

where $+\dots$ denotes a sum of terms, each of which is easily seen to be bounded by

$$C\left[C_0 + t^{1/2}A_1(t) + A_2(t) + A_1(t)^{1/2}A_2(t)^{1/2}\right],$$

plus terms which can be absorbed into the left-hand side. Performing a few more simple reductions, we obtain finally that

$$\begin{aligned}\int \big(e(x,t)-\tilde{e}\big)^2\varphi(x)dx &+ \int_0^t\int |\nabla e|^2\varphi\,dxdt' \\ &\le C\left[C_0 + t^{1/2}A_1(t) + A_2(t) + A_1(t)^{1/2}A_2(t)^{1/2}\right] \\ &\qquad + C\int_0^t\int |e-\tilde{e}|^2|u|^2\varphi\,dxdt'.\end{aligned} \tag{2.4}$$

For the case that $n=2$, we apply (1.40) and (2.3) to bound the double integral on the right by

$$\begin{aligned}&\sup_{t'\le t}\left(\int |u|^4\varphi\,dx\right)^{1/2}\int_0^t\left(\int\left[(e-\tilde{e})\varphi^{1/4}\right]^4dx\right)^{1/2}dt' \\ &\le CA_2^{1/2}\int_0^t\left(\int(e-\tilde{e})^2\varphi^{1/2}dx\right)^{1/2}\left[\left(\int|\nabla e|^2\varphi^{1/2}dx\right)^{1/2}\right. \\ &\qquad\qquad\left. + \left(\int(e-\tilde{e})^2\varphi^{-3/2}|\nabla\varphi|^2dx\right)^{1/2}\right]dt' \\ &\le CA_2^{1/2}(tA_1 + t^{1/2}A_1) \le Ct^{1/2}A_1A_2^{1/2},\end{aligned}$$

since $\varphi^{1/2} \le \varphi$. When $n = 3$, $\varphi \equiv 1$, and a similar but simpler argument yields the same bound, but with $t^{1/2}$ replaced by $t^{1/4}$. We thus obtain from (2.4) that

$$\begin{aligned}&\int \left[e(x,t) - \tilde{e}\right]^2 \varphi(x)dx + \int_0^t \int |\nabla e|^2 \varphi \, dx dt' \\ &\qquad \le C\left[C_0 + t^{1/2}A_1(t) + A_2(t) + t^{1/4}A_1(t)A_2(t)^{1/2} + A_1(t)^{1/2}A_2(t)^{1/2}\right].\end{aligned}$$

Estimating the other terms in the definition of A_1 in a similar way and taking an appropriate sup, we obtain finally that

$$A_1(t) \le C\left[C_0 + t^{1/2}A_1(t) + A_2(t) + t^{1/4}A_1(t)A_2(t)^{1/2}\right].$$

The term $t^{1/2}A_1(t)$ can then be eliminated by a simple Gronwall-type argument. This proves the lemma. //

Lemma 2.4. $A_2(t) \le C\left[C_0 + A_2(t)^{3/2} + A_1(t)A_2(t)^{1/2}\right]$, $0 \le t \le \bar{t} \wedge 1$.

Proof: We multiply the kinetic energy equation (1.12) by $|u|^2$ and rearrange to obtain

$$\begin{aligned}&\left(\tfrac14 \rho|u|^4\right)_t + \tfrac12 \varepsilon |\nabla(|u|^2)|^2 + \varepsilon|u|^2|\nabla u|^2 + \lambda|u|^2 D^2 \\ &\qquad = -\lambda D u \cdot \nabla(|u|^2) + (P - \tilde{P})\operatorname{div}(|u|^2 u) \\ &\qquad\qquad + \operatorname{div}\left[O\big(|u|^5 + |u|^3||\nabla u| + |P-\tilde{P}||u|^3\big)\right].\end{aligned}$$

Multiplying by φ and integrating, taking (2.3) into account, we then obtain that

$$\begin{aligned}&\tfrac14 \int \rho(x,\cdot)|u(x,\cdot)|^4\varphi(x)dx\Big|_0^t + \int_0^t \int \left[\tfrac12 \varepsilon|\nabla(|u|^2)|^2 + \varepsilon|u|^2|\nabla u|^2 + \lambda|u|^2D^2\right]\varphi \, dxdt' \\ &\qquad = -\lambda \int_0^t \int Du \cdot \nabla(|u|^2)\varphi \, dxdt' \qquad (2.5) \\ &\qquad\quad + O\Big(\int_0^t \int \left[|P-\tilde{P}||u|^2|\nabla u| + |u|^5 + |u|^3|\nabla u| + |P-\tilde{P}||u|^3\right]\varphi \, dxdt'.\end{aligned}$$

The first term on the right here is bounded by

$$\lambda\left[\tfrac12 \eta \iint |u|^2 D^2 \varphi + \tfrac12 \eta^{-1} \iint |\nabla(|u|^2)|^2\varphi\right]$$

for $\eta > 0$. Choosing η so that $\frac12 \lambda\eta = \lambda + \mu\varepsilon$ with $\mu \in (0,1/n)$, we then obtain the bound

$$\lambda \iint |u|^2 D^2 \varphi + n\mu\varepsilon \iint |u|^2|\nabla u|^2 \varphi + \frac{\lambda^2}{4(\lambda+\mu\varepsilon)} \iint |\nabla(|u|^2)|^2\varphi$$

(we used here the fact that $D^2 \le n|\nabla u|^2$). We thus obtain from (2.5) that

$$
\begin{aligned}
&\tfrac{1}{4}\int \rho|u|^4\varphi\,dx\Big|_0^t + \left[\frac{\varepsilon}{2} - \frac{\lambda^2}{4(\lambda+\mu\varepsilon)}\right]\iint |\nabla(|u|^2)|^2\varphi\,dxdt' \\
&\qquad + \varepsilon(1-n\mu)\iint |u|^2|\nabla u|^2\varphi\,dxdt' \\
&\quad \le C\iint \left[|P-\tilde{P}||u|^2|\nabla u| + |u|^5 + |u|^3|\nabla u| + |P-\tilde{P}||u|^3\right]\,dxdt'.
\end{aligned}
$$

The hypotheses (1.9) - (1.10) insure that the brackets on the left is nonnegative. Making some elementary reductions in the terms on the right and incorporating the estimate given in the proof of Lemma 2.3 for the term $\iint |e-\tilde{e}|^2|u|^2\varphi$, we then arrive at

$$
\begin{aligned}
&\int |u(x,t)|^4\varphi(x)dx + \int_0^t\int |u|^2|\nabla u|^2\varphi\,dxdt' \\
&\quad \le C\left[C_0^2 + tA_1(t) + tA_2(t) + t^{1/4}A_1(t)A_2(t)^{1/2} + \int_0^t\int |u|^6\varphi\,dxdt'\right].
\end{aligned}
\tag{2.6}
$$

When $n=2$ we can apply (1.40) to bound the last term here by

$$
\begin{aligned}
&\int_0^t\int (|u|^2\varphi^{1/3})^3\,dxdt' \le C\int_0^t\left(\int (|u|^2\varphi^{1/3})^2dx\right)\left(\int |\nabla(|u|^2\varphi^{1/3})|^2dx\right)^{1/2}dt' \\
&\quad \le C\int_0^{t'}\left(\int |u|^4\varphi\,dx\right)\left[\left(\int |u|^4\varphi\,dx\right)^{1/2} + \left(\int |u|^2|\nabla u|^2\varphi\,dx\right)^{1/2}\right]dt' \\
&\quad \le Ct^{1/2}A_2(t)^{3/2},
\end{aligned}
$$

where we have again used (2.3) and the fact that $\varphi^{2/3} \le \varphi$. A similar argument yields the bound $Ct^{1/4}A_2(t)^{3/2}$ for the case that $n=3$. We apply these bounds in (2.6) and take the appropriate sup to obtain

$$
A_2(t) \le C\left[C_0^2 + tA_1(t) + tA_2(t) + A_2(t)^{3/2} + A_1(t)A_2(t)^{1/2}\right].
$$

Substituting from Lemma 2.3 for the $tA_1(t)$ term, and applying an elementary Gronwall-type argument, we then obtain the result of Lemma 2.4. //

We can now combine the estimates in Lemmas 2.3 and 2.4, together with the smallness hypothesis (1.21) and the fact that A is continuous in t, to conclude that

$$
A(t) \le CC_0,\ 0 \le t \le \bar{t} \wedge 1.
\tag{2.7}
$$

To complete the proof of Theorem 2.2, we therefore need to bound $A(t)$ for times $t > \overline{t} \wedge 1$:

Lemma 2.5. *If $\overline{t} > 1$, then $A(\overline{t}) \leq CA(1)$.*

Proof: This result is derived from standard entropy considerations, which we sketch. We can define an entropy $S(\rho,e)$ by

$$\frac{\partial S}{\partial e} = \frac{1}{e},\ \frac{\partial S}{\partial \rho} = -\frac{P}{\rho^2 e} = \frac{1-\gamma}{\rho},$$

and $S(\tilde{\rho},\tilde{e}) = 0$. One then easily computes from (1.1) and (1.13) that

$$(\rho S)_t = \operatorname{div}\{\cdot\} + \frac{\nabla e \cdot \nabla T}{e^2} + \frac{\varepsilon|\nabla u|^2 + \lambda D^2}{e} + \frac{\varepsilon}{e^2}\nabla e \cdot \big((\nabla u)u - Du\big),$$

where $\operatorname{div}\{\cdot\}$ denotes any term which is a perfect divergence in x. We can also define a quasilinearization L of S about $(\tilde{\rho},\tilde{e})$ by

$$L = -\frac{\tilde{P}}{\tilde{\rho}\tilde{e}}(\rho - \tilde{\rho}) + \frac{\rho}{\tilde{e}}\left[(e - \tilde{e}) + \tfrac{1}{2}|u|^2\right],$$

so that $L = C_1 + C_2\rho + C_3\rho E$ for constants C_1, C_2, and C_3, and therefore $L_t = \operatorname{div}\{\cdot\}$. Integrating, we thus obtain that

$$\begin{aligned}\int (L - \rho S)(x,\cdot)dx\Big|_1^t + \int_1^t \int \left[\frac{\nabla e \cdot \nabla T}{e^2} + \frac{\varepsilon|\nabla u|^2 + \lambda D^2}{e}\right] dx dt' \\ \leq C \int_1^t \int |u||\nabla u||\nabla e|\, dx dt'.\end{aligned} \tag{2.8}$$

On the other hand, if we expand $\rho S(\rho,e)$ about $(\tilde{\rho},\tilde{e})$, we find that, for values (ρ,u,e) occurring here,

$$L - \rho S \geq C^{-1}\left[(\rho - \tilde{\rho})^2 + |u|^2 + (e - \tilde{e})^2\right].$$

This together with (2.8) and the smallness hypothesis of Theorem 2.2 then proves the lemma. //

§3. Higher Order Regularity

In this section we derive *a priori* bounds for certain higher-order Sobolev norms of smooth solutions of (1.1) - (1.2), (1.13). We shall prove:

Theorem 3.1. *Let* (ρ,u,e) *be a smooth solution of* (1.1) - (1.2), (1.13) *on* $\mathbb{R}^n \times [0,\bar{t}]$ *satisfying the hypotheses of Theorem 2.2, and let* B *be as defined in* (1.20). *Then*

$$B(t) \le CC_0,\ 0 \le t \le \bar{t}. \tag{3.1}$$

Also, in the polytropic case that T *is constant multiple of* e,

$$\sup_{0\le t\le \bar{t}} \left[(1\wedge t)^q \int \dot{e}(x,t)^2 dx\right] + \int_0^{\bar{t}} \int (1\wedge t)^q |\nabla \dot{e}(x,t)|^2 dx dt \le CC_0^\theta, \tag{3.2}$$

where q *and* θ *are positive constants depending only on* n, *and* C *is a constant as described in Theorem 1.1; in particular,* C *is independent of* $\bar{t}$.

To facilitate the proof we group the terms in B by defining

$$B_1(t) = \sup_{0\le t'\le t} \left[(1\wedge t')^{1-s} \int |\nabla u(x,t')|^2 dx\right] + \int_0^t \int (1\wedge t')^{1-s+\beta}(|\dot{u}|^2 + |\nabla \omega|^2) dx dt',$$

$$B_2(t) = \sup_{0\le t'\le t} \left[(1\wedge t')^{r} \int (|\dot{u}(x,t')|^2 + |\nabla \omega(x,t')|^2) dx\right] + \int_0^t \int (1\wedge t')^{r} |\nabla \dot{u}|^2 dx dt',$$

$$B_3(t) = \sup_{0\le t'\le t} \left[(1\wedge t')^{r} \int |\nabla e(x,t')|^2 dx\right] + \int_0^t \int (1\wedge t')^{r} \dot{e}^2 dx dt',$$

so that $B = B_1 + B_2 + B_3$. We shall give the proof in a sequence of lemmas, and for the $n = 3$ case only, because it is here that the special hypothesis that $u_0 \in H^s$ comes into play; the proof for $n = 2$ is considerably simpler. The hypotheses of Theorem 3.1 will be in force throughout this section.

