VISCOSITY METHODS FOR PIECEWISE SMOOTH SOLUTIONS TO SCALAR CONSERVATION LAWS

TAO TANG AND ZHEN–HUAN TENG

ABSTRACT. It is proved that for scalar conservation laws, if the flux function is strictly convex, and if the entropy solution is piecewise smooth with finitely many discontinuities (which includes initial central rarefaction waves, initial shocks, possible spontaneous formation of shocks in a future time and interactions of all these patterns), then the error of viscosity solution to the inviscid solution is bounded by $O(\epsilon|\log\epsilon|+\epsilon)$ in L^1-norm, which is an improvement of the $O(\sqrt{\epsilon})$ upper bound. If neither central rarefaction waves nor spontaneous shocks occur, the error bound is improved to $O(\epsilon)$.

0. INTRODUCTION

Consider the single hyperbolic conservation laws

$$u_t + f(u)_x = 0, \qquad -\infty < x < \infty, \quad t > 0, \tag{0.1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty. \tag{0.2}$$

The viscosity method approximating the conservation laws (0.1) and (0.2) is to solve the parabolic equation

$$(v_\epsilon)_t + f(v_\epsilon)_x = \epsilon(v_\epsilon)_{xx}, \tag{0.3}$$

subject to the same initial condition

$$v_\epsilon(x,0) = u_0(x). \tag{0.4}$$

Here, $\epsilon > 0$ is a small parameter. Viscosity methods play an important role in theoretical analysis on numerical methods for hyperbolic conservation laws. A useful technique for studying the behavior of solutions to difference equations is to model the difference equation by a differential equation called modified equation, see for example Chapter 11 of LeVeque's book [13]. The difference equation was introduced because it is easier to solve numerically than the partial differential equation (PDE). This is true if we want to generate numerical approximations, but

1991 *Mathematics Subject Classification.* 65M10, 65M05, 35L65.

Key words and phrases. hyperbolic conservation laws, error estimate, viscosity methods, piecewise smooth .

The research of this author was supported by NSERC Canada Grant OGP0105545.

The research of this author was supported by the National Natural Science Foundation of China and the Science Fund of the Education Commission of China.

on the other hand it is often easier to predict qualitative behavior of a PDE than of a system of difference equations. Since many modified equations have similar forms to equation (0.3) (more precise form is equation (6.1), see Section 6 for more discussions), a better understanding of the solution behavior of (0.3) will help us to understand many difference methods for (0.1). For existence and regularity properties for (0.3), we refer to [9, 16].

In this work, we are concerned with the accuracy of the viscosity methods. More precisely, we will investigate the L^1-convergence rate of $u - v_\epsilon$. It is well known that the L^1-norm of error decays only like $O(\sqrt{\epsilon})$ if the initial data is *piecewise smooth*. Consider an advection diffusion equation of the form

$$v_t + av_x = \epsilon v_{xx}, \qquad a = \text{constant},$$

with $u_0(x) = -sign(x)$. Its exact solution is

$$v_\epsilon(x,t) = 1 - \frac{4}{\sqrt{\pi}} \int_{-\infty}^{(x-at)/\sqrt{4\epsilon t}} e^{-y^2} dy.$$

From this and the fact that the solution to the pure advection problem is $u(x,t) = -sign(x - at)$, it can be shown that

$$\|u(\bullet,t) - v_\epsilon(\bullet,t)\|_{L^1(R)} = C\sqrt{\epsilon t}$$

for some constant C independent of ϵ and t. This indicates that the L^1-convergence rate of the viscosity methods is only one half for this simple linear problem. In practice, solutions of monotone difference methods agree very well with the true solutions of related modified equation. It is expected that monotone difference methods will have similar convergence rate to that of viscosity methods. Harten, Hyman and Lax [7] pointed out that monotone difference methods are of at most first-order accuracy and Kuznetsov [10] provided an one half order L^1-convergence rate for BV bounded initial data. Tang and Teng [19] recently proved that the best L^1-convergence rate for monotone difference methods to scalar conservation laws is one half if it includes the linear flux case. All these indicate that viscosity methods and monotone difference methods share some common properties on L^1-convergence rate.

Although it is proved that half order convergence rate is best possible for both viscosity methods and monotone difference methods, it is widely believed that these methods may have higher order of convergence rate when applying to convex conservation laws. More precisely, if the flux f is strictly convex,

$$f'' \geq \alpha > 0, \tag{0.5}$$

then higher order (greater than one half) convergence rate is expected, see Harten [6]. Lucier [14] used one of the monotone schemes, Godunov's method, to solve Riemann problems for (0.1) with a convex flux. He observed an order one convergence rate. Teng and Zhang [20] recently proved that for a special class of initial data, namely piecewise decreasing constants, both viscosity and strictly monotone difference methods have order one L^1-convergence rate, provided that the flux f is strictly convex. Likewise, Bakhvalov [1] and Harabetian [5] proved a faster convergence rate $O(\epsilon|\ln \epsilon|)$ for Riemann problems with a rarefaction wave. Teng and Zhang's work used the existence of traveling waves obtained by Jennings [8]. It is noted that Jennings' results do not cover Godunov method since his proof relies heavily on the strict monotonicity of the schemes and differentiability of the flux function. In a recent work of Fan [3], the existence and structure of traveling waves for Godunov method are investigated. The order one L^1-convergence rate is established for Godunov method on single shock solutions.

In this paper, we also consider (0.1) with a strictly convex flux, but with a more general and practical class of initial data. With this class of initial data, the entropy solution to (0.1) may include initial central rarefaction waves, initial shocks, possible spontaneous formation of shocks in a future time and interactions of all these patterns. We will establish $O(\epsilon|\ln\epsilon|)$ L^1-convergence rate for the piecewise smooth solutions with finitely many discontinuities. To the best of our knowledge, no global error estimates for piecewise smooth solutions have been obtained, although some results for Riemann problems [1, 3, 5], for piecewise constant solutions [20], and for BV bounded initial data [14, 10] are available. For piecewise smooth solutions to hyperbolic systems with finitely many *noninteracting* shocks, $O(\epsilon)$ *local* error estimates have been obtained by Goodman and Xin [4].

The remainder of the paper consists of six sections. In Section 1, we investigate the structures of the solution to (0.1) with a class of piecewise smooth initial data. Some useful estimates for first order and second order derivatives of the solution are obtained. In Section 2, we introduce a traveling wave solution and a stability lemma, which play important roles in obtaining the L^1-convergence rate. Using the stability lemma, we prove in Section 3 the following estimate:

$$\sup_{0\le t\le T}\|u(\bullet,t)-v_\epsilon(\bullet,t)\|_{L^1(R)}\le C(T)\Big(\epsilon|\log\epsilon|+\epsilon\Big),$$

provided that u_0 is piecewise smooth and that $f'(u_0)$ has finite number of inflection points. In our proof, we will use a matching method developed by Goodman and Xin [4] and Liu and Xin [12]. In Section 4, we make some remarks on optimal error estimate. It is shown that if neither initial central rarefaction waves nor spontaneous shocks occur, then

$$\sup_{0\le t\le T}\|u(\bullet,t)-v_\epsilon(\bullet,t)\|_{L^1(R)}\le C(T)\epsilon.$$

In Section 5, we derive sharp estimates for viscosity methods with a special class of initial function, which includes the initial data considered in Teng and Zhang [20]. Loosely speaking, we consider nonincreasing piecewise smooth solution with finitely many shock interactions. In this case, we obtain the following estimate:

$$\sup_{0\le t<\infty}\|u(\bullet,t)-v_\epsilon(\bullet,t)\|_{L^1(R)}\le c\epsilon,$$

for some c independent of ϵ. Some possible generalizations will be discussed in the final section.

1. Solution structures

Throughout this paper, the norm $\|\cdot\|$ denotes the standard L^1 norm, $\|\cdot\|_{L^1(R)}$; C denotes a positive constant independent of ϵ; c denotes a positive constant independent of t and ϵ, but with different values at different places.

Let $a(u) = f'(u)$. If $x = x^*$ is an inflection point of $a(u_0(x))$ and also satisfies $\dot{a}(u_0(x^*)) < 0$, then we call x^* a decreasing inflection point. If this inflection point satisfies $\dot{a}(u_0(x^*)) \geq 0$, we call it an increasing inflection point. Throughout this paper, we use the following notations:

$$\dot{a}(u_0(x)) := \frac{d}{dx}\Big(a(u_0(x))\Big); \quad \ddot{a}(u_0(x)) := \frac{d^2}{dx^2}\Big(a(u_0(x))\Big);$$
$$\dot{u}_0(x) := \frac{du_0(x)}{dx}; \quad \ddot{u}_0(x) := \frac{d^2u_0(x)}{dx^2}.$$

The behavior and structure of entropy solutions for scalar convex conservation laws have been studied for many years, see for example [2, 11, 15, 17]. It is well known that if the initial function is piecewise C^2-smooth then the entropy solutions consist of at most countable number of C^2-smooth regions. Tadmor and Tassa [18] proved that if the initial speed has a finite number of decreasing inflection points then it bounds the number of future shock discontinuities.

We make the following assumptions on the initial data:

- (A1): $u_0(x)$ is bounded and piecewise C^2 smooth with finite number of discontinuous points $\gamma_i, 1 \leq i \leq I$; $u_0(\gamma_i \pm 0)$ and $\dot{u}_0(\gamma_i \pm 0)$ exist and finite;
- (A2): $\ddot{u}_0 \in L^1(R)$;
- (A3):
$$\lim_{|x|\to\infty} \dot{a}(u_0(x)) = 0;$$
- (A4): $\ddot{a}(u_0(x))$ changes signs finite times, i.e. $a(u_0(x))$ has a finite number of inflection points.

Remark 1.1. In order to obtain an estimate for u_{xx} (Lemma 1.3), we have to assume that u_0 is piecewise C^2. The only place to use this assumption is to obtain the estimate (1.30). Otherwise, a weak assumption on u_0, i.e. u_0 is piecewise C^1, will be sufficient. Furthermore, *the existence of* $\ddot{u}_0(\gamma_i \pm 0)$ *is not required.* That is, $\ddot{u}_0(\gamma_i \pm 0)$ can be infinity. □

We now introduce following notations:

- Denote the set of points where $\dot{a}(u_0)$ has a negative minimum by $\zeta_p, 1 \leq p \leq P$. Without loss of generality, we assume that $\dot{a}(u_0(\zeta_p))$ are distinct:
$$\dot{a}(u_0(\zeta_1)) < \dot{a}(u_0(\zeta_2)) < \cdots < \dot{a}(u_0(\zeta_P)).$$
It is straightforward to extend our results to the case when there are some $\zeta_p, \zeta_q, p \neq q$ such that $\dot{a}(u_0(\zeta_p)) = \dot{a}(u_0(\zeta_q))$.
- Denote $t_0 = 0$ and $t_p = -1/\dot{a}(u_0(\zeta_p)), p = 1, \cdots, P$. For a fixed $T > t_P$, we let $t_{P+1} = T$.

Remark 1.2. A negative minimum point may form a shock in a future time. However, by no means every negative minimal point will eventually generate a new shock. It is an easier case if the negative minimal point does not lead to a shock. Without loss of generality, we assume that each point in the set $\{\zeta_p\}$ corresponds to a new shock created in a later time. The starting point of the new shock is (x_p, t_p). □

Since u_0 is assumed to be only piecewise C^2, it may be discontinuous and therefore will not have a classical derivative. We refer by $\dot{a}(u_0)$ to the generalized derivative of $a(u_0)$. In the smooth interval (γ_i, γ_{i+1}), $\dot{a}(u_0)$ is the classical derivative. In positive and negative jump points of u_0, $\dot{a}(u_0)$ is positive and negative infinity, respectively. In particular, in negative jump points (i.e. decreasing discontinuities) of u_0, $\dot{a}(u_0)$ has a negative (infinite) minimum. If $\dot{a}(u_0)$ has a continuum of negative minimal points it is considered as one minimum.

The assumption (A4) indicates that $\dot{a}(u_0)$ has a finite number of decreasing inflection point. It can be shown that under the assumptions (A1), (A3) and (A4) only a finite number of shock will occur (see, e.g. [17, 18]). The corresponding entropy solution consists of a finite number of C^2 smooth pieces. The shocks are disjoint, except for common endpoints arising from collisions, and (without loss of generality) we assume that no more than two shocks ever collide. More precisely, for each time interval $[t_p, t_{p+1}]$ there exist finite number of subintervals, $t^{(p,n)} \le t < t^{(p,n+1)}, n = 0, 1, \cdots, N_p - 1$, where $t^{(p,0)} = t_p, t^{(p,N_p)} = t_{p+1}$, such that in each interval $[t^{(p,n)}, t^{(p,n+1)})$ there are finite number of smooth curves $x = X_m^{(p,n)}(t), m = 1, \cdots, M_{p,n}$ satisfying

- (1) For $t \in (t^{(p,n)}, t^{(p,n+1)})$,

$$X_m^{(p,n)}(t) < X_{m+1}^{(p,n)}(t), \quad m = 1, \cdots, M_{p,n} - 1;$$

- (2) there exists at least one m such that

$$X_m^{(p,n)}(t^{(p,n+1)}) = X_{m+1}^{(p,n)}(t^{(p,n+1)});$$

- (3) $u(x,t)$ is smooth except on $x = X_m^{(p,n)}(t), \quad t^{(p,n)} \le t < t^{(p,n+1)}$;
- (4) $u(X_m^{(p,n)} \pm 0, t)$ exist and satisfy either $u(X_m^{(p,n)} - 0, t) > u(X_m^{(p,n)} + 0, t)$, i.e. $X_m^{(p,n)}$ is a shock, or $(X_m^{(p,n)} - 0, t) = u(X_m^{(p,n)} + 0, t)$, i.e. $X_m^{(p,n)}$ is a contact discontinuity. Moreover, for $t \in (t^{(p,n)}, t^{(p,n+1)})$,

$$\dot{X}_m^{(p,n)}(t) = \begin{cases} \frac{f\left(u(X_m^{(p,n)}+0,t)\right) - f\left(u(X_m^{(p,n)}-0,t)\right)}{u(X_m^{(p,n)}+0,t) - u(X_m^{(p,n)}-0,t)}, \\ \qquad\qquad \text{if } u(X_m^{(p,n)} + 0, t) > u(X_m^{(p,n)} - 0, t), \\ f'(u(X_m^{(p,n)}, t)), \\ \qquad\qquad \text{if } u(X_m^{(p,n)} + 0, t) = u(X_m^{(p,n)} - 0, t), \end{cases}$$

  where $\dot{X}(t) = dX(t)/dt$;
- (5) Each shock $X_m^{(p,n)}(t)$ continues to the next subinterval $[t^{(p,n+1)}, t^{(p,n+2)})$, with a possibility that it collides with another wave at $t = t^{(p,n+1)}$. Consequently, each shock curve can be extended to $t = \infty$.

