HIGHER ORDER DISCRETISATION OF INITIAL–BOUNDARY VALUE PROBLEMS FOR MIXED SYSTEMS

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Abstract. An initial-boundary value problem for a system of nonlinear partial differential equations which consists of a hyperbolic and a parabolic part is taken into consideration. Spacial derivatives are discretised by third order consistent difference operators which are constructed such that a summation–by–parts formula holds. Therefore, the space discretisation is energy bounded and algebraically stable implicit Runge-Kutta methods can be applied to integrate in time. Boundary layers arising from the artificial boundary conditions are analysed and nonlinear convergence is proved.

Key words: Higher order difference method, initial-boundary value problem, boundary layer, nonlinear hyperbolic-parabolic system, local stability, convergence.
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1. Introduction

In the field of partial differential equations, there are still many unsolved problems, like existence and uniqueness of solutions of nonlinear hyperbolic and mixed hyperbolic-parabolic systems. For numerical analysis, convergence of finite difference approximations to such generally unknown solutions plays an important role.

A whole theory has been developed during the past twenty years to prove convergence of finite difference approximations to smooth solutions of nonlinear hyperbolic and mixed systems. Initially, this technique was introduced by Strang [12] to treat pure Cauchy problems for hyperbolic systems. It was further developed to treat initial-boundary value problems for hyperbolic equations by Michelson [9] on one hand and afterwards to treat Cauchy problems for systems of mixed hyperbolic-parabolic type by Schroll [11] on the other hand. Recently, the authors proved convergence of a class of difference approximations to initial-boundary value problems for hyperbolic-parabolic systems, cf. [3]. The idea in this theory is to construct an approximation to the numerical solution, the so-called pilot function, which is highly consistent with the scheme. Convergence follows then by proving stability of a linearisation of the scheme. This is done in analogy to Lax's equivalence theorem for linear problems, which states that 'stability is equivalent to convergence' under the condition that some consistency assumption holds.

The schemes treated in [3] were constructed such that no artificial boundary conditions were necessary. For this reason they are only first order accurate. In the present paper the proof is generalised to higher order approximations.

First, we have to define artificial boundary conditions. They are chosen in such a way that a summation by parts formula holds, cf. [1]. Summation by parts is an important tool to prove stability by means of the energy method. In a further step, we have to construct the pilot function. Since the boundary-scheme is not the same as the one applied in the interior of the domain, boundary layers will occur, which have to be approximated as well. The boundary layer terms depend not only on the grid parameter but on its reciprocal as well and they thus have to be analysed carefully to insure that they are uniformly bounded as the mesh size shrinks to zero. Convergence finally follows from consistency of the pilot function together with the stability estimate for some linearisation of the scheme and some smoothness assumption on the nonlinear scheme.

2. Preliminaries and an Outline of the Convergence Theory

The mixed systems under consideration shall be written in the following block–form

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_{xx} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{1}$$

The coefficients are smooth functions depending on the unknown $u = (v, w)^T \in \mathcal{R}^n$ and $(t, x) \in \Omega$,

$$\Omega := [0, T] \times [0, 1].$$

We shall consider some fixed finite time T such that a classical solution for the initial–boundary value problem exists.

$$A, B \in C^\infty((\mathcal{R}^n \times \Omega), \mathcal{R}^{n\times n}), \quad C \in C^\infty((\mathcal{R}^n \times \Omega), \mathcal{R}^n). \tag{2}$$

The system for $v \in \mathcal{R}^m$, $m \le n$ is assumed to be strongly parabolic in the sense that

$$B_{11}(u, t, x) \in C^\infty((\mathcal{R}^n \times \Omega), \mathcal{R}^{m\times m}), \quad B_{11} + B_{11}^T \ge 2\beta I > 0. \tag{3}$$

Furthermore, we assume that all eigenvalues of A_{22} are real and distinct, i.e. the system for w is strictly hyperbolic. In fact, we will assume that A_{22} is diagonal

$$A_{22} = \Lambda = \mathrm{diag}(\lambda_i, \ i = 1, 2, \ldots, n - m). \tag{4}$$