Lemma 3.2. $\|u(\cdot,t)\|_{H^s(\mathbb{R}^n)} \le CC_0^{1/2},\ 0 \le t \le \bar{t} \wedge 1.$

Proof: We obtain this result by estimating the dual norm of the solution to the system which is dual to the momentum equation. This adjoint system is linear, so that the fractional norms of its solutions are accessible by interpolation. To do this, we write the weak form of the momentum equation (1.2) for a suitable smooth vector field $\varphi = (\varphi^1,\varphi^2,\varphi^3)$:

$$\begin{aligned}\int (\rho u \cdot \varphi)(x,\cdot) dx\Big|_0^{t_1} - \int_0^{t_1} \int \rho u^j \left[\varphi_t^j + \nabla \varphi^j \cdot u + \rho^{-1}(\varepsilon \Delta \varphi_j + \lambda \operatorname{div} \varphi_j)\right] dx dt \\ = \int_0^{t_1} \int (P - \tilde{P}) \operatorname{div} \varphi \, dx dt,\ 0 < t_1 \le \bar{t} \wedge 1.\end{aligned} \tag{3.3}$$

Now choose φ to solve the adjoint system

$$(3.4)\qquad \begin{cases} (\rho\varphi^j)_t + \mathrm{div}\,(\rho\varphi^j u) + \varepsilon\Delta\varphi^j + \lambda \mathrm{div}\,\varphi_j = 0, \ t < t_1, \\ (\rho\varphi)(x,t_1) = \chi(x), \end{cases}$$

where the components of χ are in the Schwartz class $\mathcal{S}$. Applying the mass equation (1.1), we find that the brackets in (3.3) vanishes, so that

$$(3.5)\qquad \int u(x,t_1)\cdot\chi(x)dx = \int \rho_0(x)u_0(x)\cdot\varphi(x,0)dx + \int_0^{t_1}\int (P-\tilde{P})\mathrm{div}\,\varphi\, dxdt.$$

We shall show below that the solution φ of (3.4) satisfies

$$(3.6)\qquad \|(\rho\varphi)(\cdot,t)\|_{H^{-s}} \le C\|\chi\|_{H^{-s}}, \ t \le t_1 \ ,$$

and

$$(3.7)\qquad \|\nabla\varphi(\cdot,t)\|_{L^2} \le C(t_1-t)^{-(1+s)/2}\|\chi\|_{H^{-s}}, \ t < t_1.$$

Granting this for the moment, we may then bound the first term on the right side of (3.5) by

$$C\|u_0\|_{H^s}\|(\rho\varphi)(\cdot,0)\|_{H^{-s}} \le CC_0^{1/2}\|\chi\|_{H^{-s}},$$

and the second term by

$$C\sup_t\left(\|(\rho-\tilde{\rho})(\cdot,t)\|_{L^2} + \|(e-\tilde{e})(\cdot,t)\|_{L^2}\right)\int_0^{t_1}\|\nabla\varphi(\cdot,t)\|_{L^2}dt \le CC_0^{1/2}\|\chi\|_{H^{-s}},$$

by Theorem 2.2. We thus obtain from (3.5) that

$$\left|\int u(x,t_1)\cdot\chi(x)dx\right| \le CC_0^{1/2}\|\chi\|_{H^{-s}}$$

for all χ; this proves the lemma.

To prove (3.6) we fix $t_0 \in [0,t_1)$ and choose $\psi = (\psi^1,\psi^2,\psi^3)$ to satisfy the system

$$(3.8)\qquad \begin{cases} (\rho\psi^j)_t + \mathrm{div}\,(\rho\psi^j u) = \varepsilon\Delta\psi^j + \lambda\mathrm{div}\,\psi_j, \ t > t_0 \\ \psi(x,t_0) = \Psi(x), \end{cases}$$

where the components of Ψ are in $\mathcal{S}$. Taking ψ as the test function in the weak form of the equation (3.4) for φ, and applying the mass equation (1.1), we then obtain that

$$(3.9)\qquad \int (\rho\varphi)(x,t_0)\cdot\Psi(x)dx = \int(\rho\varphi\cdot\psi)(x,t_1)dx = \int\chi(x)\cdot\psi(x,t_1)dx.$$

Now, straightforward energy estimates, which we omit, applied to (3.8), show that

$$\|\psi(\cdot,t)\|_{L^2} + (t-t_0)^{1/2}\|\nabla\psi(\cdot,t)\|_{L^2} \le C\|\Psi\|_{L^2}, \qquad t_0 \le t \le t_1\ , \tag{3.10}$$

and

$$\|\nabla\psi(\cdot,t)\|_{L^2} \le C\|\Psi\|_{H^1}, \qquad t_0 \le t \le t_1\ . \tag{3.11}$$

In particular, $\|\psi(\cdot,t)\|_{H^\alpha} \le C\|\Psi\|_{H^\alpha}$ for $\alpha = 0$ and 1. Since the solution operator $\Psi \mapsto \psi(\cdot,t)$ is linear, we may apply a suitable variant of the Riesz-Thorin theorem (see Donaghue [3]) to conclude that $\|\psi(\cdot,t)\|_{H^s} \le C\|\Psi\|_{H^s}$. It then follows from (3.9) that

$$\left|\int(\rho\varphi)(x,t_0)\cdot\Psi(x)dx\right| \le C\|\chi\|_{H^{-s}}\|\Psi\|_{H^s},$$

which proves (3.6). An alternative interpolation from (3.10) and (3.11) shows that

$$\|\psi(\cdot,t_1)\|_{H^s} \le C(t_1-t_0)^{-s/2}\|\Psi\|_{L^2},$$

so that, again by (3.9),

$$\|\varphi(\cdot,t_0)\|_{L^2} \le C(t_1-t_0)^{-s/2}\|\chi\|_{H^{-s}}. \tag{3.12}$$

We now derive (3.7) from (3.12) by the energy method, taking $t = t_0$ as the initial time. Thus from (3.4),

$$\frac{\partial}{\partial t}\left(\tfrac12\rho|\varphi|^2\right) + \operatorname{div}\left(\tfrac12\rho|\varphi|^2u\right) + \varepsilon\varphi^j\Delta\varphi^j + \lambda\varphi^j\operatorname{div}\varphi_j = 0,$$

so that, for $t < t_0$,

$$\int|\varphi(x,t)|^2dx + \int_t^{t_0}\int|\nabla\varphi|^2dxdt' \le C\int|\varphi(x,t_0)|^2dx \le C(t_1-t_0)^{-s}\|\chi\|^2_{H^{-s}}, \tag{3.13}$$

by (3.12). Next we multiply the equation

$$\rho\dot\varphi^j + \varepsilon\Delta\varphi^j + \lambda\operatorname{div}\varphi_j = 0 \tag{3.14}$$

by $(t_0-t)\dot\varphi^j$, sum over j, and integrate to obtain that

$$\begin{aligned}\int_t^{t_0}\int(t_0-t)\rho|\dot\varphi|^2dxdt' &+ \tfrac12(t_0-t)\int\left[\varepsilon|\nabla\varphi(x,t)|^2 + \lambda\big(\operatorname{div}\varphi(x,t)\big)^2\right]dx\\ &= \tfrac12\int_t^{t_0}\int\left[\varepsilon|\nabla\varphi|^2 + \lambda(\operatorname{div}\varphi)^2\right]dxdt'\\ &\quad + \int_t^{t_0}\int(t_0-t)\varphi^j_ku^k\rho\dot\varphi^j\,dxdt'.\end{aligned} \tag{3.15}$$

The first term on the right here is bounded by the right side of (3.13), and the second by (3.16)

$$\left(\sup_{t'}\int |u(x,t')|^4 dx\right)^{1/4} \int_t^{t_0} (t_0-t')\left(\int |\nabla\varphi|^4 dx\right)^{1/4}\left(\int |\dot\varphi|^2 dx\right)^{1/2} dt'$$
$$\le CC_0^{1/4}\int_t^{t_0}(t_0-t')\left(\int|\nabla\varphi|^2dx\right)^{1/8}\left(\int|D_x^2\varphi|^2dx\right)^{3/8}\left(\int|\dot\varphi|^2dx\right)^{1/2}dt',$$

by (1.41) and Theorem 2.2. Now, upon taking Fourier transforms in (3.14), we find that $\xi_j\xi_k\hat\varphi(\xi,t) = m(\xi)\widehat{(\rho\dot\varphi)}(\xi,t)$, where $m\in L^\infty(\mathbb{R}^n)$. It follows that $\|D_x^2\varphi(\cdot,t)\|_{L^2}\le C\|\dot\varphi(\cdot,t)\|_{L^2}$, so that the expression in (3.16) is bounded by

$$CC_0^{1/4}\int_t^{t_0}\left(\int|\nabla\varphi|^2dx\right)^{1/8}\left(\int(t_0-t')|\dot\varphi|^2dx\right)^{7/8}dt'$$
$$\le C\left[\eta^{-1}C_0^2\iint|\nabla\varphi|^2+\eta^{1/7}\iint(t_0-t')|\dot\varphi|^2\right]$$

for arbitrary $\eta>0$. The second term here can be absorbed into the left side of (3.15), and the first is bounded by the right side of (3.13). We thus obtain from (3.15) that

$$(t_0-t)\int|\nabla\varphi(x,t)|^2dx\le C(t_1-t_0)^{-s}\|\chi\|^2_{H^{-s}},\ 0\le t<t_0<t_1. \tag{3.17}$$

Now given $t<t_1$, we choose $t_0=\frac12(t+t_1)$ in (3.17) to obtain (3.7). //

The following lemma contains several auxiliary estimates which will be useful both here and in subsequent sections.

Lemma 3.3. *Assume that the hypotheses of Theorem 3.1 are in force, and let n be either 2 or 3. Then:*

(a) $\displaystyle\int|\nabla\omega(x,t)|^2dx\le C\int|\dot u(x,t)|^2dx,\ 0<t\le\bar t.$

(b) $$\int_0^t\int(1\wedge t')^r(\rho-\tilde\rho)^pdxdt'\le C\left[C_0+\int_0^t\int(1\wedge t')^r(|F|^p+|e-\tilde e|^p)\,dxdt'\right],$$
$$0\le t\le\bar t,$$

for any $p\in[2,\infty)$; (the constant C here may depend additionally on p).

(c) $\|\nabla u(\cdot,t)\|_{L^p}\le C\left[\|F(\cdot,t)\|_{L^p}+\|\omega(\cdot,t)\|_{L^p}+\|P(\cdot,t)-\tilde P\|_{L^p}\right],\ 0<t\le\bar t,$
where p and C are as in (b).

(d) $$\left|\int (u^{j_1}_{k_1}u^{j_2}_{k_2}u^{j_3}_{k_3})(x,t)dx\right|$$
$$\leq C\int \left[|\nabla u(x,t)|^2|w(x,t)| + |\nabla u(x,t)||w(x,t)|^2\right]\,dx$$
$$+C\|u(\cdot,t)\|_{L^4}\|\dot u(\cdot,t)\|_{L^2}\Big[\|F(\cdot,t)\|_{L^4}+\|\omega(\cdot,t)\|_{L^4}\Big],\quad 0<t\leq \bar t,$$

where $w=\{w^{j,k}\}$ *is defined by* $\widehat{w^{j,k}}(\xi,t)=|\xi|^{-2}\xi_j\xi_k\big(P(\cdot,t)-\tilde P\big)\widehat{\ }(\xi,t)$.

Proof: (a) is a consequence of the identity

$$\varepsilon|\nabla\omega^{j,k}|^2=\varepsilon\mathrm{div}\,(\omega^{j,k}\nabla\omega^{j,k})+(\rho\omega^{j,k}\dot u^k)_j-(\rho\omega^{j,k}\dot u^j)_k+\rho(\dot u^j\omega^{j,k}_k-\dot u^k\omega^{j,k}_j),$$

which is easily derived from the momentum equation (1.2).

To prove (b) we add and subtract $(\varepsilon+\lambda)^{-1}\rho(P-\tilde P)$ in the mass equation $\dot\rho=-\rho D$ (see (1.16)) and rearrange to obtain

$$(\varepsilon+\lambda)\frac{d}{dt}(\rho-\tilde\rho)+(\gamma-1)\rho\tilde e(\rho-\tilde\rho)=-\rho F-(\gamma-1)\rho^2(e-\tilde e).$$

Multiplying by $p\,\mathrm{sgn}(\rho-\tilde\rho)|\rho-\tilde\rho|^{p-1}$, we then obtain that

$$(\rho|\rho-\tilde\rho|^p)_t+\mathrm{div}\,(\rho|\rho-\tilde\rho|^p u)+C^{-1}|\rho-\tilde\rho|^p\leq C(|F|^p+|e-\tilde e|^p),$$

from which (b) follows.

To prove (c) and (d) we take Fourier transforms in the identity

$$(\varepsilon+\lambda)\Delta u^j=F_j+(\varepsilon+\lambda)\omega^{j,\ell}_\ell+P_j \tag{3.18}$$

(see (1.35)) to obtain that

$$(\varepsilon+\lambda)\widehat{u^j_k}=|\xi|^{-2}\left[\xi_j\xi_k\hat F+(\varepsilon+\lambda)\xi_k\xi_\ell\widehat{\omega^{j,\ell}}\right]+\left[|\xi|^{-2}\xi_j\xi_k(\widehat{P-\tilde P})\right]. \tag{3.19}$$

We define $\widehat{z^{j,k}}$ and $\widehat{w^{j,k}}$ to be the two terms on the right-hand side in (3.19). Then $(\varepsilon+\lambda)u^j_k=z^{j,k}+w^{j,k}$, and by the Marcinkiewicz multiplier theorem (Stein [19], pg. 96),

$$\begin{cases}\|z(\cdot,t)\|_{L^p}\leq C(p)\big(\|F(\cdot,t)\|_{L^p}+\|\omega(\cdot,t)\|_{L^p}\big),\\[2mm] \|w(\cdot,t)\|_{L^p}\leq C(p)\|P(\cdot,t)-\tilde P\|_{L^p}.\end{cases} \tag{3.20}$$

In particular, (c) holds. Observe also that $|\widehat{\nabla z}|\leq C(|\widehat{\nabla F}|+|\widehat{\nabla\omega}|)$, so that

$$\|\nabla z(\cdot,t)\|_{L^2}\leq C\big(\|\nabla F(\cdot,t)\|_{L^2}+\|\nabla\omega(\cdot,t)\|_{L^2}\big)\leq C\|\dot u(\cdot,t)\|_{L^2} \tag{3.21}$$

by (a) and the relation (1.36) that $\Delta F = \operatorname{div} \rho\dot{u}$. We then have that

$$\begin{aligned}(\varepsilon+\lambda)^{-2}\Big|\int u^{j_1}_{k_1}u^{j_2}_{k_2}u^{j_3}_{k_3}dx\Big| &= \Big|\int u^{j_1}_{k_1}(z^{j_2,k_2}+w^{j_2,k_2})(z^{j_3,k_3}+w^{j_3,k_3})dx\Big| \\ &\le C\left[\int(|\nabla u|^2|w|+|\nabla u||w|^2)dx + \Big|\int u^{j_1}(z^{j_2,k_2}z^{j_3,k_3})_{k_1}dx\Big|\right] \\ &\le C\left[\int(|\nabla u|^2|w|+|\nabla u||w|^2)dx + \|u\|_{L^4}\|z\|_{L^4}\|\nabla z\|_{L^2}\right].\end{aligned}$$