From the above results it follows that the solution of (0.1) and (0.2) is a finite combination of the cases plotted in Figure 1. To proceed, we need more information on shock and rarefaction discontinuities. Shocks will be formed in following cases:

- (i) if there is an initial negative jump;
- (ii) if $\dot{a}(u_0)$ has a negative minimum in an interval where u_0 is C^2 smooth;
- (iii) if u_0 has positive jump at a point, $x = z$ say, and $\dot{a}(u_0(z+))$ or $\dot{a}(u_0(z-))$ is a negative minimum.

A shock may also occur as a consequence of an interaction of two waves. If $x = X_m(t)$ and $x = X_{m+1}(t)$ are interacted with each other then we denote the resulting shock still as $x = X_m(t)$. It is regarded as an extension of $x = X_m(t)$ or $x = X_{m+1}(t)$, rather than a new shock.

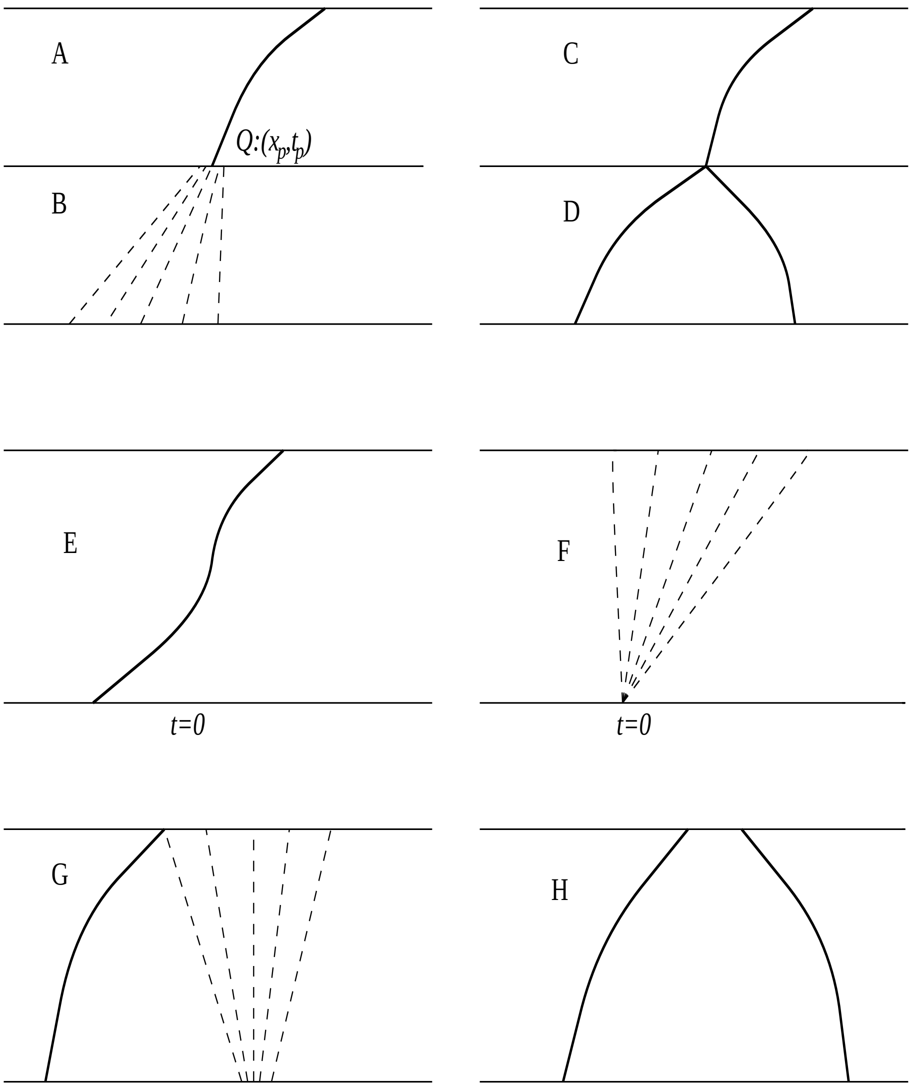

Figure 1: Solution Structures

Consider first two cases. Assume $x = X(t)$ be a shock curve. Before $x = X(t)$ is interacted by other waves, say $t_p < t < t^{(p,n)}$, we can trace two characteristic lines from $x = X(t)$ backward to $t = 0$:

$$X(t) = \zeta^- + a(u_0(\zeta^-))t = \zeta^+ + a(u_0(\zeta^+))t, \tag{1.1}$$

where $\zeta^+(t) > \zeta^-(t)$. Differentiating both sides of the first identity in (1.1) with respect to t gives

$$\dot{\zeta}^- = \frac{\dot{X}(t) - a(u_0(\zeta^-))}{1 + \dot{a}(u_0(\zeta^-))t}, \qquad \dot{\zeta}^- := \frac{d\zeta^-(t)}{dt}, \tag{1.2}$$

or equivalently

$$\frac{1}{1 + \dot{a}(u_0(\zeta^-))t} = \frac{\dot{\zeta}^-}{\dot{X}(t) - a(u_0(\zeta^-))}. \tag{1.3}$$

Observe that

$$\begin{aligned} \dot{X}(t) - a(u_0(\zeta^-)) &= \int_0^1 a(\theta u^+ + (1-\theta)u^-)d\theta - a(u^-) \\ &= \int_0^1 f''(\xi)\theta d\theta (u^+ - u^-), \end{aligned} \tag{1.4}$$

where $u^\pm = u(X(t) \pm 0, t)$, ξ donote some intermediate value. Therefore, we have

$$|\dot{X}(t) - a(u_0(\zeta^-))| \geq c|u^+ - u^-| \geq c|a(u^+) - a(u^-)| = \frac{c}{t}|\zeta^+ - \zeta^-|,$$

where c is some constant independent of t, and in the last step we have used (1.1). The above results lead to

$$\frac{1}{1 + \dot{a}(u_0(\zeta^-))t} \leq \frac{ct|\dot{\zeta}^-|}{|\zeta^+ - \zeta^-|}.$$

Graphically, it is easy to see that ζ^- is a decreasing function of t and ζ^+ is an increasing function of t. Therefore, we have $|\dot{\zeta}^-| = -\dot{\zeta}^- \leq \dot{\zeta}^+ - \dot{\zeta}^-$. It is also true that $\zeta^+ > \zeta^-$. Therefore, we have

$$\frac{1}{1 + \dot{a}(u_0(\zeta^-))t} \leq \frac{ct(\dot{\zeta}^+ - \dot{\zeta}^-)}{\zeta^+ - \zeta^-}.$$

Similar result holds for ζ^+. Consequently, we conclude for both Cases (i) and (ii) that

$$\frac{1}{1 + \dot{a}(u_0(\zeta^\pm))t} \leq \frac{ct(\dot{\zeta}^+ - \dot{\zeta}^-)}{\zeta^+ - \zeta^-}, \tag{1.5}$$

where c is a constant independent of t, and the inequality holds before $x = X(t)$ collides with another shock.

Remark 1.3. The inequality (1.5) is an important result. Assume that $x = X(t)$ is not interacted by other waves for $t \in (S, T)$. It follows from (1.5) that

$$\begin{aligned} &\int_S^T \frac{1}{1 + \dot{a}(u_0(\zeta^\pm))t} dt \\ \leq\ & C(T)\left(\left|\log(\zeta^+(T) - \zeta^-(T))\right| + \left|\log(\zeta^+(S) - \zeta^-(S))\right|\right). \end{aligned} \tag{1.6}$$

For a finite T, we have

$$\zeta^+(S) - \zeta^-(S) \le \zeta^+(T) - \zeta^-(T) \le C.$$

To bound the right hand side of (1.6), we only need to find the lower bound for $\zeta^+(S) - \zeta^-(S)$. See, for example, the step 1 in the proof of Lemma 1.1. □

In Case (iii), a new shock is generated on the boundary of a central rarefaction wave. Since one side of the resulting shock is within the rarefaction wave, one of the two characteristic lines from any point on the shock will end up at the positive jump point. Consider, for example, u_0 has positive jump at $x = z$ and $\dot{a}(u_0(z-))$ is a negative minimum in the sense that $\dot{a}(u_0(z-)) < \dot{a}(u_0(x))$ for x in the left neighborhood of z. There will have a central rarefaction wave with boundaries $x = X_L(t)$ and $x = X_R(t)$. Starting at the point $(X_L(t_z), t_z)$, where $t_z = -1/\dot{a}(u_0(z-))$, a new shock $x = X(t)$ will be generated. It satisfies

$$X(t) = \zeta^- + a(u_0(\zeta^-))t. \tag{1.7}$$

Now (1.3) holds, with

$$\dot{X}(t) = \int_0^1 a(\theta u^+ + (1-\theta)u^-)d\theta,$$

where $u^+ = a^{-1}((X(t) - z)/t), u^- = u(X(t) - 0, t)$. It follows from (1.4) and (1.7) that

$$\begin{aligned} |\dot{X}(t) - a(u_0(\zeta^-))| &\ge c|u^+ - u^-| \ge c|a(u^+) - a(u^-)| \\ &= \frac{c}{t}|(X(t) - z) - a(u_0(\zeta^-))t| = \frac{c}{t}|\zeta^- - z|. \end{aligned}$$

Therefore, we have

$$\frac{1}{1 + \dot{a}(u_0(\zeta^-))t} \le \frac{ct\dot{\zeta}^-}{\zeta^- - z}. \tag{1.8}$$

Similarly, if u_0 has positive jump at $x = z$ and $\dot{a}(u_0(z+))$ is a negative minimum, we have

$$\frac{1}{1 + \dot{a}(u_0(\zeta^+))t} \le \frac{ct\dot{\zeta}^+}{\zeta^+ - z}. \tag{1.9}$$

Remark 1.4. Similar to Remark 1.3, it follows from (1.8) that

$$\begin{aligned} &\int_S^T \frac{1}{1 + \dot{a}(u_0(\zeta^-))t} dt \\ \le\ & C(T)\left(\left|\log(\zeta^-(T) - z)\right| + \left|\log(\zeta^-(S) - z)\right|\right). \end{aligned} \tag{1.10}$$

Therefore, it is important to find a bound for $\zeta^-(S) - z$. See, for example, the step 1 in the proof of Lemma 1.1. Similar idea will be used to treat (1.9). □

From the above discussion, we can obtain the following lemma which is important in the error analysis.

Lemma 1.1. *Assume $x = X(t)$ be a shock curve satisfying*

$$\dot{X}(t) = \frac{f(u^+) - f(u_-)}{u^+ - u^-}, \qquad u^{\pm} := u(X(t) \pm 0, t). \tag{1.11}$$

Assume $a(u_0)$ be sufficiently smooth in the neighborhood of its negative minimum points. If $x = X(t)$ is formed at $t = 0$, then

$$\int_0^T \Big| u_x(X(t) \pm 0, t) \Big| dt \le C. \tag{1.12}$$

If $x = X(t)$ is formed at $t = t_p > 0$, then

$$\int_{t_p+\delta}^T \Big| u_x(X(t) \pm 0, t) \Big| dt \le C |\log \delta| + C, \tag{1.13}$$

provided that $\delta > 0$ is sufficiently small. Here $T > t_P$ is a fixed number.

Proof. The proof of this lemma consists of three steps.

Step 1: Initial negative jump. In this case, a shock is formed due to initial negative jump at a point, say $x = b$. It is obvious that this shock is not going to be interacted by other waves before a finite time $t^{(0,n)}$, $1 \le n \le N_0$. It is known that

$$u\left(\zeta^{\pm} + a(u_0(\zeta^{\pm}))t, t\right) = u_0(\zeta^{\pm}), \qquad 0 < t < t^{(0,n)}.$$

Differentiating both sides with resepect to ζ^+ or ζ^- gives

$$u_x(X(t) \pm 0, t) = \frac{\dot{u}_0(\zeta^{\pm})}{1 + \dot{a}(u_0(\zeta^{\pm}))t}. \tag{1.14}$$

Since $\dot{a}(u_0(x \pm 0))$ is bounded by a constant independent of x, there exists a constant η such that $|u_x(X(t) \pm 0, t)| \le c$ for all $0 < t < \eta$. If this shock collides with another wave at $t = t^{(0,n)}$, we choose a positive γ satisfying $\gamma < \min(\eta, t^{(0,n)})$. Obviously,

$$\int_0^\gamma |u_x(X(t) \pm 0, t)| dt \le C. \tag{1.15}$$

Again, since ζ^- is decreasing and ζ^+ is increasing, we have

$$\zeta^+(t) - \zeta^-(t) \ge \zeta^+(\gamma) - \zeta^-(\gamma) = \text{constant} > 0, \quad \text{for } t \in (\gamma, t^{(0,n)}).$$

Using Remark 1.3 and the above inequality, we can obtain that

$$\int_\gamma^{t^{(0,n)}} |u_x(X(t) \pm 0, t)| dt \le C. \tag{1.16}$$

At $t = t^{(0,n)}$, if the shock is interacted by another shock, then the resulting shock satisfies

$$\zeta^+(t) - \zeta^-(t) \ge \zeta^+(\gamma) - \zeta^-(\gamma) = \text{constant} > 0, \quad \text{for } t \in (t^{(0,n)}, t^{(0,n')}),$$

where $t^{(0,n')} \in (t^{(0,n)}, t_1)$ is the next (possible) shock interaction time. Again, using Remark 1.3 and the above inequality gives

$$\int_{t^{(0,n)}}^{t^{(0,n')}} |u_x(X(t) \pm 0, t)| dt \le C. \tag{1.17}$$

At $t = t^{(0,n)}$, if the shock is interacted by a central rarefaction wave, $u(x,t) = a^{-1}((x-z)/t)$, at $t = t^{(0,n)}$, say on the left boundary as in Figure 1(G), then we have

$$z - \zeta^-(t) \geq z - \zeta^-(t^{(0,n)}) \geq z - \zeta^-(\gamma) = \text{constant} > 0, \quad \text{for } t \in (t^{(0,n)}, t^{(0,n')}).$$

Using Remark 1.4 and the fact that

$$|u_x(X(t)+0,t)| \leq C/t \qquad \text{for} \quad t \in (t^{(0,n)}, t^{(0,n')})$$

gives the same estimate as (1.17). Repeating this procedure finite times we can obtain (1.12).