The system (1) shall be augmented with initial and boundary data

$$\begin{aligned} u(0, x) = z(x), \quad x \in [0, 1], \quad z \in C^\infty([0, 1], \mathcal{R}^n) \\ v(t, 0) = f_0(t), \quad v(t, 1) = f_1(t), \quad t \in [0, T], \quad f_i \in C^\infty([0, T], \mathcal{R}^m). \end{aligned} \tag{5}$$

For the hyperbolic component one has to prescribe the ingoing characteristic variables at the boundaries. For simplicity, we assume there is a fixed number say $q \le n - m$ characteristics travelling to the right. They will be denoted by $w^+ \in \mathcal{R}^q$. Then, we have

$$w = \begin{pmatrix} w^+ \\ w^- \end{pmatrix} \in \mathcal{R}^{n-m}.$$

The corresponding characteristic speeds are the positive and negative eigenvalues of A_{22}

$$A_{22} = \Lambda = \begin{pmatrix} \mathrm{diag}(\lambda_i > 0) & 0 \\ 0 & \mathrm{diag}(\lambda_i < 0) \end{pmatrix}.$$

At the left boundary, we need to know w^+

$$w^+(t, 0) = g_0(t), \quad g_0 \in C^\infty([0, T], \mathcal{R}^q) \tag{6}$$

and at the right boundary w^- must be given

$$w^-(t,1) = g_1(t), \quad g_1 \in C^\infty([0,T], \mathcal{R}^{n-m-q}). \tag{7}$$

For compatible data, i.e. when f_i, g_i and z vanish in the corners $(0,0)$ and $(0,1)$, one can show (cf. [3] and [7] Chap. 7) that locally in time a smooth solution exists. More precisely:

Assumption 2.1. *There is a time $T > 0$ such that the initial–boundary value problem (1) – (7) has a unique solution $\tilde{u} \in C^\infty(\Omega, \mathcal{R}^n)$.*

The purpose of the present paper is to construct a higher order finite difference method for the IBVP (1) – (7) and to prove convergence of the approximation computed by that method towards $\tilde{u}$ as the mesh size shrinks to zero.

A general concept to prove such convergence results was introduced in [11] and [3]. Let us briefly recall the main steps.

The difference method can be written as a root equation

$$\Phi_h(u) = 0 \tag{8}$$

where the discretisation operator Φ_h acts on grid functions defined on a regular grid in time and space

$$\begin{aligned} \Omega_h &:= \Omega_{\Delta t} \times \Omega_{\Delta x} \subset \Omega, \\ \Omega_{\Delta t} &:= \{t_k = k\Delta t,\ k = 0, 1, \ldots K\}, \\ \Omega_{\Delta x} &:= \{x_j = j\Delta x,\ j = 0, 1, \ldots J\}. \end{aligned}$$

Choosing $J, K \in \mathcal{N}$, the grid parameters are

$$h = \begin{pmatrix} \Delta t \\ \Delta x \end{pmatrix} = \begin{pmatrix} T/K \\ 1/J \end{pmatrix}.$$

Due to consistency reasons (see [11]), the step sizes have to be related

$$\Delta t = \mu \Delta x,$$

where μ is arbitrarily but fixed.

The concrete discretisation will be developed in the next section. The goal is to show that the nonlinear system (8) has a unique solution U and to estimate the deviation

$$\|U - \tilde{u}|_h\| = \mathcal{O}(\Delta x^p), \quad \Delta x \to 0.$$

This can be achieved by the following three steps (cf. [11] and [3]):

- Verify that $D\Phi_h$ is Lipschitz continuous near $\tilde{u}$
$$\|D\Phi_h(\tilde{u}|_h) - D\Phi_h(u)\| \le Lip_{\Delta x} \|\tilde{u}|_h - u\|.$$
- Construct an approximate solution
$$u^{pi} = \tilde{u} + \Delta x u^{(1)} + \Delta x^2 u^{(2)} + \ldots$$
which is high order consistent with the scheme
$$\|\Phi_h(u^{pi}|_h)\| = o(Lip_{\Delta x}^{-1}), \quad \Delta x \to 0.$$
- Show that the scheme linearised at u^{pi} is stable, i.e. $D\Phi_h(u^{pi}|_h)$ is regular and
$$\|D\Phi_h^{-1}(u^{pi}|_h)\| \le L. \tag{9}$$