This together with (3.20) and (3.21) proves (d). //

We now continue the proof of Theorem 3.1 for $n=3$:

Lemma 3.4. $B_1(t) \le C\left[C_0 + B_3(t) + \sum_{p>1} B(t)^p\right]$, $0 \le t \le \bar{t}$, *where the sum on the right is finite.*

Proof: We multiply the second equation in (1.16) by $\dot{u}^j$ and integrate and rearrange to obtain

$$\tfrac{1}{2}\frac{d}{dt}\int(\varepsilon|\nabla u|^2+\lambda D^2)dx + \int\rho|\dot{u}|^2dx = -\int \dot{u}\cdot\nabla P dx + O_3, \quad 0<t<\bar{t},$$

where O_3 denotes a finite sum of terms of the form $\int u^{j_1}_{k_1}u^{j_2}_{k_2}u^{j_3}_{k_3}dx$. Integrating by parts in the pressure term and substituting from (1.1) and (1.13), we then find that

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\int(\varepsilon|\nabla u|^2+\lambda D^2)dx + \int\rho|\dot{u}|^2dx &= \frac{d}{dt}\int(P-\tilde{P})Ddx + O_3 \\ &+ \int\left[\rho P_\rho D^2 - P_e D\dot{e} + O(|P-\tilde{P}||\nabla u|^2)\right]dx.\end{aligned} \tag{3.22}$$

We first restrict attention to times $t \le \bar{t}\wedge 1$. Here we make crucial use of the dissipative term $\int\rho|\dot{u}|^2dx$ and the fact that $u(\cdot,t)\in H^s$ by applying a type of Poincaré inequality, which is derived as follows. Recalling the relation (3.19), we have that, at a fixed time $t\le\bar{t}\wedge 1$ and for $R>0$,

$$\begin{aligned}\int|\nabla u|^2dx &= \int|\xi|^2|\hat{u}|^2d\xi \\ &\le R^{2-2s}\int_{|\xi|\le R}|\xi|^{2s}|\hat{u}|^2d\xi + C\|P-\tilde{P}\|^2_{L^2} + CR^{-2}\int_{|\xi|\ge R}|\xi|^2(|\hat{F}|^2+|\hat{\omega}|^2)d\xi \\ &\le C\left[R^{2-2s}\|u\|^2_{H^s} + C_0 + R^{-2}\|\dot{u}\|^2_{L^2}\right]\end{aligned}$$

by Theorem 2.2, Lemma 3.3(a), and the relation $\Delta F = \operatorname{div}\rho\dot u$. Then by Lemma 3.2,

$$\int |\dot u|^2 dx \ge C^{-1}R^2 \int (\varepsilon|\nabla u|^2 + \lambda D^2)dx - C(R^2 + R^{4-2s})C_0.$$

Substituting into (3.22), we thus obtain that, for a new C which we now fix,

$$\begin{aligned}
&\frac{d}{dt}\int(\varepsilon|\nabla u|^2+\lambda D^2)dx + C^{-1}R^2\int(\varepsilon|\nabla u|^2+\lambda D^2)dx + C^{-1}\int|\dot u|^2dx\\
&\qquad\le \frac{d}{dt}\int(P-\tilde P)Ddx + \int P_e D\dot e\,dx + O_3\\
&\qquad\quad + C\int\left[e|\nabla u|^2 + |P-\tilde P||\nabla u|^2\right]dx + C(R^2+R^{4-2s})C_0.
\end{aligned}\tag{3.23}$$

We now choose R so that $C^{-1}R^2 = (1-s+\beta)t^{-1}$. Multiplying (3.23) by $t^{1-s+\beta}$ and integrating with respect to t, we then obtain that

$$\begin{aligned}
&C^{-1}\left[t^{1-s+\beta}\int|\nabla u(x,t)|^2dx + \int_0^t\int(t')^{1-s+\beta}|\dot u|^2dxdt'\right]\\
&\qquad\le t^{1-s+\beta}\int\left[P(x,t)-\tilde P\right]D(x,t)dx + C\int_0^t\int(t')^{-s+\beta}|P-\tilde P||\nabla u|dxdt'\\
&\qquad\quad + \int_0^t (t')^{1-s+\beta}\left[O_3 - \int\rho\dot e Ddx\right]dt'\\
&\qquad\quad + \int_0^t\int(t')^{1-s+\beta}|e-\tilde e||\nabla u|^2dxdt' + t^\beta C_0.
\end{aligned}\tag{3.24}$$

We show that each term on the right here can either be absorbed into the left-hand side, or is bounded by

$$Ct^\beta\left[C_0 + C_0^{1/2}B_3(t)^{1/2} + C_0^\theta B_1(t)\right]$$

for positive θ. For example, we can apply Lemma 3.3(d) to bound the O_3 term above by

$$\begin{aligned}
&C\int_0^t\int(t')^{1-s+\beta}(|\nabla u|^2|w| + |\nabla u||w|^2)dxdt'\\
&\quad + C\int_0^t(t')^{1-s+\beta}\|u(\cdot,t')\|_{L^4}\|\dot u(\cdot,t')\|_{L^2}\big(\|F(\cdot,t')\|_{L^4} + \|\omega(\cdot,t')\|_{L^4}\big)dt'.
\end{aligned}$$

We shall estimate in detail only the term involving ω; bounds for the other terms are similar. Applying Theorem 2.2, Lemma 3.3(a), (1.41), and (1.16), we have that

$$\begin{aligned}
&\int_0^t(t')^{1-s+\beta}\|u\|_{L^4}\|\dot u\|_{L^2}\|\omega\|_{L^4}dt'\\
&\quad\le CC_0^{1/2}\int_0^t(t')^{1-s+\beta}\left(\int|\dot u|^2dx\right)^{1/2}\left(\int|\omega|^2dx\right)^{1/8}\left(\int|\nabla\omega|^2dx\right)^{3/8}dt'\\
&\quad\le CC_0^{1/2}t^{(1-s+\beta)/8}\int_0^t\left(\int|\nabla u|^2dx\right)^{1/8}\left((t')^{1-s+\beta}\int|\dot u|^2dx\right)^{7/8}dt'\\
&\quad\le Ct^\beta C_0^{5/8}B_1(t)^{7/8},
\end{aligned}$$

as required. Estimating the other terms in (3.24) in a similar way, we find that, for $t \le \bar{t} \wedge 1$,

$$t^{1-s+\beta} \int |\nabla u(x,t)|^2 dx + \int_0^t \int (t')^{1-s+\beta} |\dot{u}|^2 dx\, dt'$$
$$\le C t^\beta \Big[C_0 + C_0^{1/2} B_3(t)^{1/2} + C_0^\theta B_1(t) \Big].$$

It then follows easily that

$$B_1(t) = \sup_{0 \le t' \le t} (t')^{1-s} \int |\nabla u(x,t')|^2 dx + \int_0^t \int (t')^{1-s+\beta} |\dot{u}|^2 dx dt'$$
$$\le C \Big[C_0 + C_0^{1/2} B_3(t)^{1/2} + C_0^\theta B_1(t) \Big].$$

Since C_0 is small, we may therefore conclude that

$$B_1(t) \le C \big[C_0 + C_0^{1/2} B_3(t)^{1/2} \big], \quad 0 \le t \le \bar{t} \wedge 1. \tag{3.25}$$

We need to extend this estimate to times $t \in (1, \bar{t}]$ when $\bar{t} > 1$. To do this, we integrate (3.22) from 1 to t, eventually arriving at a bound for $B_1(t)$ in terms of $B_1(1)$ and $B_3(t)$. The details are significantly simpler than the derivation of (3.25), since initial layer considerations do not enter. It is at this point of the argument, however, that the higher-order terms $\sum_{p>1} B(t)^p$ occur. //

Lemma 3.5. $B_2(t) \le C \Big[C_0 + B_3(t) + \sum_{p>1} B(t)^p \Big], \quad 0 \le t \le \bar{t}.$

Proof. Taking the material derivative in the second equation in (1.16), we obtain that

$$\rho \frac{d}{dt} \dot{u}^j + P_{jt} + \mathrm{div}\,(P_j u) = \varepsilon \big[\Delta u_t^j + \mathrm{div}\,(\Delta u^j u) \big] + \lambda \big[D_{jt} + \mathrm{div}\,(D_j u) \big].$$

Multiplying by $(1 \wedge t')\dot{u}^j$, summing over j, and integrating, we then obtain that

$$\begin{aligned} \tfrac{1}{2}(1 \wedge t)^r \int \rho(x,t) |\dot{u}(x,t)|^2 dx &= \int_0^{1\wedge t} \int \tfrac{1}{2} r (1 \wedge t')^{r-1} \rho |\dot{u}|^2 dx dt' \\ &+ \int_0^t \int (1 \wedge t')^r \dot{u}^j \Big[- \big(P_{jt} + \mathrm{div}\,(P_j u) \big) + \varepsilon \big(\Delta u_t^j + \mathrm{div}\,(\Delta u^j u) \big) \\ &\qquad + \lambda \big(D_{jt} + \mathrm{div}\,(D_j u) \big) \Big] dx dt'. \end{aligned} \tag{3.26}$$

Now, the first term on the right here is bounded by

$$C \int_0^{1\wedge t} \int (t')^{r-1} |\dot{u}|^2 dx dt' \le C \int_0^{1\wedge t} \int (t')^{1-s+\beta} |\dot{u}|^2 dx dt'$$
$$\le C B_1(t) \le C \Big[C_0 + B_3(t) + \sum_{p>1} B(t)^p \Big]$$

by (1.18) and Lemma 3.4. Next, a short computation shows that the ε and λ terms on the right side of (3.26) may be written

$$-\iint(1\wedge t')^r(\varepsilon|\nabla\dot u|^2+\lambda\dot D^2)dxdt'+O\left(\iint(1\wedge t')^r|\nabla u|^2(|\nabla\dot u|+|\dot D|)dxdt'\right)$$
$$\le -C^{-1}\iint(1\wedge t')^r|\nabla\dot u|^2dxdt'+C\iint(1\wedge t')^r|\nabla u|^4dxdt'.$$

Substituting into (3.26), we thus obtain that

$$\begin{aligned}&(1\wedge t)^r\int|\dot u(x,t)|^2dx+C^{-1}\int_0^t\int(1\wedge t')^r|\nabla\dot u|^2dxdt'\\ &\quad\le C\left[C_0+B_3(t)+\sum_{p>1}B(t)^p+\int_0^t\int(1\wedge t')^r|\nabla u|^4dxdt'\right]\\ &\qquad+\left|\int_0^t\int(1\wedge t')^r\dot u^j\left[P_{jt}+\operatorname{div}(P_ju)\right]dxdt'\right|.\end{aligned}\tag{3.27}$$

To deal with the pressure term here, we integrate by parts in x, then apply the equations (1.1) and (1.13) for ρ and e to compute the derivatives of P. This term then becomes

$$(\gamma-1)\iint(1\wedge t')^r(\dot u^j)_j\left[-\rho eD-PD+\Delta T+O(|\nabla u|^2)\right]$$
$$+(\gamma-1)\iint(1\wedge t')^r\left[(\dot u^j)_ku^k(\rho e)_j-(\dot u^j)_ju^k(\rho e)_k\right].$$

The second integral here is easily seen to be bounded by $O\left(\iint(1\wedge t')^re|\nabla u||\nabla\dot u|\right)$, so that the pressure term in (3.27) may be written

$$(\gamma-1)\iint(1\wedge t')^r(\dot u^j)_j\Delta T+O\left(\iint(1\wedge t')^r|\nabla\dot u|(e|\nabla u|+|\nabla u|^2)\right).$$

We replace ΔT here by $\rho\dot e+PD+O(|\nabla u|^2)$, substitute back into (3.27), and apply the Cauchy-Schwartz inequality in an elementary way to obtain finally that

$$\begin{aligned}&(1\wedge t)^r\int|\dot u(x,t)|^2dx+\int_0^t\int(1\wedge t')^r|\nabla\dot u|^2dxdt'\\ &\quad\le C\left[C_0+B_3(t)+\sum_{p>1}B(t)^p+\int_0^t\int(1\wedge t')^r(\dot e^2+e^2|\nabla u|^2+|\nabla u|^4)dxdt'\right].\end{aligned}\tag{3.28}$$

We show that each term on the right here is bounded by

$$C\left[C_0+B_3(t)+\sum_{p>1}B(t)^p\right].$$

For example, by Lemma 3.3, Theorem 2.2, and (1.41),

$$
\begin{aligned}
&\iint (1\wedge t')^r |\nabla u|^4 dx dt' \\
&\quad \le C \iint (1\wedge t')^r \big[F^4 + |\omega|^4 + (e-\tilde e)^4 + (\rho-\tilde\rho)^4\big] dx\, dt' \\
&\quad \le CC_0 + C\int (1\wedge t')^r \Bigg[\left(\int F^2 dx\right)^{1/2}\left(\int |\nabla F|^2 dx\right)^{3/2} \\
&\qquad + \left(\int |\omega|^2 dx\right)^{1/2}\left(\int |\nabla\omega|^2 dx\right)^{3/2} + \left(\int (e-\tilde e)^2 dx\right)^{1/2}\left(\int |\nabla e|^2 dx\right)^{3/2}\Bigg] dt'.
\end{aligned}
$$

Since $\int |\nabla F|^2 dx \le C\int |\dot u|^2 dx$, we can bound the term on the right here which involves F by

$$
\begin{aligned}
&C\int_0^t (1\wedge t')^r \left[C_0^{1/2} + \left(\int |\nabla u|^2 dx\right)^{1/2}\right]\left(\int |\dot u|^2 dx\right)^{3/2} dt' \\
&\quad \le C \sup_{0\le t'\le t}\left[(1\wedge t')^r \int |\dot u(x,t')|^2 dx\right]^{1/2}\left[\iint (1\wedge t')^{(r-1+s)/2}|\dot u|^2 dx dt'\right] \\
&\qquad\qquad \times \left[C_0^{1/2} + \left(\sup_{0\le t'\le t}(1\wedge t')^{1-s}\int |\nabla u(x,t')|^2 dx\right)^{1/2}\right] \\
&\quad \le C B_2(t)^{1/2} B_1(t)\left[C_0^{1/2} + B_1(t)^{1/2}\right] \\
&\quad \le C\left[C_0 + \sum_{p>1} B(t)^p\right].
\end{aligned}
$$