Step 2: Negative minimum point in a smooth domain. Assume that $\zeta_p \in (z_1, z_2)$ is a negative minimum point of $\dot{a}(u_0)$, and $a(u_0)$ is sufficiently smooth in (z_1, z_2). Then there is a shock curve, $x = X(t)$, formed at (x_p, t_p), where $x_p = \zeta_p + a(u_0(\zeta_p))t_p$ with $t_p = -1/\dot{a}(u_0(\zeta_p))$. From each point away from the curve, we can trace a characteristic line backward in time to $t = 0$. Using Lax geometric condition, we know that these lines will not end up with the discontinuous points. Let $x = \zeta + a(u_0(\zeta))t$ denote characteristic line passing through (x,t) point. We point out that ζ is a function of x and t. For a fixed time t, it is easy to see that a larger value of x corresponds to a larger value of ζ. This gives

$$\frac{\partial x}{\partial \zeta} = 1 + \dot{a}(u_0(\zeta))t \geq 0, \qquad \dot{a}(u_0(\zeta)) := \frac{d}{d\zeta}\Big(a(u_0(\zeta))\Big). \tag{1.18}$$

See also Dafermos [2] for a similar result. Notice that ζ_p is a negative minimum point of $\dot{a}(u_0(\zeta))$. This suggests that there exists a positive integer s such that $a^{(k)}(u_0(\zeta_p)) = 0, 2 \leq k \leq 2s$, and $a^{(2s+1)}(u_0(\zeta_p)) > 0$, where

$$a^{(k)}(u_0(x)) := \frac{d^k}{dx^k} a(u_0(x)).$$

Moreover, there exists a constant $\gamma > 0$ such that

$$0 < a^{(2s+1)}(u_0(\zeta)) \leq 2a^{(2s+1)}(u_0(\zeta_p)), \qquad \text{for } |\zeta - \zeta_p| < \gamma. \tag{1.19}$$

From (1.18) and Taylor expansion, we obtain

$$1 + \dot{a}(u_0(\zeta_p))t + \frac{1}{(2s)!} a^{(2s+1)}(u_0(\xi))(\zeta - \zeta_p)^{2s} t \geq 0,$$

where ξ is between ζ and ζ_p. Using the fact that $\dot{a}(u_0(\zeta_p)) = -1/t_p$, we obtain that

$$1 - \frac{t}{t_p} + \frac{1}{(2s)!} a^{(2s+1)}(u_0(\xi))(\zeta - \zeta_p)^{2s} t \geq 0. \tag{1.20}$$

If $|\zeta - \zeta_p| < \gamma$, we obtain from (1.19) and (1.20) that

$$|\zeta - \zeta_p| \geq C(t - t_p)^{\frac{1}{2s}}, \qquad t_p < t < t^{(p,n)},$$

where $t^{(p,n)} \leq t_{p+1}$ is the smallest time level when $x = X(t)$ collides with other waves ($t^{(p,n)} = t_{p+1}$ if this does not happen). Therefore, we have proved that for all ζ near ζ_p,

$$|\zeta - \zeta_p| \geq \min\left\{\gamma, C(t - t_p)^{\frac{1}{2s}}\right\}, \qquad t_p < t < t^{(p,n)}. \tag{1.21}$$

Tracing two characteristic lines from $x = X(t)$ backward to $t = 0$, see (1.1), we obtain from (1.21)

$$\zeta^+(t) - \zeta^-(t) \geq \min\left\{\gamma, C(t-t_p)^{\frac{1}{2s}}\right\}, \qquad t_p < t < t^{(p,n)}. \tag{1.22}$$

From (1.14), (1.22) and Remark 1.3, we obtain

$$\int_{\delta+t_p}^{t^{(p,n)}} |u_x(X(t) \pm 0, t)| dt \leq C|\log\delta| + C,$$

where $\delta > 0$ is sufficiently small. By using the same argument as used in the step 1 of this proof, we can extend the above integration to the time interval $(t_p + \delta, T)$.

Step 3: Negative minimum point at a positive jump endpoint. Without loss of generality, we assume that u_0 has a positive jump at $x = z$, and $\dot{a}(u_0(z-))$ is a negative minimum. Let $X_L(t)$ be the left boundary of the central rarefaction wave and $t_p = -1/\dot{a}(u_0(z-))$. Then a shock $x = X(t)$ is formed at the point (x_p, t_p), where $x_p = X_L(t_p)$. Assuming that $a(u_0(\zeta))$ is sufficiently smooth as $\zeta \to z - 0$ and using Taylor expansion, we have

$$z - \zeta^-(t) \geq \min\left\{\gamma, C(t-t_p)^{\frac{1}{2s}}\right\},$$

where s is a positive constant. This, together with Remark 1.4, yield that

$$\int_{\delta+t_p}^{t^{(0,n)}} |u_x(X(t) - 0, t)| dt \leq C|\log\delta| + C,$$

where $t^{(0,n)}$ indicates the smallest time level when the shock collides with another wave. It is clear that $u(x,t) = (f')^{-1}((x-z)/t)$ for $x \to X(t) + 0$ and hence $|u_x(X(t) + 0, t)| \leq C/t$. We then can conclude that (1.13) is true for this case. □

Remark 1.5. It is observed from the proof that if the discontinuities are purely due to the initial negative jump, then the $O(|\log\delta|)$ factor is not included in the estimate for the integral of u_x, see (1.12). The optimal error estimate seems possible in the case when the solution u contains neither a central rarefaction wave nor an original shock generated at a later time. This will be investigated in Section 4. □

Remark 1.6. Assume $x = X(t)$ be a shock curve generated at $t = 0$ or $t = t_p$ and $x = \zeta^\pm + a(u_0(\zeta^\pm))t$ be the corresponding characteristic lines. We have proved in the above lemma that

$$\int_{t_p+\delta}^{T} \frac{1}{1 + \dot{a}(u_0(\zeta^\pm))t} dt \leq C|\log\delta| + C. \qquad \square \tag{1.23}$$

The partial derivative u_x will be also discontinuous at a point z if there is an initial positive jump at z, i.e. $u_0(z+0) > u_0(z-0)$. This corresponds to a central rarefaction wave. The case that the rarefaction wave collides with a shock is included in Step 1 of the proof for Lemma 1.1. Here we are interested in the central rarefaction wave before it is interacted by a shock.

Lemma 1.2. *Assume a central rarefaction wave be formed at $x = z$. Let $x = X_L(t)$ and $x = X_R(t)$ be left and right boundaries of the rarefaction wave, respectively. If $\dot{a}(u_0(z+0))$ is not a negative minimum, then*

$$|u_x(X_R(t)+0,t)| \le C, \quad |u_x(X_R(t)-0,t)| \le Ct^{-1}.$$

The above results hold before the rarefaction wave is interacted by a shock. If $\dot{a}(u_0(z+0))$ is a negative minimum, then

$$|u_x(X_R(t)+0,t)| \le C(t-t_p)^{-1}, \quad |u_x(X_R(t)-0,t)| \le Ct^{-1},$$

for $t < t_p$, where $t_p = -1/\dot{a}(u_0(z+0))$. The curve $x = X_R(t)$ will become a shock after $t = t_p$. Similar results, based on $\dot{a}(u_0(z-0))$, hold for $X_L(t)$.

Proof. Observe that

$$u(x,t) = (f')^{-1}\left(\frac{x-z}{t}\right), \qquad X_L(t) \le x \le X_R(t),$$

where $X_L(t) = z + f'(u_0(z-0))t, X_R(t) = z + f'(u_0(z+0))t$. The above result gives that

$$|u_x(X_R(t)-0,t)| \le Ct^{-1}.$$

Moreover, if $\dot{a}(u_0(z+0)) \ge 0$ or $\dot{a}(u_0(z+0)) < 0$ but not a local minimum, then we can show that $|u_x(X_R(t)+0,t)| \le C$ before $x = X_R(t)$ collides with other waves. Now consider the case that $\dot{a}(u_0(z+0)) < 0$ is a local minimum. Then a new shock will be formed at $t_p = -1/\dot{a}(u_0(z+0))$. Similar to (1.14) we have

$$u_x(X_R(t)+0,t) = \frac{\dot{u}_0(z+0)}{1+\dot{a}(u_0(z+0))t}, \qquad 0 < t < t_p.$$

This and the fact that $t_p = -1/\dot{a}(u_0(z+0))$ yield

$$|u_x(X_R(t)+0,t)| \le C(t_p - t)^{-1}, \qquad 0 < t < t_p.$$

The proof is complete. □

Remark 1.7. Let $x = X_R(t)$ be the right boundary of a central rarefaction wave. It may collide with a shock at a certain time, say $t = t^*$. We still denote the resulting shock as $X_R(t)$. Hence, we have extended the curve $x = X_R(t)$ to the whole time interval $[0,T]$. Using Lemmas 1.1 and 1.2 gives

$$\int_{t_p+\delta}^{t_{p+1}-\delta} \Big|u_x(X_R(t)\pm 0,t)\Big| dt \le C|\log\delta| + C. \tag{1.24}$$

Moreover, from each point on $x = X_R(t)$ we can trace a right characteristic line $x = \zeta^+ + a(u_0(\zeta^+))t$ back to $t = 0$. Here $\zeta^+ = z + 0$ before $X_R(t)$ is interacted by a shock. From the proofs for Lemmas 1.1 and 1.2, we can show that

$$\int_{t_p+\delta}^{t_{p+1}-\delta} \frac{1}{1+\dot{a}(u_0(\zeta^+))t} dt \le C|\log\delta| + C. \tag{1.25}$$

We may extend $x = X_L(t)$ in a similar way and trace a left characteristic line $x = \zeta^- + a(u_0(\zeta^-))t$ back to $t = 0$. Here, $\zeta^- = z - 0$ before $x = X_L(t)$ collides with a shock. We also have

$$\int_{t_p+\delta}^{t_{p+1}-\delta} \Big| u_x(X_L(t) \pm 0, t) \Big| dt \le C|\log\delta| + C, \tag{1.26}$$

$$\int_{t_p+\delta}^{t_{p+1}-\delta} \frac{1}{1 + \dot{a}(u_0(\zeta^-))t} dt \le C|\log\delta| + C. \qquad \square \tag{1.27}$$

In order to provide a sharp L^1-error bound, some properties on second order derivative are also needed, e.g. see Sections 3.2 and 3.3.

Lemma 1.3. *Assume that u_0 satisfy the assumptions* (A1)–(A4) *and let $a(u_0)$ be sufficiently smooth near its negative minimum points. Then the solution to* (0.1) *and* (0.2) *satisfies*

$$\int_{t_p+\delta}^{t_{p+1}-\delta} \|u_{xx}(\bullet, t)\| dt \le C|\log\delta| + C, \qquad 0 \le p \le P, \tag{1.28}$$

provided δ is sufficiently small.

Proof. For any fixed time $t \in (t_p, t_{p+1})$, there are finite intervals $(Z_l(t), Z_{l+1}(t))$ (the number of the intervals can be bounded by a constant independent of t) such that the transformation $x = \zeta + a(u_0(\zeta))t$ maps $(Z_l(t), Z_{l+1}(t))$ (in x space) to $(\theta_l(t), \theta_{l+1}(t))$ (in ζ space). The solution $u(x,t)$ is continuously differentiable in (Z_l, Z_{l+1}) but not at the end points $Z_l(t)$ and $Z_{l+1}(t)$. It is observed that

$$\min_l Z_l(t) = \min_l \theta_l = -\infty, \quad \text{and} \quad \max_l Z_l(t) = \max_l \theta_l = \infty.$$

In the case that $Z_l(t)$ and $Z_{l+1}(t)$ are boundaries of a central rarefaction wave, there exists an integer s such that $\theta_s(t) = \theta_{s+1}(t) = z_s$ and the solution is of the form

$$u(x,t) = (f')^{-1}\left(\frac{x - z_s}{t}\right), \qquad Z_l(t) \le x \le Z_{l+1}(t),$$

where $Z_{l+1}(t) - Z_l(t) \le [f'(u_0(z_s+)) - f'(u_0(z_s-))]t$. It follows from $f'' \ge \alpha > 0$ that $[(f')^{-1}]''(\zeta) \le C$ for $|\zeta| \le C$. Hence, we have

$$\int_{Z_l(t)}^{Z_{l+1}(t)} |u_{xx}(x,t)| dx \le C \int_{Z_l(t)}^{Z_{l+1}(t)} \frac{dx}{t^2} = C\Big(Z_{l+1}(t) - Z_l(t)\Big)t^{-2} \le Ct^{-1},$$

which leads to

$$\int_{t_p+\delta}^{t_{p+1}} \int_{Z_l(t)}^{Z_{l+1}(t)} |u_{xx}(x,t)| dx dt \le C|\log\delta| + C.$$

If $\theta_l < \theta_{l+1}$, we have $1 + \dot{a}(u_0(\zeta))t > 0$ for $\zeta \in (\theta_l, \theta_{l+1})$. Differenting the equation $u(\zeta + a(u_0(\zeta))t, t) = u_0(\zeta)$ with respect to ζ gives

$$u_{xx}(\zeta + a(u_0(\zeta))t, t)(1 + \dot{a}(u_0)t) = -\frac{\dot{u}_0 \ddot{a}(u_0)t}{(1 + \dot{a}(u_0)t)^2} + \frac{\ddot{u}_0}{1 + \dot{a}(u_0)t}.$$

It follows from the above result that

$$\begin{aligned}\int_{Z_l(t)}^{Z_{l+1}(t)} |u_{xx}(x,t)|dx &= \int_{\theta_l(t)}^{\theta_{l+1}(t)} |u_{xx}(\zeta + a(u_0(\zeta))t, t)(1+\dot{a}(u_0)t)|d\zeta \\ &\le \int_{\theta_l(t)}^{\theta_{l+1}(t)} \frac{|\dot{u}_0\ddot{a}(u_0)|t}{(1+\dot{a}(u_0)t)^2}d\zeta + \int_{\theta_l(t)}^{\theta_{l+1}(t)} \frac{|\ddot{u}_0|}{1+\dot{a}(u_0)t}d\zeta \\ &=: I_1 + I_2.\end{aligned}$$

Observe that

$$\frac{\partial}{\partial\zeta}\left(\frac{1}{1+\dot{a}(u_0)t}\right) = -\frac{\ddot{a}(u_0)t}{(1+\dot{a}(u_0)t)^2}.$$

This suggests that the maximum of $1/(1+\dot{a}(u_0)t)$ possibly occur at (i) increasing inflection points of $a(u_0)$; (ii) decreasing inflection points of $a(u_0)$; and (iii) the end points $\theta_l(t)$ or $\theta_{l+1}(t)$. For the possibility (i), we have $1/(1+\dot{a}(u_0)t) \le 1$. For the possibility (ii), noting that $1+\dot{a}(u_0)t > 0$ we have

$$\frac{1}{1+\dot{a}(u_0)t} \le \max_{t_j > t} \frac{1}{1-t/t_j}.$$

Since $t \in (t_p, t_{p+1})$, we obtain from the above result that

$$\frac{1}{1+\dot{a}(u_0)t} \le \frac{C}{t_{p+1}-t}.$$

Consequently, we obtain for the above three possibilities that

$$\frac{1}{1+\dot{a}(u_0)t} \le 1 + \frac{C}{t_{p+1}-t} + \frac{1}{1+\dot{a}(u_0(\theta_l))t} + \frac{1}{1+\dot{a}(u_0(\theta_{l+1}))t}. \tag{1.29}$$

Notice that there is no new shock be generated for $t \in (t_p, t_{p+1})$. If $x = Z_l(t)$ corresponds to a shock curve, then $\theta_l(t) = \zeta^+(t)$ following our notation in Lemma 1.1. Then using Remark 1.6 we have

$$\int_{t_p+\delta}^{t_{p+1}} \frac{1}{1+\dot{a}(u_0(\theta_l))t}dt \le C|\log\delta| + C.$$