When these requirements can be fulfilled, then it follows by the theory of López-Marcos and Sanz-Serna [8]

i) There is a constant $S > L$ and a stability radius $R_{\Delta x} = (L^{-1} - S^{-1})/Lip_{\Delta x}$ such that the nonlinear scheme is locally stable

$$\|u - v\| \leq S\|\Phi_h(u) - \Phi_h(v)\|, \quad u, v \in \mathcal{B}(u^{pi}|_h, R_{\Delta x}).$$

Here $\mathcal{B}(u, R)$ denotes the open ball of radius R, centered at u. Furthermore, we have

ii) For small h a unique solution of the scheme (8) $U \in \mathcal{B}(u^{pi}|_h, R_{\Delta x})$ exists.

An obvious consequence is that U converges to u^{pi} at high order

$$\|U - u^{pi}|_h\| \leq S\|\Phi_h(u^{pi}|_h)\|.$$

By construction, u^{pi} converges to $\tilde{u}$. The rate of this convergence is bounded by the order of consistency of the scheme. So far $\|\cdot\|$ is an arbitrary norm.

Since our linear stability theory is based on energy estimates, it is natural to measure convergence in a discrete L^2–norm in space. For grid functions on Ω_h, we use

$$\|u\|_{\infty,2} = \max_{t \in \Omega_{\Delta t}} \|u(t,\cdot)\|_2.$$

For functions on $\Omega_{\Delta x}$ the discrete L^2–norm

$$\|u\|_2^2 = \Delta x \sum_{x \in \Omega_{\Delta x}} |u(x)|^2$$

is induced by the product

$$(u, v) = \Delta x \sum_{x \in \Omega_{\Delta x}} < u(x), v(x) > .$$

Below, we shall also use the restricted product and norm

$$(u, v)^{(l,r)} := \Delta x \sum_{j=l}^{r} < u_j, v_j > \quad \text{and} \quad (\|u\|^{(l,r)})^2 := (u, u)^{(l,r)}.$$

The maximum norm is denoted by

$$\|u\|_\infty = \max_{x \in \Omega_{\Delta x}} |u(x)|.$$

Finally, for real vectors, we write

$$< u, v >= u^T v \quad \text{and} \quad |u|^2 = u^T u.$$

In the next section, the scheme is set up and the main result will be stated. Verifying Lipschitz continuity of $D\Phi_h$ is a technical and not very interesting computation. Naturally, the scheme involves Δt, Δx and Δx^2 in the denominator. As the space derivatives appear nonlinearly in the system (1), the Lipschitz constant of $D\Phi_h$ w.r.t. $\|\cdot\|_{\infty,2}$ is typically bounded by (see [11])

$$Lip_{\Delta x} \leq \frac{C}{\Delta x^{2.5}}.$$

In Section 4, the pilot function will be defined and in Section 5, linear stability is proved. Finally, the main result is established in Theorem 3.1.

3. The Finite Difference Scheme

3.1. **Space Discretisation.** In [3], a first order upwind scheme was analysed. Some investigations on second order approximations can be found, for example, in [10, 13]. Since the first order error terms of even order differences are of dispersive type, we are not interested in applying such schemes to hyperbolic equations and therefore skip to third order differences. At inner grid points, a well-known third order difference formula is

$$\frac{d}{dx}u(x_j) \approx (\mathcal{D}_-^{(3)}u)_j = \frac{u_{j-2} - 6u_{j-1} + 3u_j + 2u_{j+1}}{6\Delta x}, \quad j = 2, \dots, J-2.$$

The upper index (3) indicates that the approximation is formally of third order. $\mathcal{D}_-^{(3)}$ is of backward type. Intuitively there is more information taken from the left than from the right. In Section 5 there is a more precise definition of which differences we have to designate as backward and which as forward ones. The forward difference $\mathcal{D}_+^{(3)}$, being the almost adjoint of $-\mathcal{D}_-^{(3)}$ in the interior is given by

$$(\mathcal{D}_+^{(3)}u)_j = \frac{-2u_{j-1} + 3u_j - 6u_{j+1} + u_{j+2}}{6\Delta x}, \quad j = 2, \dots, J-2.$$