Estimating the term $\iint (1\wedge t')e^2|\nabla u|^2$ in (3.28) in a similar way and taking an appropriate sup, we conclude that

$$
B_2(t) \le C\left[C_0 + B_3(t) + \sum_{p>1} B(t)^p\right],
$$

which proves the lemma. //

Lemma 3.6. *$B(t) \le CC_0, \quad 0 \le t \le \bar t$.*

Proof. We need to derive an estimate for B_3. To do this, we multiply the energy equation $\rho\dot e + PD = \Delta T + O(|\nabla u|^2)$ by $(1\wedge t')^r \dot T$ and integrate to obtain

$$
\begin{aligned}
\int_0^t \int (1\wedge t')^r \rho\dot T\dot e\, dx dt' = \int_0^t \int (1\wedge t')^r \Big[&-\tfrac12 \frac{d}{dt}|\nabla T|^2 - T_j(T_k u^k)_j \\
&+ O\big(|\dot T|(P|\nabla u| + |\nabla u|^2)\big)\Big] dx dt'.
\end{aligned}
$$

Bounding the second term on the right by $O\left(\iint (1\wedge t')^r |\nabla T|^2 |\nabla u|\right)$ and applying the Cauchy-Schwartz inequality in an elementary way, we then obtain

$$
\begin{aligned}
&(1\wedge t)^r \int |\nabla T(x,t)|^2 dx + \int_0^t \int (1\wedge t')\dot{T}^2 dx dt' \\
&\qquad \le C \int_0^{1\wedge t} \int (1\wedge t')^{r-1} |\nabla T|^2 dx dt' \\
&\qquad\quad + C \int_0^t \int (1\wedge t')^r \left[|\nabla T|^{8/3} + e^2 |\nabla u|^2 + |\nabla u|^4 \right] dx dt'.
\end{aligned}
\tag{3.29}
$$

The last two terms on the right here occurred in (3.28) in the proof of Lemma 3.5, where we showed that

$$
\begin{aligned}
&\int_0^t \int (1\wedge t')^r \left[e^2 |\nabla u|^2 + |\nabla u|^4 \right] dx dt' \\
&\qquad \le C \left[C_0 + \sum_{p>1} B(t)^p \right].
\end{aligned}
\tag{3.30}
$$

To bound the second term on the right side of (3.29) we apply (1.41), Lemmas 3.4 and 3.5, and (3.30) to obtain that

$$
\begin{aligned}
\int_0^t \int (1\wedge t')^r |\nabla T|^{8/3} dx dt' &\le C \int_0^t (1\wedge t')^r \left(\int |\nabla T|^2 dx \right)^{5/6} \left(\int |D_x^2 T|^2 dx \right)^{1/2} dt' \\
&\le C \int_0^t (1\wedge t')^r \left(\int |\nabla T|^2 dx \right)^{5/6} \left(\int \left[\dot{e}^2 + e^2 |\nabla u|^2 + |\nabla u|^4 \right] dx \right)^{1/2} dt' \\
&\le C \left(\sup_{0\le t'\le t} (1\wedge t') \int |\nabla e(x,t')|^2 dx \right)^{1/3} \left(\int_0^t \int |\nabla e|^2 dx dt' \right)^{1/2} \\
&\qquad \times \left(\int_0^t \int (1\wedge t')^{4r/3} \left[\dot{e}^2 + e^2 |\nabla u|^2 + |\nabla u|^4 \right] dx dt' \right)^{1/2} \\
&\le C B_3(t)^{1/3} C_0^{1/2} \left[B_3(t)^{1/2} + C_0^{1/2} + \sum B(t)^{p/2} \right] \\
&\le C \left[C_0 + \sum B(t)^p \right].
\end{aligned}
$$

We thus obtain from (3.29) that

$$
(1\wedge t)^r \int |\nabla e(x,t)|^2 dx + \int_0^t \int (1\wedge t')^r \dot{e}^2 dx dt \le C \left[C_0 + \sum_{p>1} B(t)^p \right],
$$

so that, taking the sup over t,

$$B_3(t) \leq C \left[C_0 + \sum_{p>1} B(t)^p \right].$$

Combining this result with the conclusions of Lemmas 3.4 and 3.5, we then obtain

$$B(t) \leq C \left[C_0 + \sum_{p>1} B(t)^p \right].$$

An elementary argument based on the continuity of B in t and on smallness then applies to show that $B(t) \leq CC_0$. //

Proof of Theorem 3.1: The estimate (3.1) is the conclusion of Lemma 3.6, and the bounds in (3.2) for $\dot{e}$ and $\nabla \dot{e}$ in the polytropic case are obtained in a rather involved energy-method argument similar to that given in the proof of Lemma 3.5. We omit the details, since the estimates in (3.2) are not required for the main conclusions (1.22) - (1.29) of Theorem 1.1. //

§4. Pointwise Bounds

In this section we derive *a priori* pointwise bounds and Hölder continuity estimates for smooth solutions of (1.1) - (1.2), (1.13). We shall prove:

Theorem 4.1. *Let* (ρ,u,e) *be a smooth solution of* (1.1) - (1.2), (1.13) *on* $\mathbb{R}^n \times [0,\bar{t}]$ *satisfying the hypotheses and conclusions of Theorems 2.1, 2.2, and 3.1. Then there are positive constants* C, q, *and* θ, *as described in Theorem 1.1, such that*

$$\|\rho - \tilde{\rho}\|_{L^\infty(\mathbb{R}^n \times [0,\bar{t}])} \le C C_0^\theta; \tag{4.1}$$

$$\|u\|_{L^\infty(\mathbb{R}^n \times [t_0,\bar{t}])} + \langle u \rangle^{\alpha,\alpha/(n+2\alpha)}_{L^\infty(\mathbb{R}^n \times [t_0,\bar{t}])} \le C C_0^\theta (1 \wedge t_0)^{-q}, \tag{4.2}$$

where $\alpha = \frac{1}{2}$ *when* $n = 3$ *and* $\alpha \in (0,1)$ *when* $n = 2$; *(*C *may depend additionally on* α *in the* $n = 2$ *case);*

$$\|e - \tilde{e}\|_{L^\infty(\mathbb{R}^n \times [t_0,\bar{t}])} \le C C_0^{1/2} (1 \wedge t_0)^{-q}; \tag{4.3}$$

and in the polytropic case that T *is a constant multiple of* e,

$$\langle e \rangle^{\alpha,\alpha/(n+2\alpha)}_{\mathbb{R}^n \times [t_0,\bar{t}]} \le C C_0^\theta (1 \wedge t_0)^{-q}, \tag{4.4}$$

where α *and* C *are as in* (4.2).

We give the proof in a sequence of lemmas:

Lemma 4.2. $\|e - \tilde{e}\|_{L^\infty(\mathbb{R}^n \times [t_0,\bar{t}])} \le C C_0^{1/2} (1 \wedge t_0)^{-q}$.

Proof. First observe that the estimates for A and B in Theorems 2.2 and 3.1 do not provide sufficient regularity to give pointwise bounds for e via the Sobolev imbeddings. A different argument will therefore be required. Thus fix t_0, which without loss of generality we may take to be in (0,1], and let $t_2 \in (t_0,\bar{t}]$. We choose $t_1 \in \left[t_2 - \frac{2}{3}t_0, t_2 - \frac{1}{3}t_0\right]$, so that

$$t_1 \ge \tfrac{1}{3}t_0 \quad \text{and} \quad \tfrac{1}{3}t_0 \le t_2 - t_1 \le \tfrac{2}{3}t_0.$$

We shall estimate $\|e(\cdot,t_2) - \tilde{e}\|_{L^\infty}$ in terms of $\|e(\cdot,t_1) - \tilde{e}\|_{L^2}$ by exploiting the dissipation in the energy equation (1.13). To do this, we write the weak form of (1.13) for a test function φ:

$$\int \rho(e - \tilde{e})\varphi \, dx\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int \rho(e - \tilde{e})\Big[\varphi_t + \nabla\varphi \cdot u + \rho^{-1}\mathrm{div}\,\big(T'(e)\nabla\varphi\big)\Big]\, dxdt$$
$$= \int_{t_1}^{t_2}\int \Big[O(|\nabla u|^2) - PD\Big]\, dxdt.$$

Choosing φ to satisfy the adjoint system

$$\begin{cases} (\rho\varphi)_t + \operatorname{div}(\rho\varphi u) + \operatorname{div}\big(T'(e)\nabla\varphi\big) = 0 \\ \varphi(x,t_2) = \psi(x) \geq 0, \end{cases} \tag{4.5}$$

where $\psi \in \mathcal{S}$, we then obtain that

$$\begin{aligned} \Big| \int \rho(x,t_2)\big[e(x,t_2) - \tilde{e}\big]\psi(x)dx\Big| &\leq \Big| \int \rho(x,t_1)\big[e(x,t_1) - \tilde{e}\big]\varphi(x,t_1)dx\Big| \\ &+ C\int_{t_1}^{t_2}\int \big(|\nabla u|^2 + e|\nabla u|\big)|\varphi|\,dxdt. \end{aligned} \tag{4.6}$$

We shall prove below that

$$\|\varphi(\cdot,t)\|_{L^2} \leq C(t_2 - t)^{-n/4}\|\psi\|_{L^1}, \quad t_1 \leq t \leq t_2. \tag{4.7}$$

Granting this for the moment, we may then bound the first term on the right side of (4.6) by

$$\begin{aligned} C\|e(\cdot,t_1) - \tilde{e}\|_{L^2}\|\varphi(\cdot,t_1)\|_{L^2} &\leq CC_0^{1/2}(t_2 - t_1)^{-n/4}\|\psi\|_{L^1} \\ &\leq CC_0^{1/2}t_0^{-n/4}\|\psi\|_{L^1}. \end{aligned}$$

Estimating the second term on the right side of (4.6) in a similar way, we thus obtain that

$$\Big| \int \rho(x,t_2)\big[e(x,t_2) - \tilde{e}\big]\psi(x)dx\Big| \leq CC_0^{1/2}t_0^{-q}\|\psi\|_{L^1},$$

which proves (4.3).

It remains to prove the L^2 bound (4.7) for φ. Observe first that φ is nonnegative, so that, since $\frac{d}{dt}\int \rho\varphi dx = 0$,

$$\|\hat{\varphi}(\cdot,t)\|_{L^\infty} \leq C\|\varphi(\cdot,t)\|_{L^1} \leq C\|\psi\|_{L^1}. \tag{4.8}$$

Next, multiplying (4.5) by φ and integrating by parts, we obtain that

$$\frac{d}{dt}\int \rho\varphi^2 dx = 2\int T'(e)|\nabla\varphi|^2 dx. \tag{4.9}$$

However, by (4.8),

$$\int \rho\varphi^2 dx \leq C\left(\int_{|\xi|\leq R} + \int_{|\xi|\geq R}\right)|\hat{\varphi}|^2 d\xi \leq C\left[R^n\|\psi\|_{L^1}^2 + R^{-2}\int T'(e)|\nabla\varphi|^2 dx\right].$$

Substituting this into (4.9), we then obtain that

$$\frac{d}{dt}\int \rho\varphi^2 dx \geq C^{-1}R^2 \int \rho\varphi^2 dx - CR^{n+2}\|\psi\|_{L^1}^2.$$

We now fix this C and choose R so that $C^{-1}R^2 = 2(t_2-t)^{-1}$. The result is that, for a new C,

$$\frac{d}{dt}\left[(t_2-t)^2 \int \rho\varphi^2 dx\right] \geq -C(t_2-t)^{1-n/2}\|\psi\|_{L^1}^2.$$

(4.7) now follows easily. //

Lemma 4.3. $\|u\|_{L^\infty(\mathbb{R}^n\times[t_0,\bar{t}])} + \langle u\rangle^{\alpha,\alpha/(n+2\alpha)}_{\mathbb{R}^n\times[t_0,\bar{t}]} \leq CC_0^\theta(1\wedge t_0)^{-q}$, *where α and C are as in (4.2). The same result holds with u replaced by e in the polytropic case that T is a constant multiple of e.*

Proof. We apply the imbedding (1.43) with $\alpha = 1-n/p$, $p>n$, together with the result of Lemma 3.3c, to obtain that, for fixed $t>0$,

$$\langle u(\cdot,t)\rangle^\alpha \leq C\|\nabla u(\cdot,t)\|_{L^p} \leq C\Big[\|F(\cdot,t)\|_{L^p} + \|\omega(\cdot,t)\|_{L^p} + \|P(\cdot,t)-\tilde{P}\|_{L^p}\Big]. \tag{4.10}$$

We can apply Theorems 2.2 and 3.1 to show that each of the terms on the right here is bounded by $CC_0^\theta(1\wedge t)^{-q}$ for appropriate θ and q. For example, when $n=3$ and $\alpha=\frac{1}{2}$,

$$\int |\omega|^6 dx \leq C\left(\int |\nabla\omega|^2 dx\right)^3 \leq C\left(\int |\dot{u}|^2 dx\right)^3 \leq CC_0^\theta(1\wedge t)^{-q}$$

by (1.41), Lemma 3.3a, and the bound in Theorem 3.1 for B. Estimating the other two terms on the right side of (4.10) in a similar way, we then obtain that

$$\langle u(\cdot,t)\rangle^\alpha_{\mathbb{R}^n} \leq CC_0^\theta(1\wedge t)^{-q} \tag{4.11}$$

for $\alpha=\frac{1}{2}$ when $n=3$ and $\alpha\in(0,1)$ when $n=2$. It then follows that

$$\left|u(x,t) - ⨍_{B_R(x)} u(y,t)dy\right| \leq CC_0^\theta(1\wedge t)^{-q}R^\alpha, \tag{4.12}$$

where $⨍_{B_R(x)}$ denotes an average over the ball of radius R centered at x. Thus by the L^2 bound of Theorem 2.2,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq CC_0^\theta(1\wedge t)^{-q}. \tag{4.13}$$

To derive a modulus of Hölder continuity for $u(x,\cdot)$ as a function of t, we fix times $t_2 \geq t_1 \geq t_0 > 0$ and apply (4.12), (4.13), and Theorems 2.2 and 3.1 to obtain that

$$\begin{aligned}
|u^j(x,t_2) - u^j(x,t_1)| &\leq \int_{t_1}^{t_2} \fint_{B_R(x)} |u_t^j(y,t)|dydt + CC_0^\theta(1\wedge t_0)^{-q}R^\alpha \\
&\leq CR^{-n/2}(t_2-t_1)^{1/2}\left(\iint |u_t|^2 dydt\right)^{1/2} + CC_0^\theta(1\wedge t_0)^{-q}R^\alpha \\
&\leq CC_0^\theta(1\wedge t_0)^{-q}\big[R^{-n/2}(t_2-t_1)^{1/2} + R^\alpha\big],
\end{aligned}$$

since $|u_t| \leq |\dot{u}| + |u||\nabla u|$. We now choose $R = (t_2-t_1)^{1/(n+2\alpha)}$ to complete the proof that

$$\langle u(x,\cdot)\rangle_{[t_0,\bar{t}]}^{\alpha/(n+2\alpha)} \leq CC_0^\theta(1\wedge t_0)^{-q}.$$

This together with (4.11) and (4.13) proves (4.3).