If $x = Z_l(t)$ is a right boundary of a rarefaction wave and collides with a shock at $t = t^* \in (t_p, t_{p+1})$, then using Remark 1.7 gives

$$\int_{t_p+\delta}^{t_{p+1}-\delta} \frac{1}{1+\dot{a}(u_0(\theta_l))t}dt \le C|\log\delta| + C.$$

Similar results hold for the last term in (1.29). These, together with the assumption (A2), yield

$$\int_{t_p+\delta}^{t_{p+1}-\delta} |I_2|dt \le C|\log\delta| + C.$$

Observe that

$$\int_c^b \frac{\ddot{a}(u_0)t}{(1+\dot{a}(u_0)t)^2}d\zeta = \frac{1}{1+\dot{a}(u_0(c))t} - \frac{1}{1+\dot{a}(u_0(b))t},$$

provided that $a(u_0) \in C^2(b,c)$. It follows from (A3), (A4) and the above identity that

$$\int_{\theta_l(t)}^{\theta_{l+1}(t)} \frac{|\ddot{a}(u_0)|t}{(1+\dot{a}(u_0)t)^2} d\zeta \tag{1.30}$$
$$\leq \frac{1}{1+\dot{a}(u_0(\theta_l(t)))t} + \frac{1}{1+\dot{a}(u_0(\theta_{l+1}(t)))t} + \sum_j \frac{2}{1+\dot{a}(u_0(y_j))t},$$

where $y_j \in (\theta_j, \theta_{j+1})$ are inflection points of $a(u_0)$. Similar to the discussion for I_2, we have

$$\int_{t_p+\delta}^{t_{p+1}-\delta} \int_{\theta_l(t)}^{\theta_{l+1}(t)} \frac{|\ddot{a}(u_0)|t}{(1+\dot{a}(u_0)t)^2} d\zeta dt \leq C|\log \delta| + C.$$

Using (A1) and (A2), we can show that $\dot{u}_0$ is bounded almost everywhere. This and the above inequality yield that

$$\int_{t_p+\delta}^{t_{p+1}-\delta} |I_1| dt \leq C|\log\delta| + C.$$

The proof of Lemma 1.3 is thereby complete. $\square$

2. Stability and traveling wave

Our error estimates are based on a stability lemma for nonhomogeneous viscous equations and a traveling wave lemma. We first introduce the stability lemma.

Lemma 2.1. *Let $v^{(i)}(x,t), i = 1,2,$ be continuous and piecewise smooth solutions of the following equations:*

$$\left(v^{(i)}\right)_t + \left(f(v^{(i)})\right)_x - \epsilon \left(v^{(i)}\right)_{xx} = g_i(x,t), \qquad t > a, \qquad i = 1,2. \tag{2.1}$$

The above equation holds for all values of x except on some curves $X_m(t), 1 \leq m \leq M$, where $v_x^{(i)}$ may not exist. If $w := v^{(1)} - v^{(2)} \to 0$ as $x \to \infty$, then

$$\|w(\bullet,t)\| \leq \|w(\bullet,a)\| + \int_a^t \|g_1(\bullet,\tau) - g_2(\bullet,\tau)\| d\tau \tag{2.2}$$
$$+\epsilon \sum_{m=1}^{M} \int_a^t |w_x(X_m(\tau)+0,\tau) - w_x(X_m(\tau)-0,\tau)|\, d\tau.$$

Proof. It follows from (2.1) that

$$w_t + (f'(\zeta)w)_x - \epsilon w_{xx} = g_1(x,t) - g_2(x,t), \tag{2.3}$$

where ζ is some intermediate value between $v^{(1)}$ and $v^{(2)}$. If $w \geq 0$ or $w \leq 0$ for all x, then straightforward integration on the above equation gives (2.2). Let $p_0(t) < p_1(t) < p_2(t) < \cdots$ be the points such that at those points w changes signs.

Let α_j be the sign of w in (p_j, p_{j+1}). Multiplying (2.3) by α_j and integrating the resulting equation over (p_j, p_{j+1}) give

$$
\begin{aligned}
\alpha_j \int_{p_j}^{p_{j+1}} w_t dx &= \epsilon\Big(\alpha_j w_x(p_{j+1}-0,t) - \alpha_j w_x(p_j+0,t)\Big) \\
&+ \sum_{p_j<X_m<p_{j+1}} \epsilon\alpha_j \Big(w_x(X_m(t)+0,t) - w_x(X_m-0,t)\Big) \\
&+\alpha_j \int_{p_j}^{p_{j+1}} (g_1(x,t)-g_2(x,t))\,dx.
\end{aligned}
\tag{2.4}
$$

Since $w(p_j,t)=w(p_{j+1},t)=0$ and $\alpha_j w \geq 0$ for $x \in (p_j,p_{j+1})$, we have

$$
\frac{d}{dt}\int_{p_j}^{p_{j+1}} |w| dx = \alpha_j \int_{p_j}^{p_{j+1}} w_t dx.
$$

Moreover, observing that $\alpha_j w_x(p_{j+1}-0,t) \leq 0$ and $\alpha_j w_x(p_j+0,t) \geq 0$, we obtain from (2.4) that

$$
\begin{aligned}
\frac{d}{dt}\int_{p_j}^{p_{j+1}} |w| dx \quad &\leq \quad \sum_{p_j<X_m<p_{j+1}} \epsilon\Big|w_x(X_m+0,t) - w_x(X_m-0,t)\Big| \\
&\quad + \int_{p_j}^{p_{j+1}} |g_1(x,t)-g_2(x,t)|\,dx.
\end{aligned}
$$

Since the above inequality is true for all $j \geq 0$, we have

$$
\begin{aligned}
\frac{d}{dt}\int_{p_0}^{p^*} |w| dx \quad &\leq \quad \sum_{p_0<X_m<p^*} \epsilon\Big|w_x(X_m+0,t) - w_x(X_m-0,t)\Big| \\
&\quad + \int_{p_0}^{p^*} |g_1(x,t)-g_2(x,t)|\,dx,
\end{aligned}
$$

where $p^* = \sup_j p_j$. If $p^* < \infty$, using a similar method as above gives

$$
\begin{aligned}
\frac{d}{dt}\int_{p^*}^{\infty} |w| dx \quad &\leq \quad \epsilon \sum_{X_m>p^*} |w_x(X_m(t)+0,t) - w_x(X_m-0,t)| \\
&\quad + \int_{p^*}^{\infty} |g_1(x,t)-g_2(x,t)|\,dx.
\end{aligned}
$$

In obtaning the last ineuqality, we have used the fact that $w \to 0$ as $x \to \infty$. Using the above two inequalities, we obtain

$$
\begin{aligned}
\int_{p_0}^{\infty} |w(x,t)| dx \quad &\leq \quad \epsilon \sum_{X_m>p_0} \int_a^t \Big|w_x(X_m+0,\tau) - w_x(X_m-0,\tau)\Big| d\tau \\
&\quad + \int_a^t \int_{p_0}^{\infty} |g_1(x,\tau)-g_2(x,\tau)|\,dxd\tau + \int_{p_0}^{\infty} |w(x,a)| dx.
\end{aligned}
$$

A similar result which replaces p_0, ∞ by $-\infty, p_0$, respectively, in the above inequality, can be established. These results lead to (2.2). $\square$

Assume $X_m(t)$ be a smooth curve satisfying the Rankine-Hugoniot jump condition:

$$\dot{X}_m(t) = \frac{f(u_m^+) - f(u_m^-)}{u_m^+ - u_m^-}, \tag{2.5}$$

where $\dot{X}(t) := dX/dt$ is the shock speed, $u_m^+ = u(X_m(t)+,0), u_m^- = u(X_m(t)-,0)$ with $u_m^+ < u_m^-$. If $u_m^\pm$ are independent of t, it is shown in [20] that there is a traveling wave solution of the form $V_\epsilon(x-St)$, where the wave speed S is a constant, satisfying (0.3) and $V_\epsilon(-\infty) = u_m^-, V_\epsilon(\infty) = u_m^+$. In the case when $u_m^\pm$ are functions of t, we need the following generalization.

Lemma 2.2. *Let $u_m^+(t) < u_m^-(t)$ be two given functions and $X_m(t)$ be defined by* (2.5)*. Let $V(\zeta; u_m^+, u_m^-)$ be defined implicitly by*

$$\zeta = \int_{\frac{1}{2}(u_m^+ + u_m^-)}^{V} \left[\Phi(u; u_m^+, u_m^-)\right]^{-1} du, \tag{2.6}$$

$$\Phi(u; u_m^+, u_m^-) = f(u) - f(u_m^-) - \dot{X}_m(u - u_m^-). \tag{2.7}$$

With respect to ζ, the function $V(\zeta; u_m^+, u_m^-)$ is a decreasing function satisfying

(2.8)

$$V\left(-\infty; u_m^+, u_m^-\right) = u_m^-, \quad V\left(0; u_m^+, u_m^-\right) = \frac{1}{2}(u_m^+ + u_m^-), \quad V\left(\infty; u_m^+, u_m^-\right) = u_m^+.$$

It also satisfies the following inequalities

$$\left|V\left(\zeta; u_m^+, u_m^-\right) - H(\zeta; u_m^+,, u_m^-)\right| \le (u_m^- - u_m^+) \exp\{-\alpha(u_m^- - u_m^+)|\zeta|/2\}, \tag{2.9}$$

$$\left\|V^{(\epsilon)}\left(\bullet - X_m; u_m^+, u_m^-\right) - H\left(\bullet - X_m; u_m^+, u_m^-\right)\right\| \le \frac{4}{\alpha}\epsilon, \tag{2.10}$$

where α is defined by (0.5)*. In the above inequalities, $V^{(\epsilon)}(\zeta; u_m^+, u_m^-) := V(\zeta/\epsilon; u_m^+, u_m^-)$, H is the so-called Heaviside function defined by*

$$H(x; u_m^+, u_m^-) = \begin{cases} u_m^+ & \text{if} \quad x \ge 0, \\ u_m^- & \text{if} \quad x < 0\ . \end{cases} \quad \Box$$

Denote $V^{(\epsilon)}(x - X_m(t); u_m^+, u_m^-)$ by $V_m^{(\epsilon)}$. Direct calculation shows that

$$\left(V_m^{(\epsilon)}\right)_t + f\left(V_m^{(\epsilon)}\right)_x - \epsilon\left(V_m^{(\epsilon)}\right)_{xx} = \left(V_m^{(\epsilon)}\right)_{u_m^+} \dot{u}_m^+ + \left(V_m^{(\epsilon)}\right)_{u_m^-} \dot{u}_m^-, \tag{2.11}$$

where $\left(V_m^{(\epsilon)}\right)_{u_m^\pm}$ indicate the partial derivatives for $V^{(\epsilon)}$ with respect to the parameters $u_m^\pm$, respectively. Since $V_m^{(\epsilon)} \to H(x - X_m; u_m^+, u_m^-)$ as $\epsilon \to 0+$ (see (2.10)), it is expected that the right hand side of (2.11) approaches $H(x - X_m(t); \dot{u}_m^+, \dot{u}_m^-)$ as ϵ is sufficiently small.

Lemma 2.3. *Let $V^{(\epsilon)}(\zeta; u_m^+, u_m^-)$ be defined by that in Lemma 2.2. We have*

$$\begin{aligned} &\left\|\left(V^{(\epsilon)}\right)_{u_m^+}(\bullet; u_m^+, u_m^-)\dot{u}_m^+ + \left(V^{(\epsilon)}\right)_{u_m^-}(\bullet; u_m^+, u_m^-)\dot{u}_m^- - H(\bullet; \dot{u}_m^+, \dot{u}_m^-)\right\| \\ \le\ & C\epsilon\Big(|u_x(X_m + 0, t)| + |u_x(X_m - 0, t)|\Big), \end{aligned} \tag{2.12}$$

where $\dot{u}_m^\pm = d(u_m^\pm)/dt$.

We defer the proofs of the above two lemmas to the Appendix.

3. Main result and outline of proof

In the preceding sections we were concerned with the solution structure, traveling wave solution and stability lemma. Using these results, we are now ready to state and prove the main result of this paper.

Theorem 3.1. *Let the flux f be strictly convex. Assume that the initial data u_0 satisfies the requirements* (A1)-(A4) *stated in Section* 1 *and that $a(u_0)$ is sufficiently smooth near its negative minimum points. If v_ϵ is solution to* (0.3) *and* (0.4) *and u is solution to* (0.1) *and* (0.2)*, then the following error estimate holds:*

$$\sup_{0\le t\le T}\|v_\epsilon(\bullet,t)-u(\bullet,t)\|\le C(T)\Big(\epsilon|\log\epsilon|+\epsilon\Big), \tag{3.1}$$

where $C(T)$ is a constant independent of ϵ.

Before giving a proof for the above estimate, we point out a direct application of Theorem 3.1.

Corollary 3.1. *Let the flux f be strictly convex and $u_0\in C_0^2(R)$. If $a(u_0)$ has a finite number of inflection points and is sufficiently smooth at the decreasing inflection points, then the following error estimate holds:*

$$\sup_{0\le t\le T}\|v_\epsilon(\bullet,t)-u(\bullet,t)\|\le C(T)\Big(\epsilon|\log\epsilon|+\epsilon\Big). \tag{3.2}$$

Remark 3.1. Loosely speaking, the above result implies that for scalar convex conservation laws, if the initial data u_0 is smooth and $f'(u_0)$ has a finite number of inflection points then L^1 convergence rate of viscosity methods is $O(\epsilon|\log\epsilon|)$. The class of initial data described in Corollary 3.1 is quite general, and is widely used in practical computations. □

We will use Lemma 2.1 to prove Theorem 3.1. An essential requirement in Lemma 2.1 is that the solution to (2.1) should be continuous. It is known that the solution of (0.3) is continuous for $t>0$ if the initial data is piecewise smooth. However, the solution of (0.1) may be discontinuous for $t>0$ even if the initial data is smooth. Therefore, we cannot apply Lemma 2.1 directly to equations (0.1) and (0.3).

As discussed before, there are two kinds of discontinuity, namely rarefaction wave and shock wave for solutions of (0.1) and (0.2). Across the rarefaction region, the solution is continuous. But on the shock curves, the solution is discontinuous. We need to construct reasonable approximation to u so that it can get rid of the shock discontinuities.