In [1], it is proved that it is possible to find artificial boundary conditions of first order and a norm matrix H, which has to be positive definite, such that the summation by parts formula holds

$$(u, \mathcal{D}_-^{(3)}v)_H = -(\mathcal{D}_+^{(3)}u, v)_H + u_J v_J - u_0 v_0. \tag{10}$$

The above scalar product is defined as

$$(u, v)_H = (u, Hv).$$

From (10), we can easily deduce a summation by parts formula for the sum of the two approximations

$$(u, \frac{1}{2}(\mathcal{D}_-^{(3)} + \mathcal{D}_+^{(3)})v)_H = -(\frac{1}{2}(\mathcal{D}_-^{(3)} + \mathcal{D}_+^{(3)})u, v)_H + u_J v_J - u_0 v_0. \tag{11}$$

The full difference operators, including boundary points, read

$$\Delta x \mathcal{D}_-^{(3)} = \begin{pmatrix} -1 & 1 & & & & & & & & \\ -\frac{9}{13} & \frac{5}{13} & \frac{4}{13} & & & & & & & \\ \frac{1}{6} & -1 & \frac{1}{2} & \frac{1}{3} & & & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & & & \\ & & \frac{1}{6} & -1 & \frac{1}{2} & \frac{1}{3} & & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & \frac{1}{6} & -1 & \frac{1}{2} & \frac{1}{3} & & \\ & & & & & \frac{2}{13} & -\frac{12}{13} & \frac{5}{13} & \frac{5}{13} & \\ & & & & & & \frac{2}{5} & -\frac{9}{5} & \frac{7}{5} & \end{pmatrix}$$

and

$$\Delta x \mathcal{D}_+^{(3)} = \begin{pmatrix} -\frac{7}{5} & \frac{9}{5} & -\frac{2}{5} & & & & & & & \\ -\frac{5}{13} & -\frac{5}{13} & \frac{12}{13} & -\frac{2}{13} & & & & & & \\ & -\frac{1}{3} & -\frac{1}{2} & 1 & -\frac{1}{6} & & & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & & \\ & & & -\frac{1}{3} & -\frac{1}{2} & 1 & -\frac{1}{6} & & & \\ & & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & & -\frac{1}{3} & -\frac{1}{2} & 1 & -\frac{1}{6} \\ & & & & & & -\frac{4}{13} & -\frac{5}{13} & \frac{9}{13} \\ & & & & & & & -1 & 1 \end{pmatrix}$$

The corresponding norm matrix is

$$H = \operatorname{diag}(\tfrac{5}{12}, \tfrac{13}{12}, 1, 1, \dots, 1, \tfrac{13}{12}, \tfrac{5}{12}). \tag{12}$$

To apply these operators to systems, we have to multiply each entry of $\mathcal{D}_-^{(3)}$ and $\mathcal{D}_+^{(3)}$ by an identity matrix I. The dimension of I is given by the number of unknowns in the system to which $\mathcal{D}_-^{(3)}$ and $\mathcal{D}_+^{(3)}$ are applied. In fact, we have to build the tensor products $\mathcal{D}_-^{(3)} \otimes I$ and $\mathcal{D}_+^{(3)} \otimes I$. The dimension of I is n, when $\mathcal{D}_\pm^{(3)}$ is applied to u, it is m or $n-m$ when $\mathcal{D}_\pm^{(3)}$ is applied to v or w respectively, and finally it is q or $n-m-q$, when $\mathcal{D}_\pm^{(3)}$ is applied to w^+ or to w^-. In order not to overload notation, we shall omit these identity matrices whenever it is clear which one has to be taken. For example, we write $(\mathcal{D}_-^{(3)} u_k)_j$ for the j-th block of $(\mathcal{D}_-^{(3)} \otimes I_n) u(t_k)$.