For the polytropic case we instead apply (3.2) in the energy equation (1.13) to obtain the bound $\|\Delta T(\cdot,t)\|_{L^2} \leq CC_0^\theta(1\wedge t)^{-q}$. It then follows from (1.40) and (1.41) that $\|\nabla T(\cdot,t)\|_{L^p} \leq CC_0^\theta(1\wedge t)^{-q}$ for the same values of p as in (4.10). The Hölder continuity estimate (4.4) for e then follows exactly as in the proof of (4.2) for u. //

The following lemma contains several technical facts required for the proof of (4.1):

Lemma 4.4. *Let (ρ,u,e) be the smooth solution of Theorem 4.1. Then for $n=2$ and $\alpha \in (0,1)$,*

(a) $$\int_0^1 \langle u(\cdot,t)\rangle^\alpha dt \leq CC_0^{1/2}\ ,$$

where C may depend on α; for $n=3$,

(b) $$\int_0^1 \langle u(\cdot,t)\rangle^{1/2}\|u(\cdot,t)\|_{L^7}dt \leq CC_0^\theta\ ;$$

and when n is either 2 or 3,

(c) $$\int_0^1 \|e(\cdot,t)-\tilde{e}\|_{L^\infty}dt \leq CC_0^\theta.$$

Proof. We prove (b) and omit the proof of (a), which is easier. First, applying the imbedding (1.41) to the function $|u|^2$, we obtain that, for $t \in [0,1]$,

$$\int |u|^7 dx \leq C\left(\int |u|^4 dx\right)^{5/8}\left(\int |u|^2|\nabla u|^2 dx\right)^{9/8} \leq CC_0^{5/8}\left(\int |u|^2|\nabla u|^2 dx\right)^{9/8},$$

by Theorem 2.2. Then from (4.10),

$$\begin{aligned}&\int_0^1 \langle u(\cdot,t)\rangle^{1/2}\|u(\cdot,t)\|_{L^7}dt \\ &\quad\le C\int_0^1 \left(\|F(\cdot,t)\|_{L^6}+\|\omega(\cdot,t)\|_{L^6}+\|P(\cdot,t)-\tilde{P}\|_{L^6}\right)\|u(\cdot,t)\|_{L^7}dt.\end{aligned}\tag{4.14}$$

Applying (1.41) again, we can bound the term here involving F by

$$\begin{aligned}&CC_0^\theta\int_0^1\left(\int|\nabla F|^2dx\right)^{1/2}\left(\int|u|^2|\nabla u|^2dx\right)^{9/56}dt\\ &\quad\le CC_0^\theta\int_0^1 t^{-(1-s+\beta)/2}\left(t^{1-s+\beta}\int|\dot{u}|^2dx\right)^{1/2}\left(\int|u|^2|\nabla u|^2dx\right)^{9/56}dt\\ &\quad\le CC_0^\theta\left(\int_0^1 t^{-28(1-s+\beta)/19}dt\right)^{19/56}\left(\iint t^{1-s+\beta}|\dot{u}|^2dxdt\right)^{1/2}\\ &\qquad\qquad\times\left(\iint|u|^2|\nabla u|^2dxdt\right)^{9/56}\\ &\quad\le CC_0^\theta\end{aligned}$$

by Theorems 2.2 and 3.1 and the restrictions (1.18) on s and β. The other terms in (4.14) are estimated in a similar way. This proves (b).

We shall give the proof of (c) for $n=3$, because it is here that the requirement that $r<2$ plays a role; the proof for $n=2$ is somewhat simpler. We choose a number $p>3$ so that

$$r<\frac{2p}{3p-6},\tag{4.15}$$

which is possible because $r<2$. We then replace the L^∞ norm in (c) by the $W^{1,p}$ norm, and since integrability in a neighborhood of $t=0$ is the real issue, omit the estimate for $\int_0^1\|e(\cdot,t)-\tilde{e}\|_{L^p}dt$. Applying (1.41), we can bound the integral of the other term in the $W^{1,p}$ norm as follows:

$$\begin{aligned}\int_0^1\|\nabla e(\cdot,t)\|_{L^p}dt&\le C\int_0^1\left(\int|\nabla e|^2dx\right)^{(6-p)/4p}\left(\int\Delta T^2dx\right)^{(3p-6)/4p}dt\\ &\le C\left(\int_0^1 t^{-r(3p-6)/2p}dt\right)^{1/2}\left(\iint|\nabla e|^2dxdt\right)^{(6-p)/4p}\\ &\qquad\qquad\times\left(\iint t^r\Delta T^2dxdt\right)^{(3p-6)/4p}\\ &\le CC_0^\theta\end{aligned}$$

by (4.15) and the fact that

$$\iint t^r \Delta T^2 dxdt \le C \iint t^r\big(\dot e^2 + e^2|\nabla u|^2 + |\nabla u|^4\big)dxdt \le CC_0^\theta,$$

which follows from estimates given previously in the proof of Lemma 3.6. //

Lemma 4.5. $\|\rho - \tilde\rho\|_{L^\infty(\mathbb{R}^n\times[0,\overline{t}])} \le CC_0^\theta$.

Proof. We first restrict attention to times $t \le \overline{t}\wedge 1$. Dividing the mass equation $\dot\rho = -\rho D$ by ρ and adding and subtracting the pressure difference $P - \tilde P$, we obtain that

$$(\varepsilon+\lambda)\frac{d}{dt}\log(\rho/\tilde\rho) = -F - (P-\tilde P)\ ,$$

so that

$$\begin{aligned}(\varepsilon+\lambda)\log(\rho/\tilde\rho)\Big|_{(x(0),0)}^{(x(t),t)} &= -\int_0^t F\big(x(t'),t'\big)dt' \\ &\quad + O\left(\int_0^t \big[\|e(\cdot,t') - \tilde e\|_{L^\infty} + \|\rho(\cdot,t') - \tilde\rho\|_{L^\infty}\big]dt'\right)\ ,\end{aligned} \tag{4.16}$$

where $x(t)$ is a particle path (that is, $\dot x(t) = u(x(t),t)$). To deal with the first term on the right here we apply the relation (1.36) $\Delta F = \operatorname{div}\rho\dot u$ to express F in terms of the fundamental solution Γ of the Laplacian:

$$F = \Gamma * \operatorname{div}\rho\dot u = \Gamma_j * \Big((\rho u^j)_t + \operatorname{div}(\rho u^j u)\Big).$$

Thus

$$\begin{aligned}\int_0^t F\big(x(t'),t'\big)dt' &= \int_0^t\int \Gamma_j\big(x(t')-y\big)\Big[(\rho u^j)_t(y,t') + \operatorname{div}(\rho u^j u)(y,t')\Big]dydt' \\ &= \nabla\Gamma * (\rho u)\Big|_{(x(0),0)}^{(x(t),t)} \\ &\quad + \int_0^t\int \Gamma_{jk}\big(x(t')-y\big)\Big[u^k(y,t') - u^k\big(x(t'),t'\big)\Big](\rho u^j)(y,t')dydt'.\end{aligned} \tag{4.17}$$

We apply (1.44) to bound the first term on the right here by

$$C\sup_t\Big(\|u(\cdot,t')\|_{L^p} + \|u(\cdot,t')\|_{L^4}\Big),$$

where $p\in(1,n)$. When $n=3$ we choose $p=2$ and apply Theorem 2.2 to obtain the bound CC_0^θ. For the $n=2$ case we recall from Theorem 2.2 the weighted L^2-norm bound

$$\sup_{0\le t\le 1}\int |u(x,t)|^2(1+|x|^2)^\ell dx \le CC_0.$$

We can therefore choose $p \in (1,2)$ so that $(1+|x|^2)^{-\ell p/(2-p)} \in L^1$. Then

$$\begin{aligned}\int |u|^p dx &\le \int |u|^p (1+|x|^2)^{\ell p/2}(1+|x|^2)^{-\ell p/2} dx \\ &\le \left(\int |u|^2(1+|x|^2)^{\ell}\right)^{p/2} \left(\int (1+|x|^2)^{-\ell p/(2-p)} dx\right)^{(2-p)/2} \\ &\le CC_0^\theta.\end{aligned}$$

Thus in either case, the first term on the right-hand side of (4.17) is bounded by CC_0^θ. To deal with the double integral on the right side of (4.17), we let $I(y,t')$ be its integrand and note that, since $|\Gamma_{jk}(x)| \le C$ for $|x| \ge 1$,

$$\int_0^t \int_{|y-x(t')|\ge 1} I(y,t')dydt' \le CC_0^\theta$$

by the results of Theorem 2.2 and Lemma 4.4. To estimate the integral of I near the singular set $\{y = x(t')\}$ in the $n = 2$ case, we choose $\alpha \in \left(\frac{1}{2},1\right)$ and apply Lemma 4.4a to obtain the bound

$$\begin{aligned}&\int_0^t \int_{|y-x(t')|\le 1} I(y,t')dydt' \\ &\quad\le C \int_0^t \int_{|y-x(t')|\le 1} |x(t')-y|^{\alpha-2} \langle u(\cdot,t')\rangle^\alpha |u(y,t')| dydt' \\ &\quad\le C \sup_{t'} \|u(\cdot,t')\|_{L^4} \left(\int_{|z|\le 1} |z|^{4(\alpha-2)/3} dz\right)^{3/4} \int_0^t \langle u(\cdot,t')\rangle^\alpha dt' \\ &\quad\le CC_0^\theta .\end{aligned}$$

For $n = 3$ we bound this term instead by

$$\begin{aligned}C\int_0^t \int |x(t')-y|^{-5/2}\langle u(\cdot,t)\rangle^{1/2} |u(y,t')|dydt' \\ \le C\int_0^t \langle u(\cdot,t')\rangle^{1/2} \|u(\cdot,t')\|_{L^7} dt' \\ \le CC_0^\theta\end{aligned}$$

by Lemma 4.4b. We have thus shown that the first term on the right side of (4.16) is bounded by CC_0^θ, and Lemma 4.4c gives the same bound for the second term. Therefore

$$|\rho(x(t),t) - \tilde{\rho}| \le C\left[C_0^\theta + \int_0^t \|\rho(\cdot,t') - \tilde{\rho}\|_{L^\infty} dt'\right].$$

It follows easily that

$$\|\rho - \tilde\rho\|_{L^\infty(\mathbb{R}^n\times[0,1\wedge\bar t])} \le CC_0^\theta. \tag{4.18}$$

It remains to extend the bound (4.18) to times $t \in (1,\bar t\,]$ when $\bar t > 1$. To do this, we again substitute for D in terms of $F = (\varepsilon+\lambda)D - P + \tilde P$ in the mass equation $\dot\rho = -\rho D$ to obtain

$$(\varepsilon+\lambda)\frac{d}{dt}(\rho-\tilde\rho) + (\gamma-1)\tilde e\rho(\rho-\tilde\rho) = -\rho F - (\gamma-1)\rho^2(e-\tilde e),$$

so that

$$\frac{d}{dt}(\rho-\tilde\rho)^2 + C^{-1}(\rho-\tilde\rho)^2 \le C\big[F^2 + (e-\tilde e)^2\big].$$

Integrating along the particle path $\big(x(t),t\big)$, we then get that, for $t\in(1,\bar t]$,

$$\begin{aligned}\Big[\rho\big(x(t),t\big) - \tilde\rho\Big]^2 &\le e^{(1-t)/C}\Big[\rho\big(x(1),1\big)-\tilde\rho\Big]^2\\ &\qquad + C\int_1^t e^{(t'-t)/C}\Big(\|F(\cdot,t')\|^2_{L^\infty} + \|e(\cdot,t)-\tilde e\|^2_{L^\infty}\Big)dt'.\end{aligned} \tag{4.19}$$

We apply (4.18) to bound the first term on the right, and (4.3) to bound the last term on the right, both by CC_0^θ. For the term involving F, we replace the L^∞ norm by the $W^{1,4}$ norm and apply Lemma 3.3c, Theorems 2.2 and 3.1, the relation $\Delta F = \operatorname{div}\rho\dot u$, and the Marcinkiewicz theorem to obtain the bound

$$\begin{aligned}&\int_1^t e^{(t'-t)/C}\bigg[\Big(\int F^2 dx\Big)^{1/4}\Big(\int|\dot u|^2dx\Big)^{3/4}\\ &\qquad\qquad + \Big(\int|\dot u|^2dx\Big)^{1/4}\Big(\int|\nabla\dot u|^2dx\Big)^{3/4}\bigg]dt'\\ &\le CC_0^\theta\end{aligned}$$

when $n=3$. A similar bound holds when $n=2$. We thus obtain from (4.19) that

$$\|\rho(\cdot,t)-\tilde\rho\|_{L^\infty} \le CC_0^\theta, \quad 1<t<\bar t,$$

which, together with (4.18), proves the lemma. //

§5. Existence and Regularity

In this section we apply the *a priori* bounds of sections 2 - 4 to obtain the existence of the solution (ρ,u,e) described in Theorem 1.1, together with its regularity properties (1.22) - (1.26) and (1.28) - (1.31). The proof of the asymptotic behavior result (1.27) will be deferred to section 6.