We will consider the cases when u possesses shock discontinuity. For any interval $(t_p,t_{p+1}), 0\le p\le P$ (here $t_0=0, t_{P+1}=T$), we will construct $\bar{v}_{\epsilon,p}(x,t)$, an approximate function to $u(x,t)$ such that

- (B1): $\bar{v}_{\epsilon,p}(x,t)\in C(R\times(t_p,t_{p+1})), \bar{v}_{\epsilon,p}(x,t)-v_\epsilon\to 0$, as $x\to\pm\infty$;

- (B2): $\bar{v}_{\epsilon,p}(x,t)$ is piecewise smooth and satisfies

$$(\bar{v}_{\epsilon,p})_t + (f(\bar{v}_{\epsilon,p}))_x - \epsilon(\bar{v}_{\epsilon,p})_{xx} = \bar{g}(x,t) \tag{3.3}$$

  in its smooth regions; where the right hand side function satisfies

$$\int_{t_p+\epsilon}^{t_{p+1}-\epsilon} \|\bar{g}(\bullet,t)\| dt \le C\epsilon|\log\epsilon| + C\epsilon; \tag{3.4}$$

- (B3): u_x and $(\bar{v}_{\epsilon,p})_x$ are discontinuous on the same curves, namely, they are discontinuous on $X_m(t), 1 \le m \le M_p$ for $(x,t) \in (R \times (t_p, t_{p+1}))$ (see Section 1 for the definition of X_m; here for ease of notations we ignore the superscript p and n). Moreover,

$$\begin{aligned} &\sum_m |(\bar{v}_{\epsilon,p})_x(X_m(t)+0,t) - (\bar{v}_{\epsilon,p})_x(X_m(t)-0,t)| \\ = &\sum_m |u_x(X_m(t)+0,t) - u_x(X_m(t)-0,t)|\,; \end{aligned} \tag{3.5}$$

- (B4): For $t \in (t_p, t_{p+1})$,

$$\|\bar{v}_{\epsilon,p}(\bullet,t) - u(\bullet,t)\| \le C\epsilon. \tag{3.6}$$

It is known that v_ϵ, the solution to (0.3) and (0.4), belongs to $C^1(R\times(t_p,t_{p+1}))$. If $\bar{v}_{\epsilon,p}$ satisfies (B1), (B2) and (B3), using Lemma 2.1 gives

$$\begin{aligned} \|v_\epsilon(\bullet,t) - \bar{v}_{\epsilon,p}(\bullet,t)\| &\le \|v_\epsilon(\bullet,t_p+\epsilon) - \bar{v}_{\epsilon,p}(\bullet,t_p+\epsilon)\| + \int_{t_p+\epsilon}^{t} \|\bar{g}(\bullet,\tau)\| d\tau \\ &+\epsilon\sum_m \int_{t_p+\epsilon}^{t} |u_x(X_m(\tau)+0,\tau) - u_x(X_m(\tau)-0,\tau)| d\tau, \end{aligned}$$

for $t \in (t_p+\epsilon, t_{p+1}-\epsilon)$. Using Lemmas 1.1, 1.2 and (3.4) gives

$$\|v_\epsilon(\bullet,t) - \bar{v}_{\epsilon,p}(\bullet,t)\| \le \|v_\epsilon(\bullet,t_p+\epsilon) - \bar{v}_{\epsilon,p}(\bullet,t_p+\epsilon)\| + C\epsilon|\log\epsilon| + C\epsilon,$$

for $t \in (t_p+\epsilon, t_{p+1}-\epsilon)$. It follows from the above inequlaity and (B4) that

$$\|v_\epsilon(\bullet,t) - u(\bullet,t)\| \le \|v_\epsilon(\bullet,t_p+\epsilon) - u(\bullet,t_p+\epsilon)\| + C\epsilon|\log\epsilon| + C\epsilon,$$

for $t \in (t_p+\epsilon, t_{p+1}-\epsilon)$. Since u and v_ϵ satisfy the following stability results

$$\|u(\cdot,s_1) - u(\cdot,s_2)\| \le C|s_1-s_2|, \qquad \|v_\epsilon(\cdot,s_1) - v_\epsilon(\cdot,s_2)\| \le C|s_1-s_2|,$$

we obtain

$$\|v_\epsilon(\bullet,t) - u(\bullet,t)\| \le \|v_\epsilon(\bullet,t_p) - u(\bullet,t_p)\| + C\epsilon|\log\epsilon| + C\epsilon, \qquad t \in [t_p, t_{p+1}].$$

Noting above inequality is true for all $0 \le p \le P$, we obtain that for any $t \in [0,T]$,

$$\begin{aligned} \|v_\epsilon(\bullet,t) - u(\bullet,t)\| &\le \|v_\epsilon(\bullet,0) - u(\bullet,0)\| + C\epsilon|\log\epsilon| + C\epsilon \\ &= C\epsilon|\log\epsilon| + C\epsilon. \end{aligned}$$

To prove Theorem 3.1, the remaining work is to construct $\bar{v}_{\epsilon,p}$, $0 \le p \le P$, satisfying (B1)–(B4). We will concentrate on (t_p, t_{p+1}), one of the intervals in $\cup_{p=0}^{P}(t_p, t_{p+1})$.

Remark 3.2. Once again we point out that the main difficulty in using Lemma 2.1 is that the solution u may be discontinuous. This is happened when u has shock discontinuities. If the solution u contains contact discontinuities only (i.e. rarefaction waves), the solution is still continuous and Lemma 2.1 can be used directly. Therefore, the construction of $\bar{v}_\epsilon$ is to remove shock discontinuities and no efforts need to be made to remove the contact discontinuities. □

For ease of notation, we denote $\bar{v}_{\epsilon,p}$ by $\bar{v}_\epsilon$ in the remaining of this section.

3.1. Zero shock. In this case, there is no shock occur in the domain $R\times(t_p, t_{p+1})$. We simply choose $\bar{v}_\epsilon = u$. Since there is no shock discontinuity for u, we have $\bar{v}_\epsilon = u \in C(R\times(t_p,t_{p+1}))$. It is straightforward to verify that this $\bar{v}_\epsilon$ satisfies the requirements (B1), (B3) and (B4). It is also obvious that $\bar{g}(x,t)$ in (3.3) equals to $-\epsilon u_{xx}$. Using Lemma 1.3 we can verify that this $\bar{g}$ satisfies (3.4). Therefore, the requirement (B2) is also satisfied.

3.2. One shock. In this case, we assume that u is smooth in $R\times(t_p,t_{p+1})$, except on $x = X_m(t)$. Here the function $X_m(t)$ satisfies (2.5). As mentioned before, Lemma 2.1 cannot be applied directly since $u \notin C(R\times(t_p,t_{p+1}))$. In order to overcome the difficulty, we introduce the following approximate solution:

$$\bar{v}_\epsilon(x,t) = u(x,t) + \left[V^{(\epsilon)}\left(x - X_m; u_m^+, u_m^-\right) - H\left(x - X_m; u_m^+, u_m^-\right)\right]. \tag{3.7}$$

We will verify that this $\bar{v}_\epsilon$ satisfies the requirements (B1)-(B4). First, we observe that

$$\bar{v}_\epsilon(X_m+0,t) - \bar{v}_\epsilon(X_m-0,t) = V^{(\epsilon)}(0+;u_m^+,u_m^-) - V^{(\epsilon)}(0-;u_m^+,u_m^-) = 0,$$

which indicates that $\bar{v}_\epsilon \in C(R\times(t_p,t_{p+1}))$. It is also easy to see that $\bar{v}_\epsilon - u \to 0$ as $x\to\pm\infty$, which implies that $\bar{v}_\epsilon - v_\epsilon \to 0$ as $x\to\pm\infty$. Hence, (B1) is true. We can also show that (B3) is true. Using Lemma 2.2 will give (B4). We now need to show that $\bar{v}_\epsilon$ satisfies (B2).

Proof of (B2). We denote $V^{(\epsilon)}\left(x - X_m; u_m^+, u_m^-\right)$ by $V_m^{(\epsilon)}$, and $H\left(x - X_m; u_m^+, u_m^-\right)$ by H_m. It follows from (2.11) that

$$\left(V_m^{(\epsilon)}\right)_t + f\left(V_m^{(\epsilon)}\right)_x - \epsilon\left(V_m^{(\epsilon)}\right)_{xx} = \left(V_m^{(\epsilon)}\right)_{u_m^+}\dot{u}_m^+ + \left(V_m^{(\epsilon)}\right)_{u_m^-}\dot{u}_m^-.$$

This, together with (0.1), gives

$$(\bar{v}_\epsilon)_t + (f(\bar{v}_\epsilon))_x - \epsilon(\bar{v}_\epsilon)_{xx} = \bar{g}(x,t), \qquad t\in(t_p,t_{p+1}),$$

where

$$\bar{g}(x,t) = I_1 + I_2 - \epsilon u_{xx}, \qquad I_1 = -(f(u))_x - f\left(V_m^{(\epsilon)}\right)_x + f(\bar{v}_\epsilon)_x,$$
$$I_2 = \left(V_m^{(\epsilon)}\right)_{u_m^+}\dot{u}_m^+ + \left(V_m^{(\epsilon)}\right)_{u_m^-}\dot{u}_m^- - H\left(x - X_m(t); \dot{u}_m^+, \dot{u}_m^-\right).$$

Notice that $[H(x-X_m(t);u_m^+,u_m^-)]_x=0$ for all x except on the curve $x=X_m(t)$. Away from $x=X_m(t)$, we have

$$\begin{aligned}
I_1 &= f'(\bar v_\epsilon)\left(u_x+\left(V_m^{(\epsilon)}\right)_x\right)-f'(u)u_x-f'\left(V_m^{(\epsilon)}\right)\left(V_m^{(\epsilon)}\right)_x\\
&= \left(f'(\bar v_\epsilon)-f'(u)\right)u_x+\left(f'(\bar v_\epsilon)-f'\left(V_m^{(\epsilon)}\right)\right)\left(V_m^{(\epsilon)}\right)_x\\
&= f''(\xi_1)\left(V_m^{(\epsilon)}-H_m\right)u_x+f''(\xi_2)(u-H_m)\left(V_m^{(\epsilon)}\right)_x\\
&=: I_1^{(1)}+I_1^{(2)}.
\end{aligned}$$

where ξ_1 and ξ_2 are some intermediate values. For $x>X_m(t)$, although there is no shock curves across the interval (X_m,x), there may have finite number of central rarefaction waves in the interval. Let $x=Y_l(t)$, $1\le l\le L$, denote left or right boundaries of the rarefaction waves for $t\in(t_p,t_{p+1})$. If $x>X_m(t)$ but $x\neq Y_l(t)$,

$$u_x(x,t)=u_x(X_m(t)+0,t)+\sum_l\left(u_x(Y_l(t)+0,t)-u_x(Y_l(t)-0,t)\right)+\int_{X_m}^x u_{xx}(x,t)dx.$$

It follows from the above result that, for $x>X_m(t)$,

$$|u_x(x,t)|\le|u_x(X_m(t)+0,t)|+\sum_l|u_x(Y_l(t)\pm0,t)|+\|u_{xx}(\bullet,t)\|. \tag{3.8}$$

Similarly, we can show that, for $x<X_m(t)$,

$$|u_x(x,t)|\le|u_x(X_m(t)-0,t)|+\sum_l|u_x(Y_l(t)\pm0,t)|+\|u_{xx}(\bullet,t)\|. \tag{3.9}$$

The above results and (2.10) lead to

$$\|I_1^{(1)}(\bullet,t)\|\le C\epsilon\left(|u_x(X_m(t)\pm0,t)|+\sum_l|u_x(Y_l(t)\pm0,t)|+\|u_{xx}(\bullet,t)\|\right).$$

An application of Lemmas 1.1, 1.2 and 1.3 yields

$$\int_{t_p+\epsilon}^{t_{p+1}-\epsilon}\|I_1^{(1)}(\bullet,t)\|dt\le C\epsilon|\log\epsilon|+C\epsilon.$$

Using the facts that $[H(x-X_m(t);u_m^+,u_m^-)]_x=0, a.e.$ and $\left(V^{(\epsilon)}\right)_x\le0$, we have

$$\|I_1^{(2)}(\bullet,t)\|\le C\int_{-\infty}^{\infty}|u-H_m|\left(H_m-V_m^{(\epsilon)}\right)_x dx. \tag{3.10}$$

For any $x\neq X_m(t)$, we can show that

$$|u(x,t)-H_m|\le\max_{z\in R}|u_x(z\pm0,t)||x-X_m(t)|.$$

For $x\neq X_m(t)$, using (3.8) and (3.9) yields

$$|u(x,t)-H_m|\le\left(|u_x(X_m(t)\pm0,t)|+\sum_l|u_x(Y_l(t)\pm0,t)|+\|u_{xx}(\bullet,t)\|\right)|x-X_m(t)|.$$

Hence, from (3.10) we have

$$\begin{aligned}\|I_1^{(2)}(\bullet,t)\| &\le C\left(|u_x(X_m(t)\pm 0,t)| + \sum_l |u_x(Y_l(t)\pm 0,t)| + \|u_{xx}(\bullet,t)\|\right)\\ &\qquad\times\int_{-\infty}^{\infty}|x-X_m(t)|\left(H_m - V_m^{(\epsilon)}\right)_x dx.\end{aligned}$$

Using integration by parts and Lemma 2.2, we can show that the integral on the right hand side can be bounded by $C\epsilon$. Further, using Lemmas 1.1, 1.2 and 1.3 gives

$$\int_{t_p+\epsilon}^{t_{p+1}-\epsilon}\|I_1^{(2)}(\bullet,t)\|dt \le C\epsilon|\log\epsilon| + C\epsilon.$$

We have therefore obtained an estimate for I_1. An application of Lemmas 1.1 and 2.3 yields

$$\int_{t_p+\epsilon}^{t_{p+1}-\epsilon}\|I_2(\bullet,t)\|dt \le C\epsilon|\log\epsilon| + C\epsilon.$$

Using the estimates for I_1 and I_2, together with Lemma 1.3, we establish the inequality (3.4). □

Remark 3.3. From the definition of $\bar{g}(x,t)$, we find that it is necessary to estimate u_{xx}. Therefore, Lemma 1.3 is useful in obtaining our error estimates. □

3.3. Two shocks. In this case, we have two smooth shock curves, $X_m(t)$ and $X_{m+1}(t)$, satisfying one of the following possibilities:

- (P1): $X_{m+1}(t) > X_m(t)$ for $t\in[t_p,t_{p+1}]$, see Case H in Figure 1;
- (P2): $X_{m+1}(t) > X_m(t)$ for $t\in[t_p,t_{p+1})$ and $X_m(t_{p+1}) = X_{m+1}(t_{p+1})$, see Case D in Figure 1;
- (P3): there exists a time level $t^{(p,1)}\in(t_p,t_{p+1})$ such that there have two shock waves for $t\in(t_p,t^{(p,1)})$, but the two shocks meet at $t=t^{(p,1)}$ and form one shock for $t\in[t^{(p,1)},t_{p+1})$, see Cases C and D in Figure 1.

For the first two possibilities, we introduce the following approximate solution:

$$\begin{aligned}\bar{v}_\epsilon(x,t) = u(x,t) &+ \left[V^{(\epsilon)}\left(x-X_m(t);u_m^+,u_m^-\right) - H\left(x-X_m(t);u_m^+,u_m^-\right)\right]\\ &+ \left[V^{(\epsilon)}\left(x-X_{m+1}(t);u_{m+1}^+,u_{m+1}^-\right) - H\left(x-X_{m+1}(t);u_{m+1}^+,u_{m+1}^-\right)\right].\end{aligned}\tag{3.11}$$

We can verify that $\bar{v}_\epsilon$ satisfies the requirements (B1), (B3) and (B4). Again, we will show that $\bar{v}_\epsilon$ satisfies (B2).