To begin, we let (ρ_0,u_0,e_0) be initial data as described in Theorem 1.1, and $\rho_0^\delta = j_\delta * \rho_0$, $u_0^\delta = j_\delta * u_0$, $e_0^\delta = j_\delta * e_0$, where j_δ is the standard mollifier. We shall show in Lemma 5.1 that there is a global solution $(\rho^\delta,u^\delta,e^\delta)$ corresponding to this mollified data; in Lemma 5.2 we extract a subsequence $\delta_k \to 0$ for which $(\rho^{\delta_k},u^{\delta_k},e^{\delta_k}) \to (\rho,u,e)$ in an appropriate sense; and in Lemma 5.3 we show that $\left(\rho,u,e+\frac{1}{2}|u|^2\right)$ is indeed a weak solution of (1.1) - (1.3) with initial data $(\rho_0,u_0,e_0+\frac{1}{2}|u_0|^2)$.

Lemma 5.1. *Assume that the system parameters P, T, ε, and λ, and the constants $r, s, \beta, \tilde\rho$, $\tilde e$, e_1, and $\underline{e}$, are given as in Theorem 1.1. Then there are positive numbers a and C depending on these parameters and constants such that: given initial data (ρ_0,u_0,e_0) satisfying*

$$C_0 \le a \quad \text{and} \quad \text{ess inf}\, e_0 \ge e_1,$$

there is a global smooth solution $(\rho^\delta,u^\delta,e^\delta)$ of (1.1) - (1.2), (1.13) *with initial data $(\rho_0^\delta,u_0^\delta,e_0^\delta)$, satisfying the regularity conditions* (1.23), (1.26), (1.28) - (1.29), *and in the polytropic case,* (1.30) *and* (1.31). *In addition*

$$\rho^\delta(\cdot,t)-\tilde\rho,\ u^\delta(\cdot,t),\ e^\delta(\cdot,t)-\tilde e \in H^\infty(\mathbb{R}^n)$$

for all $t \ge 0$.

Proof. The local existence of solutions is proved in Kawashima [11] (see Racke [14] for an exposition of Kawashima's result). We let C_0^δ be just as in (1.17), but with (ρ_0,u_0,e_0) replaced by $(\rho_0^\delta,u_0^\delta,e_0^\delta)$, and we fix δ. The hypotheses of Theorems 2.1, 2.2, 3.1, and 4.1 then hold for a positive time $\bar t_\delta$, possibly depending on δ, so that, in particular,

$$\|\rho^\delta-\tilde\rho\|_{L^\infty(\mathbb{R}^n\times[0,\bar t_\delta])} \le C(C_0^\delta)^\theta \le CC_0^\theta, \tag{5.1}$$

$$\inf e^\delta(\cdot,t) > \underline{e},\ t\in[0,\bar t_\delta], \tag{5.2}$$

and if $\bar t_\delta \ge \tau$, where τ is as in Theorem 2.1,

$$\|e^\delta-\tilde e\|_{L^\infty(\mathbb{R}^n\times[\tau,\bar t_\delta])} \le C(C_0^\delta)^\theta\tau^{-q} \le CC_0^\theta. \tag{5.3}$$

Thus if C_0 is sufficiently small,

$$\rho^\delta(\cdot,\overline{t}_\delta) \in (\underline{\rho},\overline{\rho}),\ \inf e^\delta(\cdot,\overline{t}_\delta) > \underline{e},$$

and $\rho^\delta(\cdot,\overline{t}_\delta) - \tilde{\rho},\ u^\delta(\cdot,\overline{t}_\delta),\ e^\delta(\cdot,\overline{t}_\delta) - \tilde{e} \in H^\infty(\mathbb{R}^n)$. We can therefore reapply Kawashima's local existence result at the new initial time $\overline{t}_\delta$ to extend the solution $(\rho^\delta,u^\delta,e^\delta)$, eventually to all of $t > 0$. We omit the details, which are routine, but we note especially here the Gronwall-type inequality Theorem C.1(i) in [14], which enters the argument in an important way.

The pointwise bounds (5.1) - (5.3) then hold for all t, and therefore the conclusions of Theorems 2.2, 3.1, and 4.1 hold for all t as well. This proves that $(\rho^\delta,u^\delta,e^\delta)$ satisfies the bounds (1.23), (1.26), (1.28) - (1.29), and in the polytropic case, (1.30) and (1.31). //

Lemma 5.2. *Let $(\rho^\delta,u^\delta,e^\delta)$ be as in Lemma 5.1. There is a sequence $\delta = \delta_k \to 0$ and a limiting triple (ρ,u,e) such that*

$$\rho^\delta(\cdot,t) - \rho(\cdot,t) \rightharpoonup 0 \text{ weakly in } L^2(\mathbb{R}^n),\ t \geq 0; \tag{5.4}$$

$$\begin{cases} u^\delta \to u \text{ uniformly on compact sets in } \{t > 0\}, \\ u^\delta(\cdot,t) \to u(\cdot,t) \text{ in } L^2_{loc}(\mathbb{R}^n),\ t \geq 0, \\ u^\delta(\cdot,t) \rightharpoonup u(\cdot,t) \text{ weakly in } L^2(\mathbb{R}^n),\ t \geq 0; \end{cases} \tag{5.5}$$

$$\begin{cases} e^\delta(\cdot,t) - e(\cdot,t) \to 0 \text{ in } L^2_{loc}(\mathbb{R}^n),\ t \geq 0 \\ e^\delta(\cdot,t) - e(\cdot,t) \rightharpoonup 0 \text{ weakly in } L^2(\mathbb{R}^n),\ t \geq 0; \end{cases} \tag{5.6}$$

and in the polytropic case,

$$e^\delta \to e \text{ uniformly on compact sets in } \{t > 0\}. \tag{5.7}$$

In addition,

$$u^\delta - u,\ e^\delta - e \to 0 \text{ in } L^2_{loc}(\mathbb{R}^n \times [0,\infty)); \tag{5.8}$$

$$\rho^\delta - \rho,\ \nabla(u^\delta - u),\ \nabla(T^\delta - T),\ P^\delta - P \rightharpoonup 0 \text{ weakly in } L^2_{loc}(\mathbb{R}^n \times [0,\infty)), \tag{5.9}$$

where $P^\delta = P(\rho^\delta,e^\delta),\ T^\delta = T(e^\delta),\ P = P(\rho,e)$, and $T = T(e)$;

$$F^\delta(\cdot,t) - F(\cdot,t),\ \omega^\delta(\cdot,t) - \omega(\cdot,t) \to 0 \text{ in } L^2_{loc}(\mathbb{R}^n),\ t > 0, \tag{5.10}$$

where $F^\delta = (\varepsilon+\lambda)\mathrm{div}\, u^\delta - P^\delta + \tilde{P}$. *Finally, the limit* (ρ,u,e) *satisfies the conclusions* (1.22) - (1.26) *and* (1.28) - (1.29) *of Theorem 1.1, and in the polytropic case,* (1.30) *and* (1.31).

Proof. The proof consists in applying the bounds in (1.23), (1.26), and (1.28) - (1.29), which hold for the approximate solutions $(\rho^\delta,u^\delta,e^\delta)$ uniformly in δ, to obtain the necessary compactness. We shall sketch the arguments for a few of the less obvious results. To begin, let $\{t_j\}$ be a countable dense subset of $[0,\infty)$. Then since $\{\rho^\delta(\cdot,t_j)-\tilde{\rho}\}_\delta$ is bounded in $L^2(\mathbb{R}^n)$, uniformly in δ, we can apply a diagonal process to obtain a sequence $\delta_k \to 0$ for which $\rho^\delta(\cdot,t_j)-\tilde{\rho}$ converges weakly in $L^2(\mathbb{R}^n)$, say to $\rho(\cdot,t_j)-\tilde{\rho}$, for each t_j. However, given $\varphi\in H^1(\mathbb{R}^n)$, we have from (1.1) and the bounds in (1.23) and (1.26) for $(\rho^\delta,u^\delta,e^\delta)$ that

$$
\begin{aligned}
(5.11)\qquad \Big|\int \rho^\delta(x,\cdot)\varphi(x)dx\Big|_{t'}^{t''}\Big| &= \Big|\int_{t'}^{t''}\int \rho^\delta u^\delta\cdot\nabla\varphi\, dxdt\Big| \\
&\le CC_0^{1/2}\|\nabla\varphi\|_{L^2}|t''-t'|^{1/2}.
\end{aligned}
$$

It follows easily that $\rho^{\delta_k}(\cdot,t)-\tilde{\rho}$ converges weakly to an element of H^{-1}, which we denote by $\rho(\cdot,t)-\tilde{\rho}$, for *every* $t\in[0,\infty)$. On the other hand, the uniform bound $\|\rho^{\delta_k}(\cdot,t)-\tilde{\rho}\|_{L^2}\le CC_0^{1/2}$ shows that every subsequence of $\{\delta_k\}$ has in turn a subsequence which converges weakly in $L^2(\mathbb{R}^n)$, necessarily to $\rho(\cdot,t)-\tilde{\rho}$. Thus $\rho(\cdot,t)-\tilde{\rho}\in L^2(\mathbb{R}^n)$, $\|\rho(\cdot,t)-\tilde{\rho}\|_{L^2(\mathbb{R}^n)}\le CC_0^{1/2}$, and the pointwise bounds in (1.23) and (1.28) concerning ρ hold. This proves (5.4), and (5.11) proves the assertion in (1.22) that $\rho-\tilde{\rho}\in C\big([0,\infty);\ H^{-1}(\mathbb{R}^n)\big)$.

The compactness results (5.5) - (5.7) for u^δ and e^δ are proved in a similar way. To prove that $u\in C\big((0,\infty);\ L^2(\mathbb{R}^n)\big)$, we compute that, for $t''>t'>0$,

$$
\begin{aligned}
\|u^\delta(\cdot,t'')-u^\delta(\cdot,t')\|_{L^2}^2 &= \int\Big|\int_{t'}^{t''} u_t^\delta(x,t)dt\Big|^2 dx \\
&\le (t''-t')\iint |u_t^\delta|^2 dxdt \\
&\le (t''-t')\iint \big[|\dot{u}^\delta| + |u^\delta||\nabla u^\delta|\big]^2 dxdt \\
&\le CC_0^\theta(1\wedge t')^{-q}(t''-t'),
\end{aligned}
$$

by the bounds in (1.26) and (1.28) for u^δ. We then let $\delta=\delta_k$ go to zero to obtain the result. A similar argument proves that $e-\tilde{e}\in C\big((0,\infty);\ L^2(\mathbb{R}^n)\big)$.

Next we observe that, since $u^{\delta_k}(\cdot,t)\to u(\cdot,t)$ in $\mathcal{D}'(\mathbb{R}^n)$, $\nabla u^{\delta_k}(\cdot,t)\to\nabla u(\cdot,t)$ in $\mathcal{D}'(\mathbb{R}^n)$ as well. On the other hand, $\|\nabla u^\delta(\cdot,t)\|_{L^2}\le CC_0^{1/2}(1\wedge t)^{(s-1)/2}$ by (1.26), so that, in fact, $\nabla u^\delta(\cdot,t)\rightharpoonup\nabla u(\cdot,t)$ weakly in L^2, $\nabla u(\cdot,t)\in L^2$, and therefore $u(\cdot,t)\in H^1$, for each $t>0$.

A similar argument shows that $\nabla\omega^\delta(\cdot,t) \rightharpoonup \nabla\omega(\cdot,t)$ in L^2, $\omega^\delta(\cdot,t) \to \omega(\cdot,t)$ in L^2_{loc}, and $\omega(\cdot,t) \in H^1$, for each $t > 0$.

As for the pressure, we have that

$$P^\delta(\cdot,t) = (\gamma-1)\rho^\delta(\cdot,t)e^\delta(\cdot,t) \rightharpoonup (\gamma-1)\rho(\cdot,t)e(\cdot,t) = P(\cdot,t)$$

weakly in L^2_{loc}, because $\rho^\delta(\cdot,t) \rightharpoonup \rho(\cdot,t)$ and $e^\delta(\cdot,t) \to e(\cdot,t)$. Then since $F^\delta = (\varepsilon+\lambda)\mathrm{div}\, u^\delta - P^\delta + \tilde{P}$, we obtain that $F^\delta(\cdot,t) \rightharpoonup F(\cdot,t) = (\varepsilon+\lambda)\mathrm{div}\, u(\cdot,t) - P(\cdot,t) + \tilde{P}$ weakly in $L^2_{loc}(\mathbb{R}^n)$, for each $t > 0$. On the other hand, the bound

$$\|\nabla F^\delta(\cdot,t)\|_{L^2} \le C\|\dot{u}^\delta(\cdot,t)\|_{L^2} \le CC_0^{1/2}(1\wedge t)^{-q}$$

shows that $F^\delta(\cdot,t) \to F(\cdot,t)$ strongly in L^2_{loc}, and that $F(\cdot,t) \in H^1(\mathbb{R}^n)$, for each $t > 0$.