Proof of (B2). We denote $V^{(\epsilon)}\left(x-X_m(t);u_m^+,u_m^-\right)$ and $H\left(x-X_m(t);u_m^+,u_m^-\right)$ by $V_m^{(\epsilon)}$ and H_m, respectively. It follows from (2.11) that

$$\begin{aligned}&\left(V_m^{(\epsilon)}\right)_t + f\left(V_m^{(\epsilon)}\right)_x - \epsilon\left(V_m^{(\epsilon)}\right)_{xx} + \left(V_{m+1}^{(\epsilon)}\right)_t + f\left(V_{m+1}^{(\epsilon)}\right)_x - \epsilon\left(V_{m+1}^{(\epsilon)}\right)_{xx}\\ =\;& \left(V_m^{(\epsilon)}\right)_{u_m^+}\dot{u}_m^+ + \left(V_m^{(\epsilon)}\right)_{u_m^-}\dot{u}_m^- + \left(V_{m+1}^{(\epsilon)}\right)_{u_{m+1}^+}\dot{u}_{m+1}^+ + \left(V_{m+1}^{(\epsilon)}\right)_{u_{m+1}^-}\dot{u}_{m+1}^-.\end{aligned}$$

This, together with (0.1), gives

$$(\bar{v}_\epsilon)_t + f(\bar{v}_\epsilon)_x - \epsilon(\bar{v}_\epsilon)_{xx} = \bar{g}(x,t), \qquad t\in(t_p,t_{p+1}),$$

where

$$
\begin{aligned}
(3.12)\qquad & \bar{g}(x,t) = I_1 + I_2 - \epsilon u_{xx}, \\
& I_1 = -f(u)_x - f\left(V_m^{(\epsilon)}\right)_x - f\left(V_{m+1}^{(\epsilon)}\right)_x + f(\bar{v}_\epsilon)_x, \\
& I_2 = \left(V_m^{(\epsilon)}\right)_{u_m^+} \dot{u}_m^+ + \left(V_m^{(\epsilon)}\right)_{u_m^-} \dot{u}_m^- - H\left(x - X_m(t); \dot{u}_m^+, \dot{u}_m^-\right) \\
& \quad + \left(V_{m+1}^{(\epsilon)}\right)_{u_{m+1}^+} \dot{u}_{m+1}^+ + \left(V_{m+1}^{(\epsilon)}\right)_{u_{m+1}^-} \dot{u}_{m+1}^- \\
& \quad - H\left(x - X_{m+1}(t); \dot{u}_{m+1}^+, \dot{u}_{m+1}^-\right).
\end{aligned}
$$

We only need to estimate $\|I_1\|$; other terms on the right hand side of (3.12) can be handled by Lemmas 2.3, 1.1, 1.2 and 1.3. Observe that

$$
\begin{aligned}
I_1 &= f'(\bar{v}_\epsilon)\left(u_x + \left(V_m^{(\epsilon)}\right)_x + \left(V_{m+1}^{(\epsilon)}\right)_x\right) - f'(u)u_x \\
&\quad - f'\left(V_m^{(\epsilon)}\right)\left(V_m^{(\epsilon)}\right)_x - f'\left(V_{m+1}^{(\epsilon)}\right)\left(V_{m+1}^{(\epsilon)}\right)_x \\
&= f''(\xi_1)\left(\bar{v}_\epsilon - u\right)u_x + f''(\xi_2)\left(\bar{v}_\epsilon - V_m^{(\epsilon)}\right)\left(V_m^{(\epsilon)}\right)_x \\
&\quad + f''(\xi_3)\left(\bar{v}_\epsilon - V_{m+1}^{(\epsilon)}\right)\left(V_{m+1}^{(\epsilon)}\right)_x \\
&=: I_1^{(1)} + I_1^{(2)} + I_1^{(3)},
\end{aligned}
$$

where ξ_1, ξ_2, ξ_3 are some intermediate values. Similar to the proof of (3.8) and (3.9), we have

$$
\begin{aligned}
(3.13)\qquad |u_x(x \pm 0, t)| &\le |u_x(X_m \pm 0, t)| + |u_x(X_{m+1} \pm 0, t)| \\
&\quad + \sum_l |u_x(Y_l(t) \pm 0, t)| + \|u_{xx}(\bullet, t)\|,
\end{aligned}
$$

for any $(x,t) \in R \times (t_p, t_{p+1})$, where $x = Y_l(t)$, $1 \le l \le L$, denote left or right boundaries of (possible) rarefaction waves. This, together with Lemma 2.2 (more precisely (2.10)), leads to

$$
\int_{t_p+\epsilon}^{t_{p+1}-\epsilon} \|I_1^{(1)}(\bullet, t)\| dt \le C\epsilon|\log \epsilon| + C\epsilon.
$$

Observe that

$$
\begin{aligned}
(3.14)\qquad \left|I_1^{(2)}\right| &\le C|u - H_m|\left(-V_m^{(\epsilon)}\right)_x + C|V_{m+1}^{(\epsilon)} - H_{m+1}|\left(-V_m^{(\epsilon)}\right)_x \\
&=: J_1 + J_2.
\end{aligned}
$$

From (2.9) we have

$$\begin{aligned}
\|J_2(\bullet,t)\| &= C\int_{-\infty}^{\infty}|V_{m+1}^{(\epsilon)}-H_{m+1}|\left(-V_m^{(\epsilon)}\right)_x dx \\
&\le C\beta_{m+1}\int_{-\infty}^{\infty}\exp\left(-\beta_{m+1}\frac{|x-X_{m+1}(t)|}{2\epsilon}\right)\left(-V_m^{(\epsilon)}\right)_x dx \\
&= C\beta_{m+1}\left[\int_{|x-X_{m+1}(t)|\le\delta(t)/2}+\int_{|x-X_{m+1}(t)|\ge\delta(t)/2}\right] \\
&\le C\beta_{m+1}\left[\int_{|x-X_{m+1}|\le\delta/2}\left(-V_m^{(\epsilon)}\right)_x dx+\exp\left(-\beta_{m+1}\frac{\delta(t)}{4\epsilon}\right)\int_{-\infty}^{\infty}\left(-V_m^{(\epsilon)}\right)_x dx\right] \\
&\le C\beta_{m+1}\left[V\left(\frac{\delta(t)}{2\epsilon};u_m^+,u_m^-\right)-V\left(\frac{3\delta(t)}{2\epsilon};u_m^+,u_m^-\right)+\beta_m\exp\left(-\beta_{m+1}\frac{\delta(t)}{4\epsilon}\right)\right],
\end{aligned}$$

where $\beta_m = u_m^- - u_m^+$, $\beta_{m+1} = u_{m+1}^- - u_{m+1}^+$, $\delta(t) = X_{m+1}(t) - X_m(t)$. It follows from (2.9) that

$$\begin{aligned}
& V\left(\frac{\delta(t)}{2\epsilon};u_m^+,u_m^-\right)-V\left(\frac{3\delta(t)}{2\epsilon};u_m^+,u_m^-\right) \\
\le\ & V\left(\frac{\delta(t)}{2\epsilon};u_m^+,u_m^-\right)-u_m^+\le\beta_m\exp\left(-\beta_m\frac{\delta(t)}{4\epsilon}\right).
\end{aligned}$$

Consequently, we obtain

$$\|J_2(\bullet,t)\|\le C\beta_{m+1}\beta_m\left\{\exp\left(-\beta_m\frac{\delta(t)}{4\epsilon}\right)+\exp\left(-\beta_{m+1}\frac{\delta(t)}{4\epsilon}\right)\right\}. \tag{3.15}$$

Consider the following function

$$z(t)=\frac{t_{p+1}-t}{X_{m+1}(t)-X_m(t)},\qquad t\in[t_p,t_{p+1}].$$

It is obvious that $z(t)\in C[t_p,t_{p+1})$. For the first possibility, (P1), $X_{m+1}(t)-X_m(t)>0$ for all $t\in[t_p,t_{p+1}]$, which indicates that $z\in C[t_p,t_{p+1}]$. For the second possibility, (P2), we have $u_m^+(t_{p+1}) = u_{m+1}^-(t_{p+1})$. It follows from the definition (2.5)

$$\dot X_m(t_{p+1})-\dot X_{m+1}(t_{p+1})=\int_0^1 f''(\xi)\theta d\theta\left(u_m^-(t_{p+1})-u_{m+1}^+(t_{p+1})\right), \tag{3.16}$$

where ξ is an intermediate value. Since $X_{m+1}(t_{p+1}) = X_m(t_{p+1})$, we have

$$\begin{aligned}
& 0<\zeta_{m+1}^+(t_{p+1})-\zeta_m^-(t_{p+1})=t_{p+1}\Big(a(u_m^-(t_{p+1}))-a(u_{m+1}^+(t_{p+1}))\Big) \\
\le\ & C\Big(u_m^-(t_{p+1})-u_{m+1}^+(t_{p+1})\Big).
\end{aligned}$$

This and (3.16) lead to

$$\dot X_m(t_{p+1})-\dot X_{m+1}(t_{p+1})\ge C>0.$$

From this and the L'Hospital's rule, we have

$$\lim_{t\to t_{p+1}-0} z(t) = \frac{1}{\dot{X}_m(t_{p+1}) - \dot{X}_{m+1}(t_{p+1})} \le C.$$

This indicates that $z \in C[t_p, t_{p+1}]$. Therefore, for both possibilities (P1) and (P2), there exists a constant $\gamma > 0$, such that $|z(t)| \le \gamma$ for all $t \in [t_p, t_{p+1}]$. It follows from the definition of $z(t)$

$$\delta(t) = X_{m+1}(t) - X_m(t) \ge C(t_{p+1} - t), \qquad t_p \le t \le t_{p+1}.$$

From this and (3.15), we have

$$\|J_2(\bullet, t)\| \le C\beta_{m+1}\beta_m \Big\{ \exp\left(-C\beta_{m+1}(t_{p+1}-t)/\epsilon\right) + \exp\left(-C\beta_m(t_{p+1}-t)/\epsilon\right) \Big\}.$$

This yields

$$\int_{t_p}^{t_{p+1}} \|J_2(\bullet, t)\| dt \le C\epsilon.$$

The estimate for J_1 is a little simpler than that for J_2. It follows from the facts

$$|u(x,t) - H_m| \le \begin{cases} \max_{z<X_{m+1}(t)} |u_x(z \pm 0, t)||x - X_m(t)|, & \text{if} \quad x < X_{m+1}(t), \\ C, & \text{if} \quad x \ge X_{m+1}(t), \end{cases}$$

that

$$\begin{aligned} \|J_1(\bullet, t)\| &= C \int_{-\infty}^{\infty} |u - H_m| \left(-V_m^{(\epsilon)}\right)_x dx \\ \le\ & C \max_{z<X_{m+1}(t)} |u_x(z \pm 0, t)| \int_{-\infty}^{X_{m+1}} |x - X_m(t)| \left(-V_m^{(\epsilon)}\right)_x dx \\ & +C \int_{X_{m+1}}^{\infty} \left(-V_m^{(\epsilon)}\right)_x dx \\ \le\ & C \max_{z<X_{m+1}(t)} |u_x(z \pm 0, t)| \int_{-\infty}^{\infty} |x - X_m(t)| \left(H_m - \left(-V_m^{(\epsilon)}\right)\right)_x dx \\ & +C\Big(V\left(\delta(t)/\epsilon; u_m^+, u_m^-\right) - u_m^+\Big). \end{aligned}$$

Using integration by parts and Lemma 2.2, we can show that the last integral can be bounded by $C\epsilon$. Similar to the proof for J_2, we have

$$0 < V\left(\delta(t)/\epsilon; u_m^+, u_m^-\right) - u_m^+ \le \beta_m \exp\left(-C\beta_m(t_{p+1}-t)/\epsilon\right).$$

These results, together with (3.13), yield

$$\int_{t_p+\epsilon}^{t_{p+1}-\epsilon} \|J_1(\bullet, t)\| dt \le C\epsilon|\log\epsilon| + C\epsilon.$$

Therefore, for both possibilities (P1) and (P2), we have

$$\int_{t_p+\epsilon}^{t_{p+1}-\epsilon} \|I_1^{(2)}(\bullet, t)\| dt \le C|\log\epsilon| + C\epsilon.$$

In a similar manner we can show that the above estimate holds also for $I_1^{(3)}$. Hence, we have proved that in cases (P1) and (P2) the function $\bar{v}_\epsilon$ given by (3.11) satisfied the requirement (B2).

If the possibility (P3) is the case, namely there are two shock curves, $X_m(t)$ and $X_{m+1}(t)$, for $t \in (t_p, t^{(p,1)})$ and one shock, $X_m(t)$, for $t \in (t^{(p,1)}, t_{p+1})$, we construct $\bar{v}_\epsilon$ in the following way:

(3.17)
$$\bar{v}_\epsilon = \begin{cases} u(x,t) + \left(V_m^{(\epsilon)} - H_m\right) + \left(V_{m+1}^{(\epsilon)} - H_{m+1}\right), & \text{if} \quad t \in (t_p, t^{(p,1)}), \\ u(x,t) + \left(V_m^{(\epsilon)} - H_m\right), & \text{if} \quad t \in [t^{(p,1)}, t_{p+1}). \end{cases}$$

This case can be regarded as a combination of possibility (P2) analyzed above and the one shock case analyzed in the last subsection. It is then not difficult to verify that this $\bar{v}_\epsilon$ satisfies the requirements (B1)-(B4). □

3.4. More shocks. Assume that we have M shock curves $X_m(t), 1 \le m \le M$ for $t \in (t_p, t^{(p,1)})$, where $t^{(p,1)} \le t_{p+1}$. These shocks may emerge and there will have less number of shocks for $t \in (t^{(p,1)}, t_{p+1})$. Similar to the last two subsections, we let

(3.18)
$$\bar{v}_\epsilon(x,t) = u(x,t) + \sum_{m=1}^{M} \left(V^{(\epsilon)}(x - X_m(t); u_m^+, u_m^-) - H(x - X_m(t); u_m^+, u_m^-) \right),$$

for $(x,t) \in R \times (t_p, t^{(p,1)})$. We found that it is straightforward to extend the method of proof in the last subsection to show that the function $\bar{v}_\epsilon$ given by (3.18) satisfies the requirements (B1)-(B4). It is found that the following estimates are particularly useful:

$$\text{(3.19)} \quad \|\bar{v}_\epsilon(\bullet,t) - u(\bullet,t)\| \le \sum_{m=1}^{M} \|V_m^{(\epsilon)} - H_m\| \le C\epsilon;$$

$$\text{(3.20)} \quad |u_x(x \pm 0,t)| \le \sum_{m=1}^{M} |u_x(X_m(t) \pm 0,t)| + \sum_l |u_x(Y_l(t) \pm 0,t)| + \|u_{xx}(\bullet,t)\|;$$

$$\begin{aligned} \text{(3.21)} \quad & \left\| \left(\bar{v}_\epsilon(\bullet,t) - V_k^{(\epsilon)}\right) \left(V_k^{(\epsilon)}\right)_x \right\| \\ \le \quad & \left\| (u(\bullet,t) - H_k) \left(V_k^{(\epsilon)}\right)_x \right\| + \sum_{m \ne k} \left\| \left(V_m^{(\epsilon)} - H_m\right) \left(V_k^{(\epsilon)}\right)_x \right\| \\ \le \quad & \max_{X_{k-1} < z < X_{k+1}} |u_x(z \pm 0,t)| C\epsilon + C\beta_k \exp\left(-C\beta_k(t_{p+1} - t)/\epsilon\right) \\ & \quad + \sum_{m \ne k} \beta_m \beta_k \Big\{ \exp\left(-C\beta_m(t_{p+1} - t)/\epsilon\right) + \exp\left(-C\beta_k(t_{p+1} - t)/\epsilon\right) \Big\}, \end{aligned}$$

for $1 \le k \le M$, where $\beta_m = u_m^- - u_m^+, \beta_k = u_k^- - u_k^+$. Here we have set $X_0(t) = -\infty, X_{M+1}(t) = \infty$.