These and similar arguments prove the convergence results (5.4) - (5.10), as well as the properties (1.22) - (1.26) and (1.30) - (1.31) of the limit (ρ,u,e), with the exception of (1.25) and the bounds in (1.26) and (1.30) for the time derivatives $\dot{u}$ and $\dot{e}$. For these we must anticipate the results of Lemma 5.3 below, that (ρ,u,e) is indeed a weak solution of (1.1) - (1.3). (Of course, the proof of Lemma 5.3 is independent of the above estimates for these time derivatives.) Assuming this for the present, then, we let $\varphi \in \mathcal{D}(\mathbb{R}^n \times (0,\infty))$ and apply the identity (1.35) to write the weak form of the momentum equation:

$$\begin{aligned}
&\iint (F\varphi_j + \varepsilon\omega^{j,k}\varphi_k)dxdt = \iint (\rho u^j\varphi_t + \rho u^j u\cdot\nabla\varphi)dxdt \\
(5.12)\qquad &= \iint (\rho^\delta u^j \varphi_t + \rho^\delta u^j u^\delta\cdot\nabla\varphi)dxdt \\
&\qquad + \iint \left[(\rho-\rho^\delta)u^j\varphi_t + u^j(\rho u - \rho^\delta u^\delta)\cdot\nabla\varphi\right]dxdt.
\end{aligned}$$

The second term on the right here approaches zero as $\delta = \delta_k \to 0$; and because the product rule holds on $\mathcal{D}\cdot\mathcal{D}'$, the first term may be written

$$-\iint \rho^\delta(u^j_t + \nabla u^j \cdot u^\delta)\varphi\, dxdt.$$

This term approaches

$$-\iint \rho(u^j_t + \nabla u^j\cdot u)\varphi\, dxdt = -\iint \rho\dot{u}^j\varphi\, dxdt$$

as $\delta \to 0$, since $\rho^\delta \rightharpoonup \rho$, $u^\delta \to u$, and $\varphi\nabla u \in L^2$. We thus have from (5.12) that

$$\rho\dot{u}^j = F_j + \varepsilon\omega^{j,k}_k$$

in $\mathcal{D}'\big(\mathbb{R}^n \times (0,\infty)\big)$. We have already shown that $F(\cdot,t), \omega(\cdot,t) \in H^1(\mathbb{R}^n)$ for all $t > 0$. Thus since $\rho^{-1}(\cdot,t) \in L^\infty$, we conclude that $\dot{u}(\cdot,t) \in L^2$, and

$$\begin{aligned}\|\dot{u}(\cdot,t)\|_{L^2} &\le C\big(\|\nabla F(\cdot,t)\|_{L^2} + \|\nabla\omega(\cdot,t)\|_{L^2}\big)\\ &\le C \liminf_{\delta\to 0}\big(\|\nabla F^\delta(\cdot,t)\|_{L^2} + \|\nabla\omega^\delta(\cdot,t)\|_{L^2}\big)\\ &\le C \liminf_{\delta\to 0}\|\dot{u}^\delta(\cdot,t)\|_{L^2}\\ &\le C C_0^{1/2}(1\wedge t)^{-r/2}.\end{aligned}$$

//

Lemma 5.3. *The limit* $\big(\rho,u,e+\frac{1}{2}|u|^2\big)$ *obtained in Lemma 5.2 satisfies the weak equations* (1.4) - (1.6) *for times* $t_2 > t_1 \ge 0$, *as well as the weak equation* (1.14) *for times* $t_2 > t_1 > 0$.

Proof. The proof consists in writing these weak forms for the approximate solutions $(\rho^\delta,u^\delta,e^\delta)$, then letting $\delta = \delta_k \to 0$ and taking appropriate limits. The arguments are completely routine; we shall therefore content ourselves with examining two of the terms in (1.14). To do this, we rewrite the energy equation (1.13) in the form

$$(\rho e)_t + \operatorname{div}(\rho u e) = \Delta T + (\sigma^{j,k}u^j)_k - \nabla F\cdot u - \varepsilon\omega_k^{j,k}u^j,$$

where σ is the stress tensor defined in (1.15). Thus for appropriate test functions φ,

$$\begin{aligned}&\int(\rho^\delta e^\delta\varphi)(x,\cdot)dx\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int(\rho^\delta e^\delta\varphi_t + \rho^\delta e^\delta u^\delta\cdot\nabla\varphi)dxdt\\ &\quad= \int_{t_1}^{t_2}\int\Big[-\nabla T^\delta\cdot\nabla\varphi - (\sigma^\delta)^{j,k}(u^\delta)^j\varphi_k + F^\delta(u^\delta\cdot\nabla\varphi + D^\delta\varphi)\\ &\qquad\qquad + \varepsilon(\omega^\delta)^{j,k}\big((u^\delta)^j\varphi_k + (u^\delta)_k^j\varphi\big)\Big]dxdt.\end{aligned} \tag{5.13}$$

We show that, in the limit as $\delta = \delta_k \to 0$, each term here converges to the corresponding term with $(\rho^\delta,u^\delta,e^\delta)$ replaced by (ρ,u,e). For example, $\iint \rho^\delta e^\delta u^\delta\cdot\nabla\varphi \to \iint \rho e u\cdot\nabla\varphi$ because $\rho^\delta \rightharpoonup \rho$ weakly in L^2_{loc}, $e^\delta \to e$ strongly in L^2_{loc}, and $u^\delta \to u$ pointwise uniformly. To show that the term $\iint F^\delta D^\delta\varphi$ converges to $\iint FD\varphi$, we triangulate as follows:

$$\iint(F^\delta D^\delta - FD)\varphi = \iint(F^\delta - F)D^\delta\varphi + \iint F(D^\delta - D)\varphi.$$

The second integral on the right goes to zero because $D^\delta \rightharpoonup D$ weakly in $L^2\big(\mathbb{R}^n\times[0,\infty)\big)$. For the first term on the right, we observe that, for each fixed $t > 0$,

$$\begin{aligned}\Big|\int(F^\delta - F)D^\delta\varphi\,dx\Big| &\le C\|F^\delta(\cdot,t) - F(\cdot,t)\|_{L^2(\operatorname{supp}\varphi(\cdot,t))}\|\nabla u^\delta(\cdot,t)\|_{L^2}\\ &\le CC_0(1\wedge t)^{-q}\|F^\delta(\cdot,t) - F(\cdot,t)\|_{L^2(\operatorname{supp}\varphi(\cdot,t))},\end{aligned} \tag{5.14}$$

which approaches zero as $\delta = \delta_k \to 0$, by (5.10). The above bound, together with (1.26), also shows that

$$\left| \int (F^\delta - F) D^\delta \varphi \, dx \right| \le C C_0 (1 \wedge t)^{-q}. \tag{5.15}$$

Therefore $\displaystyle\int_{t_1}^{t_2} \int (F^\delta - F) D^\delta \varphi \, dx dt \to 0$ by the bounded convergence theorem, provided that $t_1 > 0$.

Similar arguments apply to the other terms in (5.13), thus proving that (5.13) holds with $(\rho^\delta, u^\delta, e^\delta)$ replaced by (ρ, u, e). A somewhat careful resubstitution must be done to recover the weak form (1.14). To do this we write the conclusion (5.13) for (ρ, u, e) in the form

$$\begin{aligned} \int \rho e \varphi \, dx \Big|_{t_1}^{t_2} - \iint \rho e (\varphi_t + \nabla\varphi \cdot u) + \iint [\nabla T \cdot \nabla \varphi - \sigma^{j,k} u^j_k \varphi] \\ = \iint \big[- \sigma^{j,k} u^j \varphi_k + F \mathrm{div}\,(\varphi u) + \varepsilon \omega^{j,k} (u^j \varphi)_k \big]. \end{aligned} \tag{5.16}$$

Substituting for σ and simplifying, we find that the right-hand side here is

$$\iint \big[(u^j \varphi)_j u^k_k - (u^j \varphi)_k u^k_j \big].$$

If $u(\cdot, t)$ were in H^2 for $t > 0$, a simple integration by parts would then show that this integral is zero. Instead, we observe that, for each $t > 0$, $u^j \varphi$, u^k_k, $u^k_j \in L^2(\mathbb{R}^n)$, so that the Plancherel theorem can be applied to achieve the same result. We thus obtain from (5.16) that

$$\int \rho e \varphi \, dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int \rho e (\varphi_t + \nabla \varphi \cdot u) dx dt = \int_{t_1}^{t_2} \int [-\nabla T \cdot \nabla \varphi + \sigma^{j,k} u^j_k \varphi] dx dt, \tag{5.17}$$

which is precisely (1.14). //

Remark. A more careful application of (1.26) in (5.15) for the case $n = 3$ shows that

$$\left| \int (F^\delta - F) D^\delta \varphi \, dx \right| \le C C_0 (1 \wedge t)^{s-1},$$

which is therefore in $L^1([t_1, t_2])$, even when $t_1 = 0$. Thus the weak form (1.14) can be shown to hold in the $n = 3$ case considered here for times $t_2 > t_1 \ge 0$. The same result would apply in the $n = 2$ case if we were to assume, in addition to the hypotheses of Theorem 1.1, that $u_0 \in H^{s_1}$ for some $s_1 > 0$.

§6. Large-Time Behavior

In this section we complete the proof of Theorem 1.1 by establishing the large-time behavior result (1.27). We fix the solution (ρ,u,e) obtained in section 5, which has now been shown to satisfy the conclusions (1.22) - (1.26) and (1.28) - (1.29) of Theorem 1.1. Again, we divide the proof into a sequence of lemmas:

Lemma 6.1. $\lim_{t\to\infty}\big(\|u(\cdot,t)\|_{L^p}+\|e(\cdot,t)-\tilde{e}\|_{L^p}\big)=0, \quad p\in(2,\infty).$

Proof. The proof for u is essentially identical to that given in [9] for the isothermal case. We therefore present here the proof for e, which is similar. Thus define

$$b^\delta(t)=\int|\nabla T^\delta(x,t)|^2dx,$$

where $(\rho^\delta,u^\delta,e^\delta)$ is the approximate solution from section 5, and $T^\delta=T(e^\delta)$. Then by Theorems 2.2 and 3.1,

$$\begin{cases} 0\le b^\delta(t)\le CC_0,\ t\ge 1\\ \displaystyle\int_1^\infty b^\delta(t)dt\le CC_0\ . \end{cases} \tag{6.1}$$

Dropping δ's temporarily, we also obtain from the estimates in Theorems 2.2 and 3.1 that

$$\begin{aligned}\int_1^\infty\Big|\frac{db^\delta}{dt}(t)\Big|dt&=2\int_1^\infty\Big|\int|\nabla T\cdot\nabla T_t\,dx\Big|dt\\ &\le C\int_1^\infty\int|\Delta T||e_t|dxdt\\ &\le C\int_1^\infty\int\big(|\dot{e}|+|\nabla u|+|\nabla u|^2\big)\big(|\dot{e}|+|u||\nabla e\big)dxdt\\ &\le CC_0.\end{aligned}$$

This shows that the functions b^δ are uniformly bounded and have uniformly bounded variations on $[1,\infty)$. There is therefore a subsequence δ' of $\delta=\delta_k$ such that $b^{\delta'}\to b$ a.e. on $[1,\infty)$. The limiting function b is in BV, and so has a limit as $t\to\infty$. But for $t\ge 1$,

$$\int_1^t b(t')dt'=\lim\int_1^t b^{\delta'}(t')dt'\le CC_0$$

by (6.1). Thus $b\in L^1\big([1,\infty)\big)$, and therefore its limit at infinity is zero:

$$\lim_{t\to\infty}b(t)=0. \tag{6.2}$$

On the other hand, by the arguments in section 5, $\nabla T^{\delta'}(\cdot,t) \rightharpoonup \nabla T(\cdot,t)$ weakly in L^2 for each $t \geq 1$, so that

$$\int |\nabla T(x,t)|^2 dx \leq \underline{\lim} \int |\nabla T^{\delta'}(x,t)|^2 dx = b(t)$$

for almost all $t \geq 1$. For such t, then, by (1.41),

$$\int |e(x,t) - \tilde{e}|^4 dx \leq C \left(\int |e(x,t) - \tilde{e}|^2 dx \right)^{1/2} \left(\int |\nabla T(x,t)|^2 dx \right)^{3/2} \leq C C_0^{1/2} b(t)^{3/2} \tag{6.3}$$

for $n = 3$, and similarly for $n = 2$. On the other hand, $(e - \tilde{e}) \in C\big((0,\infty);\ L^4(\mathbb{R}^n)\big)$ by (1.22) and the pointwise bound (1.28). This together with (6.3) and (6.2) then proves that $\|e(\cdot,t) - \tilde{e}\|_{L^4} \to 0$ as $t \to \infty$, and the convergence in L^p for $p \in (2,\infty)$ follows by interpolation from our time-independent bounds (1.26) and (1.28) for $e(\cdot,t) - \tilde{e}$ in $L^2 \cap L^\infty$. //

Lemma 6.2. $\lim_{t\to\infty} \big(\|u(\cdot,t)\|_{L^\infty} + \|e(\cdot,t) - \tilde{e}\|_{L^\infty} \big) = 0$.

Proof. To prove the result for u we apply the imbedding $W^{1,4} \subseteq L^\infty$ to the function $|u|^2$ to obtain that

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty}^2 &\leq C\big(\|u(\cdot,t)\|_{L^8}^2 + \||u(\cdot,t)|\nabla u(\cdot,t)\|_{L^4}\big) \\ &\leq C\big(\|u(\cdot,t)\|_{L^8}^2 + \|u(\cdot,t)\|_{L^{12}} \|\nabla u(\cdot,t)\|_{L^6}\big). \end{aligned}$$

Now, by Theorem 3.3c, (1.41), and various of the bounds in (1.26), we obtain that

$$\|\nabla u(\cdot,t)\|_{L^6} \leq C\big(\|F(\cdot,t)\|_{L^6} + \|\omega(\cdot,t)\|_{L^6} + \|P(\cdot,t) - \tilde{P}\|_{L^6}\big) \leq C C_0^\theta,\ t \geq 1.$$

Thus

$$\|u(\cdot,t)\|_{L^\infty}^2 \leq C\big(\|u(\cdot,t)\|_{L^8}^2 + C_0^\theta \|u(\cdot,t)\|_{L^{12}}^2\big),$$

which approaches zero as $t \to \infty$, by Lemma 6.1.