For $t \in [t^{(p,1)}, t_{p+1})$, the number of shock curves is less than M and we may use mathematical induction (based on the number of shock curves) to show that an appropriate $\bar{v}_\epsilon$ can be found (cf. [20]). Therefore, Theorem 3.1 is proved. □

Remark 3.4. Under the present framework, the factor $O(|\log\epsilon|)$ cannot be removed from our results. We found the factor is due to the estimates for u_x and u_{xx}, see Lemmas 1.1 to 1.3. In some special cases, these estimates can be improved so that optimal convergence rates can be obtained. $\square$

4. Remarks on sharp estimate

In this section, we consider the case when the initial function u_0 satisfies the following conditions:

- (C1): $u_0(x)$ is bounded and piecewise C^2 smooth with finite number of discontinuous points $\gamma_i, 1 \le i \le I$; $\dot u_0(\gamma_i \pm 0)$ exist and finite; $u_0(\gamma_i - 0) > u_0(\gamma_i + 0)$;
- (C2): $\ddot u_0 \in L^1(R)$;
- (C3):
$$\lim_{|x|\to\infty} \dot a(u_0(x)) = 0;$$
- (C4): $a(u_0)$ has no decreasing inflection points, but may have a finite number of increasing inflection points.

The assumption (C4) implies that there is no original shock generated after $t = 0$. The assumption (C1) implies that there is no central rarefaction waves formed at $t = 0$. Therefore, only discontinuities are initial shocks and their interactions. Assume $x = X(t)$ be a shock curve. Following Lemma 1.1, we can show that

$$\int_0^T |u_x(X(t) \pm 0, t)| dt \le C, \tag{4.1}$$

$$\int_0^T \frac{1}{1 + \dot a(u_0(\zeta^\pm))t} dt \le C. \tag{4.2}$$

We now use notations introduced in Section 1. The interval $[0, T]$ can be divided into several subintervals: $0 = t^{(0)} < t^{(1)} < \cdots < t^{(N)} = T$. In each interval $[t^{(n)}, t^{(n+1)})$, there are finite number of smooth curves $x = X_m^{(n)}(t)$ satisfying

$$X_m^{(n)}(t) < X_{m+1}^{(n)}(t), \qquad m = 1, \cdots, M_n - 1.$$

From each point on $x = X_m^{(n)}(t)$, we can trace two characteristic lines back to $t = 0$: $X_m^{(n)}(t) = \zeta_m^\pm + a(u_0(\zeta_m^\pm))t$. Using (1.29) and the assumption (C4) gives

$$\frac{1}{1 + \dot a(u_0(\zeta))t} \le 1 + \frac{1}{1 + \dot a(u_0(\zeta_m^+))t} + \frac{1}{1 + \dot a(u_0(\zeta_{m+1}^-))t},$$

for $\zeta_m^+ \le \zeta \le \zeta_{m+1}^-$ with $m = 0, \cdots, M_n$, where we have set $\zeta_0^+ = -\infty, \zeta_{M_n+1}^- = \infty$. It follows from (4.2) that the integral for the right hand side functions over $[0, T]$ can be bounded by a constant C. Following the proof for Lemma 1.3 we can show that

$$\int_0^T \|u_{xx}(\bullet, t)\| dt \le C. \tag{4.3}$$

Using (4.1)–(4.3) and repeating the procedures in the last section yield the following theorem.

Theorem 4.1. *Let the flux f be strictly convex. Assume that the initial data u_0 satisfies the requirements* (C1)-(C4) *stated above. If v_ϵ is the solution to* (0.3) *and* (0.4) *and u is the solution to* (0.1) *and* (0.2), *then the following error estimate holds:*

$$\sup_{0\le t\le T} \|v_\epsilon(\bullet,t) - u(\bullet,t)\| \le C(T)\epsilon, \tag{4.4}$$

where $C(T)$ is a constant independent of ϵ.

It is easy to show that $O(\epsilon)$ convergence rate is the best possible one for viscosity methods. The above result indicates that if the solution to (0.1) and (0.2) includes neither central rarefaction waves nor spontaneous shocks, then viscosity methods yield the optimal convergence rate.

5. Results on nonincreasing initial data

Throughout this section, c denotes a positive constant independent of t and ϵ, but with different values at different places. We consider a special case when the initial function satisfies the following conditions:

- (D1): $u_0(x)$ is bounded, nonincreasing and piecewise C^2 smooth with a finite number of discontinuous points $\gamma_1 < \cdots < \gamma_I$; $\dot u_0(\gamma_i \pm 0)$ exist and bounded;
- (D2): $\ddot u_0(x)$ does not change sign when $|x| \ge \Lambda$, with Λ a positive constant;
- (D3):
$$\lim_{|x|\to\infty} \dot a(u_0(x)) = 0;$$
- (D4): $\ddot a(u_0)$ has no inflection points.

Since $\ddot a(u_0)$ does not change sign in each of its smooth domain, we obtain from (D1) and (D3) that $\ddot a(u_0) \in L^1(R)$. The assumption (D1) also indicates that $\dot u_0(x\pm 0)$ is bounded for $|x| \le \Lambda$. Observe that

$$\ddot a(u_0) = a''(u_0)(\dot u_0)^2 + a'(u_0)\ddot u_0,$$

where $a'(u_0) = f''(u_0) \ge \alpha > 0$. The above results yield that $\ddot u_0 \in L^1(-\Lambda,\Lambda)$. This, together with (D2) and (D3), imply that $\ddot u_0 \in L^1(R)$.

The assumption (D1) also indicates that $u_0(\gamma_i - 0) > u_0(\gamma_i + 0)$. Since u_0 is nonincreasing, it is known that there exists a time level T^* such that there is only one shock after $t = T^*$. The solution is smooth away from this shock for $t > T^*$. From each point on the shock curve, we can trace two characteristic lines $X(t) = \zeta^\pm + a(u_0(\zeta^\pm))$ back to $t = 0$. It is obvious that

$$\zeta^-(t) < \gamma_1, \qquad \zeta^+(t) > \gamma_I, \qquad T^* \le t < \infty. \tag{5.1}$$

It follows from (D1) that

$$u_0(\zeta^-) \ge u_0(\gamma_1 - 0), \qquad u_0(\zeta^+) \le u_0(\gamma_I + 0), \qquad T^* \le t < \infty,$$

which leads to

$$|u_0(\zeta^-) - u_0(\zeta^+)| \ge u_0(\gamma_1 - 0) - u_0(\gamma_I + 0) = \text{constant} > 0, \tag{5.2}$$

for $T^* \le t < \infty$. From (1.2), (1.4) and (5.2), we have

$$|\dot{\zeta}^-| \ge \frac{c}{1+\dot{a}(u_0(\zeta^-))t} \ge c > 0, \tag{5.3}$$

where we have used the fact $\dot{a}(u_0) \le 0$ (due to (D1)). Since $\dot{\zeta}^- < 0$, we obtain that

$$\lim_{t\to\infty} \zeta^-(t) = -\infty. \tag{5.4}$$

Moreover, from (D1), (D3) and (D4), we have

$$\ddot{a}(u_0(\zeta)) \le 0, \quad \zeta \in (-\infty, \gamma_1), \quad \text{and} \quad \ddot{a}(u_0(\zeta)) \ge 0, \quad \zeta \in (\gamma_I, \infty). \tag{5.5}$$

We need to prove the following results for some finite values $T_1, T_2 > T^*$:

$$\int_{T_1}^{\infty} |u_x(X(t)-0,t)|dt \le c, \qquad \int_{T_2}^{\infty} |u_x(X(t)+0,t)|dt \le c; \tag{5.6}$$

$$\int_{T_1}^{\infty}\int_{-\infty}^{\zeta^-(t)} \frac{|\dot{u}_0\ddot{a}(u_0)|t}{(1+\dot{a}(u_0)t)^2} d\zeta dt \le c, \qquad \int_{T_2}^{\infty}\int_{\zeta^+(t)}^{\infty} \frac{|\dot{u}_0\ddot{a}(u_0)|t}{(1+\dot{a}(u_0)t)^2} d\zeta dt \le c; \tag{5.7}$$

$$\int_{T_1}^{\infty}\int_{-\infty}^{\zeta^-(t)} \frac{|\ddot{u}_0|}{1+\dot{a}(u_0)t} d\zeta dt \le c, \qquad \int_{T_2}^{\infty}\int_{\zeta^+(t)}^{\infty} \frac{|\ddot{u}_0|}{1+\dot{a}(u_0)t} d\zeta dt \le c. \tag{5.8}$$

It follows from (5.4) that there exists a time level $t = T_1 > T^*$ such that $\zeta^-(t) \le -\Lambda$ for $t \ge T_1$. From (1.14) and (5.3), we obtain

$$\begin{aligned} \int_{T_1}^{\infty} |u_x(X(t)-0,t)|dt &\le c\int_{T_1}^{\infty} |\dot{\zeta}^-(t)\dot{u}_0(\zeta^-)|dt \\ &\le c\Big(u_0(-\infty) - u_0(\zeta^-(T_1))\Big) \le c, \end{aligned} \tag{5.9}$$

where in the second inequality we have used the facts that $\dot{\zeta}^- \le 0$ and $\dot{u}_0(\zeta) \le 0$. Hence the first inequality in (5.6) is proved. Further, using (5.5) gives

$$\int_{-\infty}^{\zeta^-(t)} \frac{|\ddot{a}(u_0)|t}{(1+\dot{a}(u_0)t)^2} d\zeta = \frac{1}{1+\dot{a}(u_0(\zeta^-))t} - 1 \le \frac{1}{1+\dot{a}(u_0(\zeta^-))t}.$$

It follows from (D2) and (D3) that

$$|\dot{u}_0(\zeta)| \le |\dot{u}_0(\zeta^-)| \quad \text{for} \quad \zeta \in (-\infty, \zeta^-).$$

The above two inequalities lead to

$$\int_{-\infty}^{\zeta^-(t)} \frac{|\dot{u}_0\ddot{a}(u_0)|t}{(1+\dot{a}(u_0)t)^2} d\zeta \le \frac{|\dot{u}_0(\zeta^-)|}{1+\dot{a}(u_0(\zeta^-))t} = |u_x(X(t)-0,t)|.$$

This and (5.9) yield the first inequality in (5.7). Using (D2) and (D3) gives

$$\int_{-\infty}^{\zeta^-(t)} |\ddot{u}_0| d\zeta = -\dot{u}_0(\zeta^-), \qquad \text{for} \quad t \ge T_1.$$

This and (5.5) yield

$$\int_{-\infty}^{\zeta^-(t)} \frac{|\ddot{u}_0|}{1+\dot{a}(u_0)t} d\zeta \leq \frac{|\dot{u}_0(\zeta^-)|}{1+\dot{a}(u_0(\zeta^-))t} = |u_x(X(t)-0,t)| \,.$$

Consequently, the first inequality in (5.8) is proved. The second inequalities in (5.6)–(5.8) can be proved in a similar way.

Let $T = \max\{T_1, T_2\}$. Using (5.6)–(5.8) and following the proofs for Lemmas 1.1 and 1.3, we can show that

$$\int_T^\infty |u_x(X(t)\pm 0,t)|dt \leq c, \qquad \int_T^\infty \|u_{xx}(\bullet,t)\|dt \leq c. \tag{5.10}$$

Repeating the proof in Section 3.1, we can obtain

$$\|v_\epsilon(\bullet,t) - u(\bullet,t)\| \leq \|v_\epsilon(\bullet,T) - u(\bullet,T)\| + c\epsilon, \qquad t \in [T,\infty), \tag{5.11}$$

where c is independent of t and ϵ. Since the conditions (C1)–(C4) include (D1)–(D4), Theorem 4.1 implies that

$$\|v_\epsilon(\bullet,t) - u(\bullet,t)\| \leq C(T)\epsilon, \qquad t \in [0,T]. \tag{5.12}$$

Combining (5.11) and (5.12) gives the following theorem.

Theorem 5.1. *Let the flux f be strictly convex. Assume that the initial data u_0 satisfies the requirements* (D1)-(D4) *stated above. If v_ϵ is the solution to* (0.3) *and* (0.4) *and u is the solution to* (0.1) *and* (0.2)*, then the following error estimate holds:*

$$\sup_{0\leq t<\infty} \|v_\epsilon(\bullet,t) - u(\bullet,t)\| \leq c\epsilon, \tag{5.13}$$

where c is a constant independent of ϵ.

6. Generalizations

In this section, we will briefly mention two possible generalizations. The first one is a straightforward extension. The ideas in this paper can be applied to a more general viscosity equation of the form

$$(v_\epsilon)_t + f(v_\epsilon)_x = \epsilon(B(v_\epsilon)v_\epsilon)_{xx}, \tag{6.1}$$

where $B(u) > 0$ for all $u \in R$ and $B \in C^1(R)$. We only need to change Φ defined by (2.7) to

$$\Phi(u; u_m^+, u_m^-) = [f(u) - f(u_m^-) - \dot{X}_m(u - u_m^-)]/B(u). \tag{6.2}$$

With this definition of Φ, it can be shown that the corresponding $V_m^{(\epsilon)} := V^{(\epsilon)}(x - X_m(t); u_m^+, u_m^-)$ satisfies

$$\left(V_m^{(\epsilon)}\right)_t + f\left(V_m^{(\epsilon)}\right)_x - \epsilon\left(B\left(V_m^{(\epsilon)}\right)V_m^{(\epsilon)}\right)_{xx} = \left(V_m^{(\epsilon)}\right)_{u_m^+}\dot{u}_m^+ + \left(V_m^{(\epsilon)}\right)_{u_m^-}\dot{u}_m^-. \tag{6.3}$$

Furthermore, the stability lemma, i.e. Lemma 2.1, can be modified to give an inequality similar to (2.2). It is then straightforward to extend our results and proofs in Sections 3 to 5 to the solution of (6.1).