A similar argument applies to e in the polytropic case, since (1.30) gives a time-independent bound for $T(\cdot,t) \in H^2$ (for $t \geq 1$), hence for $\nabla T(\cdot,t) \in L^6$. A more involved argument is required in the nonpolytropic case, however. We thus fix times $t_2 > t_1 = t_2 - 1 \geq 1$ and recall the weak form (5.17) of the energy equation

$$\int \big(\rho(e - \tilde{e})\varphi\big)(x,\cdot) dx \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int \big[\rho(\tilde{e} - e)(\varphi_t + \nabla\varphi \cdot u) - \nabla T \cdot \nabla\varphi + \sigma^{j,k} u_k^j \varphi\big] dx dt, \tag{6.4}$$

where σ is the stress tensor given by (1.15). (We have also applied here the weak form of the mass equation to replace e by $e-\tilde{e}$.) We now choose φ to solve the adjoint system

$$\begin{cases} \rho^{\eta_1}\varphi_t + (\rho u)^{\eta_1}\cdot\nabla\varphi + \operatorname{div}\big(T'(e^{\eta_2})\nabla\varphi\big) = 0,\ t\le t_2, \\ \varphi(x,t_2) = j_{\eta_0}(x-x_0), \end{cases} \tag{6.5}$$

where $j_{\eta_0}(x-x_0) = \eta_0^{-n} j\left(\dfrac{x-x_0}{\eta_0}\right)$ is the standard approximate delta-function, and ρ^{η_1}, $(\rho u)^{\eta_1}$, and e^{η_2} are smooth approximations to ρ, ρu, and e obtained by space-time convolution with standard mollifiers of widths η_1 and η_2. Multiplying the pde in (6.5) by $e-\tilde{e}$ and integrating, we obtain that

$$\int_{t_1}^{t_2}\int \big[(e-\tilde{e})\big(\rho^{\eta_1}\varphi_t + (\rho u)^{\eta_1}\cdot\nabla\varphi\big) - T'(e^{\eta_2})\nabla e\cdot\nabla\varphi\big]dxdt = 0,$$

and combining with (6.4),

$$\begin{aligned} \int \rho(x,t_2)\big[e(x,t_2)-\tilde{e}\big]j_{\eta_0}(x-x_0)dx &= \int \big(\rho(e-\tilde{e})\varphi\big)(x,t_1)dx \\ &+ \int_{t_1}^{t_2}\int \Big[(e-\tilde{e})(\rho-\rho^{\eta_1})\varphi_t + (e-\tilde{e})\big(\rho u - (\rho u)^{\eta_1}\big)\cdot\nabla\varphi\Big]dxdt \\ &+ \int_{t_1}^{t_2}\int \big[T'(e^{\eta_2}) - T'(e)\big]\nabla e\cdot\nabla\varphi\, dxdt \\ &+ \int_{t_1}^{t_2}\int \sigma^{j,k}u^j_k\varphi\, dxdt. \end{aligned} \tag{6.6}$$

Now, the adjoint system (6.5) is essentially identical to (4.5), so that, just as in (4.7),

$$\begin{cases} \|\varphi(\cdot,t)\|_{L^1} \le C, \\ \|\varphi(\cdot,t)\|_{L^2} \le C(t_2-t)^{-n/4}, \quad t\le t_2. \end{cases} \tag{6.7}$$

In addition, we can easily show by routine energy estimates that

$$\begin{cases} \|\varphi(\cdot,t)\|^2_{L^2} + \displaystyle\int_t^{t_2}\int |\nabla\varphi|^2 dxdt \le C\eta_0^{-n}, \\ \|\nabla\varphi(\cdot,t)\|^2_{L^2} + \displaystyle\int_t^{t_2}\int \varphi_t^2 dxdt \le \mathcal{C}(\eta_0,\eta_2), \quad t\le t_2, \end{cases} \tag{6.8}$$

where $\mathcal{C}$ denotes a function which is bounded on compact sets in $(0,1)^2$, which is independent of η_1, but which may become unbounded as $\eta_0\to 0$ or $\eta_2\to 0$. (We point out, however, that the proofs of these bounds depend upon the approximate mass equation

$\rho_t^{\eta_1} + \mathrm{div}\,(\rho u)^{\eta_1} = 0$; it was for this reason that we chose to mollify ρu rather than u.) We now fix η_0 and η_2 and bound each term on the right side of (6.6), then let $\eta_1 \to 0$. The first double integral goes to zero as $\eta_1 \to 0$ by the strong convergence of ρ^{η_1} and $(\rho u)^{\eta_1}$ and by the bounds (6.8), which are independent of η_1. The single integral on the right side of (6.6) is bounded by

$$C\|e(\cdot,t_1) - \tilde{e}\|_{L^4}\|\varphi(\cdot,t_1)\|_{L^{4/3}} \le C\|e(\cdot,t_1) - \tilde{e}\|_{L^4}$$

by (6.7). The temperature term in (6.6) is bounded (in $n = 3$, say) by

$$\begin{aligned}
&C\left(\iint |e^{\eta_2} - e|^5 dxdt\right)^{1/5}\left(\iint |\nabla T|^{10/3} dxdt\right)^{3/10}\left(\iint |\nabla\varphi|^2 dxdt\right)^{1/2} \\
&\quad \le C\eta_0^{-3/2}\left[\int_{t_1}^{t_2}\left(\int |\nabla T|^2 dx\right)^{2/3}\left(\int \Delta T^2 dx\right) dt\right]^{3/10}\left[\int_{t_1}^{t_2}\int |e^{\eta_2} - e|^5 dxdt\right]^{1/5} \\
&\quad \le CC_0^\theta \eta_0^{-3/2}\left(\int_{t_1}^{t_2}\int |e^{\eta_2} - e|^5 dxdt\right)^{1/5},
\end{aligned}$$

by (1.41) and the bound in (1.26) for $\iint \dot{e}^2$, which shows that $\iint \Delta T^2 \le CC_0^\theta$, by way of the pde (1.13). Finally, the stress tensor term in (6.6) is bounded by

$$C\left(\iint \varphi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int [P^2 + |\nabla u|^2]|\nabla u|^2 dxdt\right)^{1/2}.$$

We now let $\eta_2 \to 0$ with η_0 fixed, then let $\eta_0 \to 0$ with x_0 a fixed Lebesgue point of $e(\cdot,t_2)$. Taking the essential sup over x_0 and replacing t_2 by t, we conclude that

$$\|e(\cdot,t) - \tilde{e}\|_{L^\infty} \le C\left[\|e(\cdot,t) - \tilde{e}\|_{L^4} + \left(\int_{t-1}^{t}\int [|\nabla u|^2 + |\nabla u|^4] dxdt\right)^{1/2}\right],$$

which approaches zero as $t \to \infty$, by Lemma 6.1 and the fact that $\nabla u \in (L^2 \cap L^4)(\{t \ge 1\})$.
//

Lemma 6.3. $\lim_{t\to\infty} \|\rho(\cdot,t) - \tilde{\rho}\|_{L^p} = 0, \quad p \in (2,\infty]$.

Proof. It suffices to prove the result for $p = \infty$, since $\|\rho(\cdot,t) - \tilde{\rho}\|_{L^2} \le CC_0^{1/2}$ for all t, by (1.23). We apply here a result of DiPerna-Lions [2] that, if $\rho^h(\cdot,t) = j_h * \rho(\cdot,t)$, where j_h is the standard spatial mollifier, then

$$\rho_t^h + \mathrm{div}\,(\rho^h u) = S^h, \tag{6.9}$$

where $S^h \to 0$ in $L^1_{loc}(\{t \geq 1\})$. This result requires that

$$\int_1^t \|\mathrm{div}\, u(\cdot,t')\|_{L^\infty} dt' < \infty, \tag{6.10}$$

which is equivalent to the condition $\displaystyle\int_1^t \|F(\cdot,t')\|_{L^\infty} dt' < \infty$. To prove the latter we apply the imbeddings (1.40) - (1.41), the bound $\|\nabla F(\cdot,t)\|_{L^4} \leq C\|\dot{u}(\cdot,t)\|_{L^4}$, which follows from the relation (1.36) $\Delta F = \mathrm{div}\, \rho\dot{u}$, and the fact that $\nabla\dot{u}(\cdot,t) \in L^2$ for almost all t, to compute that

$$\begin{aligned}
\|F(\cdot,t)\|_{L^\infty}^4 &\leq C\int \left[F(x,t)^4 + |\dot{u}(x,t)|^4\right] dx \\
&\leq C\Bigg[\left(\int F^2\, dx\right)^{2(1-\alpha)}\left(\int |\dot{u}|^2\, dx\right)^{2\alpha} \\
&\qquad\qquad + \left(\int |\dot{u}|^2\, dx\right)^{2(1-\alpha)}\left(\int |\nabla\dot{u}|^2\, dx\right)^{2\alpha}\Bigg] \\
&\leq CC_0^\theta (1\wedge t)^{-q}\left[\int (|\dot{u}|^2 + |\nabla\dot{u}|^2) dx\right]^{2\alpha}
\end{aligned} \tag{6.11}$$

for almost all t, where $\alpha = \frac{1}{2}$ or $\frac{3}{4}$ when $n = 2$ or 3. It then follows from (1.26) that $\displaystyle\int_1^t \|F(\cdot,t')\|_{L^\infty} dt' < \infty$ for every $t > 1$, as required.

Writing the weak form of (6.9) for a suitable test function φ, we then obtain that

$$\int (\rho^h(x,\cdot) - \tilde{\rho})\varphi(x,\cdot) dx\Big|_1^t - \int_1^t \int \left[(\rho^h - \tilde{\rho})\varphi_t + \rho^h u\cdot\nabla\varphi\right] dx dt' = \int_1^t \int S^h\varphi\, dx dt'.$$

We compute

$$u\cdot\nabla\varphi = \mathrm{div}\,(\varphi u) - \varphi\,\mathrm{div}\, u = \mathrm{div}\,(\varphi u) - (\varepsilon+\lambda)^{-1}\varphi(F + P - \tilde{P})$$

and substitute $P = (\gamma - 1)\rho e$ to obtain

(6.12)

$$\begin{aligned}
\int (\rho^h(x,\cdot) - \tilde{\rho})\varphi(x,\cdot) dx\Big|_1^t &- \int_1^t \int (\rho^h - \tilde{\rho})\left[\varphi_t + \mathrm{div}\,(\varphi u) - \frac{\gamma-1}{\varepsilon+\lambda}\rho^h\varphi\right] dx dt' \\
= \left(\frac{\gamma-1}{\varepsilon+\lambda}\right)&\iint \rho^h\varphi\left[\tilde{e}(\rho - \rho^h) + \rho(e - \tilde{e})\right] dx dt' + (\varepsilon+\lambda)^{-1}\iint \rho^h\varphi F\, dx dt' \\
&+ \int_1^t \int S^h\varphi\, dx dt'.
\end{aligned}$$

We now choose φ to solve the adjoint system

$$\begin{cases} \varphi_t + \operatorname{div}(u^\eta \varphi) - \dfrac{\gamma - 1}{\varepsilon + \lambda}\rho^\eta \varphi = 0 \\ \varphi(x,t) = j_{\eta_0}(x - x_0), \end{cases} \tag{6.13}$$

where again j_{η_0} is the standard approximate delta function, and ρ^η and u^η denote space-time mollifiers of ρ and u. Multiplying the pde in (6.13) by $\rho^h - \tilde{\rho}$ and combining with (6.12), we obtain finally that

$$\begin{aligned} \int (\rho^h(x,t) - \tilde{\rho}) j_{\eta_0}(x - x_0)dx &= \int (\rho^h(x,1) - \tilde{\rho})\varphi(x,1)dx \\ &+ \int_1^t \int (\rho^h - \tilde{\rho}) \left[\operatorname{div}((u - u^\eta)\varphi) + \left(\frac{\gamma - 1}{\varepsilon + \lambda} \right) (\rho^h - \rho^\eta)\varphi \right] dx dt' \\ &+ O\left(\int_1^t \int \left[|\varphi||\rho^h - \rho| + |\varphi|(|F| + |e - \tilde{e}|) \right] dx dt' \right) \\ &+ \int_1^t \int S^h \varphi \, dx dt'. \end{aligned} \tag{6.14}$$

Now, it is clear from (6.13) that $\varphi \geq 0$ and that

$$\frac{d}{dt} \int \varphi \, dx \geq C^{-1} \int \varphi \, dx,$$

so that

$$\|\varphi(\cdot,t')\|_{L^1} \leq e^{(t'-t)/C} \|\varphi(\cdot,t)\|_{L^1} = e^{(t'-t)/C}, \quad 1 \leq t' \leq t. \tag{6.15}$$

In addition, by integrating along characteristics and applying (6.10), we can easily prove that

$$\|\varphi(\cdot,t')\|_{L^\infty}, \ \|\varphi(\cdot,t')\|_{L^2} \leq \mathcal{C}(\eta_0,t), \tag{6.16}$$

where $\mathcal{C}$ is a function which is bounded on compact subsets of $(0,1] \times [1,\infty)$, but which may become unbounded as $\eta_0 \to 0$ or $t \to \infty$. We apply (6.15) and (6.16) to bound each term in (6.14), letting $\eta \to 0$ with η_0 and h fixed, then letting $h \to 0$ with η_0 fixed. The result is that

$$\left| \int (\rho(x,t) - \tilde{\rho}) j_{\eta_0}(x - x_0) dx \right| \leq Ce^{-t/C} + C \int_1^t e^{(t'-t)/C} (\|F(\cdot,t')\|_{L^\infty} + \|e(\cdot,t') - \tilde{e}\|_{L^\infty}) dt',$$

so that, substituting from (6.11),

$$\|\rho(\cdot,t) - \tilde{\rho}\|_{L^\infty} \le C\left[e^{-t/C} + \int_{t/2}^{t} e^{(t'-t)/C}\left[\int (|\dot{u}|^2 + |\nabla\dot{u}|^2)\,dx\right]^{\alpha/2} dt' + \sup_{t'\ge t/2} \|e(\cdot,t') - \tilde{e}\|_{L^\infty}\right]$$

$$\le C\left[e^{-t/C} + \left(\int_{t/2}^{t}\int (|\dot{u}|^2 + |\nabla\dot{u}|^2)\,dx\,dt'\right)^{\alpha/2} + \sup_{t'\ge t/2} \|e(\cdot,t') - \tilde{e}\|_{L^\infty}\right].$$

Since $\|e(\cdot,t) - \tilde{e}\|_{L^\infty} \to 0$ by Lemma 6.2, and $\dot{u},\ \nabla\dot{u} \in L^2\big(\mathbb{R}^n \times [1,\infty)\big)$ by (1.26), we obtain finally that $\|\rho(\cdot,t) - \tilde{\rho}\|_{L^\infty} \to 0$ as $t \to \infty$. This proves Lemma 6.3, which, together with Lemmas 6.1 and 6.2, establishes the large-time behavior result (1.27). //

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