The second extension is concerned with finite difference approximations for the hyperbolic conservation laws (0.1). We only point out some possible results, and their proofs could be very complicated. As mentioned in the first section that viscosity methods and monotone difference schemes share some common properties when they are applied to approximate solutions to hyperbolic conservation laws. It is expected that Theorem 3.1 can be extended to some monotone difference schemes. More precisely, assuming that $u_{\Delta x}(x,t)$ are solutions of monotone difference schemes consistent with the equation (0.1) and the initial condition (0.2), and assuming that u_0 satisfies the conditions stated in Theorem 3.1, we expect the following estimate:

$$\sup_{0\le t\le T} \|u_{\Delta x}(\bullet,t)-u(\bullet,t)\| \le C(T)\Big(\Delta x|\log \Delta x| + \Delta x\Big),$$

where Δx is the discrete meshsize in space and $C(T)$ is a constant independent of Δx. Similar extensions for Theorems 4.1 and 5.1 are also expected to be true. For decreasing piecewise constant initial data, a rigorous analysis, based on Jennings traveling wave results [8], shows a uniform error bound similar to that in Theorem 5.1 holds for strictly monotone difference schemes [20]. However, for piecewise smooth solutions, this extension could be quite difficult.

Appendix A.

Proof of Lemma 2.2. The entropy solution u of (0.1) and (0.2) satisfies $\|u\| \le \|u_0\|_{L^\infty(R)}$. We let

$$\beta = \max_{|u|\le\|u_0\|_{L^\infty(R)}} f''(u). \tag{A.1}$$

From (0.5) and (A.1) we have $\alpha \le f'' \le \beta$. We observe that

$$\begin{aligned}
&\Phi(u;u_m^+,u_m^-) \\
&= \left[\int_0^1 f'(\theta u+(1-\theta)u_m^-)d\theta - \int_0^1 f'(\theta u_m^+ + (1-\theta)u_m^-)d\theta\right](u-u_m^-) \\
&= \int_0^1 f''(u^*)\theta d\theta (u-u_m^+)(u-u_m^-),
\end{aligned}\tag{A.2}$$

where u^* is some intermediate value. Since $f''\ge\alpha>0$, it can be obtained from (2.6), (2.7) and (A.2) that $V(-\infty;u_m^+,u_m^-)=u_m^-, V(\infty;u_m^+,u_m^-)=u_m^+$. It is easy to see that $V_\zeta=\Phi(V;u_m^+,u_m^-)\le 0$, which indicates that V is a decreasing function with respect to ζ. From (A.2), we have

$$\frac{1}{2}\beta(u-u_m^+)(u-u_m^-) \le \Phi(V;u_m^+,u_m^-) \le \frac{1}{2}\alpha(u-u_m^+)(u-u_m^-). \tag{A.3}$$

This together with (2.6) and (2.7) imply that for $\zeta\le 0$ (in this case $V\ge\frac{1}{2}(u_m^+ + u_m^-)$)

$$\frac{2}{\alpha}\frac{1}{u_m^- - u_m^+}\ln\left[\frac{u_m^- - V}{V-u_m^+}\right] \le \zeta(V,u_m^+,u_m^-) \le \frac{2}{\beta}\frac{1}{u_m^- - u_m^+}\ln\left[\frac{u_m^- - V}{V-u_m^+}\right].$$

Therefore we have

$$\tilde V(\alpha\zeta/2; u_m^+, u_m^-) \le V \le \tilde V(\beta\zeta/2; u_m^+, u_m^-), \qquad \zeta \le 0,$$

where

$$\tilde V(\zeta; u_m^+, u_m^-) = \frac{u_m^- + u_m^+ \exp\{\zeta(u_m^- - u_m^+)\}}{1 + \exp\{\zeta(u_m^- - u_m^+)\}}.$$

It follows from the above results that, for $\zeta \le 0$,

$$|V(\zeta; u_m^+, u_m^-) - u_m^-| \le (u_m^- - u_m^+) \exp\{\alpha\zeta(u_m^- - u_m^+)/2\}. \tag{A.4}$$

Similarly, we can show that

$$\tilde V(\beta\zeta/2; u_m^+, u_m^-) \le V \le \tilde V(\alpha\zeta/2; u_m^+, u_m^-), \qquad \zeta \ge 0,$$

which gives that, for $\zeta \ge 0$,

$$|V(\zeta; u_m^+, u_m^-) - u_m^+| \le (u_m^- - u_m^+) \exp\{-\alpha\zeta(u_m^- - u_m^+)/2\}. \tag{A.5}$$

It follows from (A.4) and (A.5) that

$$\|V(\bullet; u_m^+, u_m^-) - H(\bullet; u_m^+, u_m^-)\| \le \frac{4}{\alpha}. \tag{A.6}$$

Notice that $H(x - X_m; u_m^+, u_m^-) = H((x - X_m)/\epsilon; u_m^+, u_m^-)$. Using integration by substitution and (A.6) gives the inequality (2.10). $\square$

Proof of Lemma 2.3. Differentiating (2.6) with respect to the parameter u_m^+ gives

$$0 = \frac{V_{u_m^+}}{\Phi(V; u_m^+, u_m^-)} - \frac{1}{2}\frac{1}{\Phi\left(\frac{1}{2}(u_m^+ + u_m^-); u_m^+, u_m^-\right)} - \int_{\frac{1}{2}(u_m^+ + u_m^-)}^{V} \frac{(\Phi(u; u_m^+, u_m^-))_{u_m^+}}{[\Phi(u; u_m^+, u_m^-)]^2} du.$$

We obtain from the above equation that

$$\begin{aligned} V_{u_m^+} - 1 = &-\frac{1}{2}\frac{\Phi(V; u_m^+, u_m^-)}{\Phi\left(\frac{1}{2}(u_m^+ + u_m^-); u_m^+, u_m^-\right)} \\ &+\Phi(V; u_m^+, u_m^-) \int_{\frac{1}{2}(u_m^+ + u_m^-)}^{V} \frac{(\Phi(u; u_m^+, u_m^-))_{u_m^+} + (\Phi(u; u_m^+, u_m^-))_u}{[\Phi(u; u_m^+, u_m^-)]^2} du. \end{aligned} \tag{A.7}$$

Direction calculation using (2.7) and (2.5) gives

$$\begin{aligned} &\Phi_{u_m^+} + \Phi_u \\ = \;& -(\dot X_m)_{u_m^+}(u - u_m^-) + f'(u) - \dot X_m \\ = \;& -\frac{f'(u_m^+)(u_m^+ - u_m^-) - (f(u_m^+) - f(u_m^-))}{(u_m^+ - u_m^-)^2}(u - u_m^-) + f'(u) - \frac{f(u_m^+) - f(u_m^-)}{u_m^+ - u_m^-} \\ = \;& -\frac{f'(u_m^+)(u_m^+ - u_m^-) - (f(u_m^+) - f(u_m^-))}{(u_m^+ - u_m^-)^2}(u - u_m^+) + f'(u) - f'(u_m^+) \\ = \;& -\frac{f''(\zeta_1)}{2}(u - u_m^+) + f''(\zeta_2)(u - u_m^+), \end{aligned}$$

where $u_m^+ < \zeta_1, \zeta_2 < u_m^-$. Therefore, we have obtained that $|\Phi_{u_m^+} + \Phi_u| \le \beta |u - u_m^+|$, where β is given in (A.1). For $\zeta \ge 0$, we have $u_m^+ \le V \le \frac{1}{2}(u_m^+ + u_m^-)$. From (A.7) and (A.3), we have

$$\begin{aligned} &\left| V_{u_m^+} - 1 \right| \\ \le\ & C \frac{|V - u_m^+|}{u_m^- - u_m^+} + C|V - u_m^+|(u_m^- - u_m^+) \int_{\frac{1}{2}(u_m^+ + u_m^-)}^{V} \frac{1}{|u - u_m^+|(u_m^- - u_m^+)^2} du \\ \le\ & C \frac{|V - u_m^+|}{u_m^- - u_m^+} \left(1 + \ln \left(\frac{\frac{1}{2}(u_m^- - u_m^+)}{V - u_m^+} \right) \right), \end{aligned}$$

where C is some constant depending only on α and β. Since $x(1 + \ln(A/x))$ is an increasing function for $0 < x \le A$, we obtain from (2.9) that

$$\begin{aligned} &\int_0^\infty \left| V_{u_m^+}(\zeta; u_m^+, u_m^-) - 1 \right| d\zeta \\ \le\ & C \int_0^\infty e^{-(u_m^- - u_m^+)\zeta/2} \left[1 + \ln \left(\frac{1}{2} e^{(u_m^- - u_m^+)\zeta/2} \right) \right] d\zeta \le \frac{C}{u_m^- - u_m^+}. \end{aligned}$$

In a similar way we obtain that

$$V_{u_m^-} = \frac{1}{2} \frac{\Phi(V; u_m^+, u_m^-)}{\Phi\left(\frac{1}{2}(u_m^+ + u_m^-); u_m^+, u_m^-\right)} + \Phi(V; u_m^+, u_m^-) \int_{\frac{1}{2}(u_m^+ + u_m^-)}^{V} \frac{(\Phi(u; u_m^+, u_m^-))_{u_m^-}}{[\Phi(u; u_m^+, u_m^-)]^2} du,$$

which also gives

$$\int_0^\infty \left| V_{u_m^-}(\zeta; u_m^+, u_m^-) \right| d\zeta \le \frac{C}{u_m^- - u_m^+}.$$

Combining the above two inequalities yields

$$\begin{aligned} &\int_0^\infty \left| \left(V^{(\epsilon)}\right)_{u_m^+} (x; u_m^+, u_m^-) \dot{u}_m^+ + \left(V^{(\epsilon)}\right)_{u_m^-} (x; u_m^+, u_m^-) \dot{u}_m^- - H\left(x; \dot{u}_m^+, \dot{u}_m^-\right) \right| dx \\ &= \epsilon \int_0^\infty \left| V_{u_m^+}(\zeta; u_m^+, u_m^-) \dot{u}_m^+ + V_{u_m^-}(\zeta; u_m^+, u_m^-) \dot{u}_m^- - H\left(\zeta; \dot{u}_m^+, \dot{u}_m^-\right) \right| d\zeta \\ &\le C\epsilon \frac{|\dot{u}_m^+| + |\dot{u}_m^-|}{u_m^- - u_m^+}, \end{aligned}$$

where we have used the change of variable $\zeta = x/\epsilon$. It can be verified that

$$|\dot{u}_m^\pm| = \left| \left(\dot{X}_m - a(u_m^\pm) \right) u_x(X_m \pm 0, t) \right| \le C|u_m^+ - u_m^-||u_x(X_m \pm 0, t)|.$$

The above results lead to

$$\begin{aligned} &\int_0^\infty \left| \left(V^{(\epsilon)}\right)_{u_m^+} (x; u_m^+, u_m^-) \dot{u}_m^+ + \left(V^{(\epsilon)}\right)_{u_m^-} (x; u_m^+, u_m^-) \dot{u}_m^- - H\left(x; \dot{u}_m^+, \dot{u}_m^-\right) \right| dx \\ &\le C\epsilon \Big(|u_x(X_m + 0, t)| + |u_x(X_m - 0, t)| \Big). \end{aligned}$$

Similar estimate holds for the integral with same integrand over $(-\infty, 0)$. This completes the proof of the lemma. $\square$

Acknowledgement. We thank the referees for helpful comments and suggestions.

References

1. N. S. Bakhvalov, *Estimation of the error of numerical integration of a first-order quasilinear equation*, Zh. Vychisl. Mat. i Mat. Fiz., 1 (1961), pp. 771-783; English transl. in USSR Comput. Math. and Math. Phys., 1 (1962), pp. 926-938.
2. C. M. Dafermos, *Generalized characteristics and the structure of solutions of hyperbolic conservation laws*, Indiana Univ. Math. J., 26 (1977), pp. 1097-1119.
3. H. Fan, *Existence of discrete traveling waves and error estimates for Godunov schemes of conservation laws*, Preprint (1996).
4. J. Goodman and Z. Xin, *Viscous limits for piecewise smooth solutions to systems of conservation laws*, Arch. Rational Mech. Anal., 121 (1992), pp. 235-265.
5. E. Harabetian, *Rarefactions and large time behavior for parabolic equations and monotone schemes*, Comm. Math. Phys., 114 (1988), pp. 527-536.
6. A. Harten, *The artificial compression method for computation of shocks and contact discontinuities*, Comm. Pure Appl. Math., 30 (1977), pp. 611-638.
7. A. Harten, J. M. Hyman and P. D. Lax, *On finite difference approximations and entropy conditions for shocks*, Comm. Pure Appl. Math., 29 (1976), pp. 297–322.
8. G. Jennings, *Discrete shocks*, Comm. Pure Appl. Math. 27 (1974), pp. 25–37.
9. H-O Kreiss and J. Lorenz, Initial-Boundary Value Problems and the Navier-Stokes Equations, Academic Press, 1989.
10. N. N. Kuznetsov, *Accuracy of some approximate methods for computing the weak solutions of a first–order quasi–linear equation*, USSR Comput. Math. and Math. Phys., 16 (1976), pp. 105–119.
11. P. D. Lax, *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math., 10 (1957), pp. 537-566.
12. J. Liu and Z. Xin, L^1 *-stability of stationary discrete shocks*, Math. Comp., 60 (1993), pp. 233-244.
13. R. J. LeVeque, Numerical Methods for Conservation Laws, Birkhauser, 2nd ed., 1992.
14. B. J. Lucier, *Error bounds for the methods of Glimm, Godunov and LeVeque*, SIAM J. Numer. Anal., 22 (1985), pp. 1074-1081.
15. O. A. Oleinik, *Discontinuous solutions of non-linear differential equations*, Amer. Math. Soc. Transl., 26 (1963), pp. 95-172.
16. J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, 1983.
17. D. G. Schaeffer, *A regularity theorem for conservation laws*, Adv. in Math., 11 (1973), pp. 368-386.
18. E. Tadmor and T. Tassa, *On the piecewise smoothness of entropy solutions to scalar conservation laws*, Commun. in PDEs, 18 (1993), pp. 1631-1652.
19. T. Tang and Z.-H. Teng, *The sharpness of Kuznetsov's* $O(\sqrt{\Delta x})$ L^1*-error estimate for monotone difference schemes*. Math. Comp. (1995), pp.581-589.
20. Z.-H. Teng and P. W. Zhang, *Optimal* L^1*-rate of convergence for viscosity method and monotone scheme to piecewise constant solutions with shocks*, 1994. To appear in SIAM J. Numer. Anal.

Tao Tang, Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6.
E-mail address: ttang@sfu.ca

Zhen–huan Teng, Department of Mathematics, Peking University, Beijing 100871, China.
E-mail address: tengzh@sxx0.math.pku.edu.cn