Having this, we can write down the spatial discretisation with the centered second order approximations for the derivatives of the v-components and with the third order approximation $\mathcal{D}_\mp^{(3)}$ for the derivatives of the $w^\pm$-components. The time variable t is omitted in the notation.

$$P_{\Delta x}^{(3)}(u) = \begin{pmatrix} -(f_0)_t \\ -(g_0)_t \\ (\tilde{A}_{21}^-)_0 (\mathcal{D}_0^{(2)} v)_0 + \tilde{\Lambda}_0^- (\mathcal{D}_+^{(3)} w^-)_0 - (\tilde{C}_2^-)_0 \\ A_j(\mathcal{D}_0^{(2)} v, \mathcal{D}_-^{(3)} w^+, \mathcal{D}_+^{(3)} w^-)_j - B_j (D_+ D_- u)_j - C_j \\ -(f_1)_t \\ (\tilde{A}_{21}^+)_J (\mathcal{D}_0^{(2)} v)_J + \tilde{\Lambda}_J^+ (\mathcal{D}_-^{(3)} w^+)_J - (\tilde{C}_2^+)_J \\ -(g_1)_t \end{pmatrix}. \tag{13}$$

The middle line has to be repeated for $j = 1, 2, \dots, J-1$. To be more precise, we should have written $(\mathcal{D}_0^{(2)} v, \mathcal{D}_-^{(3)} w^+, \mathcal{D}_+^{(3)} w^-)$ as $((\mathcal{D}_0^{(2)} v)^T, (\mathcal{D}_-^{(3)} w^+)^T, (\mathcal{D}_+^{(3)} w^-)^T)^T$.

The matrices A and C are partitioned according to the decomposition of u as

$$A_{21} = \begin{pmatrix} \widetilde{A}_{21}^{+} \\ \widetilde{A}_{21}^{-} \end{pmatrix}, \quad A_{22} = \Lambda = \Lambda^{+} + \Lambda^{-} = \begin{pmatrix} \widetilde{\Lambda}^{+} & 0 \\ 0 & \widetilde{\Lambda}^{-} \end{pmatrix},$$
$$A_{12} = (\widetilde{A}_{12}^{+}, \widetilde{A}_{12}^{-}) \quad \text{and} \quad C_2 = \begin{pmatrix} \widetilde{C}_2^{+} \\ \widetilde{C}_2^{-} \end{pmatrix}.$$

$D_{\pm}$ are the two point forward and backward differences, and $\mathcal{D}_0^{(2)}$ is defined by

$$\Delta x \mathcal{D}_0^{(2)} = \frac{1}{2} \begin{pmatrix} -2 & 2 & & & & & & \\ -1 & 0 & 1 & & & & & \\ & -1 & 0 & 1 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -1 & 0 & 1 & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & -1 & 0 & 1 \\ & & & & & & -2 & 2 \end{pmatrix}.$$

3.2. **Time Stepping by High Order Runge-Kutta Methods.** For the time discretisation, we apply a high order implicite Runge-Kutta method, in order to get an overall scheme that is accurate of at least second order in the interiour.

The semidiscretisation reads

$$u_t + P_{\Delta x}^{(3)}(u,t) = 0, \quad u(0) = z. \tag{14}$$

An implicit Runge-Kutta method for solving (14) is given by

$$\begin{gathered} u_0 = z|_{\Delta x}, \\ \frac{u_{k_l} - u_{k-1}}{\Delta t} + \sum_{m=1}^{s} a_{l,m} P_{\Delta x}^{(3)}(u_{k_m}, t_{k_m}) = 0, \\ \frac{u_k - u_{k-1}}{\Delta t} + \sum_{l=1}^{s} b_l P_{\Delta x}^{(3)}(u_{k_l}, t_{k_l}) = 0, \\ k = 1, 2, \ldots, K. \end{gathered} \tag{15}$$

Here, u_{k_l} is an approximation to the solution at the intermediate gridpoint

$$t_{k_l} = t_{k-1} + c_l \Delta t \ \in [t_{k-1}, t_k], \quad l = 1, 2, \ldots, s$$

The goal in Section 4 is to construct a function, which is of third order consistent with the fully-discrete problem. For this purpose, a Runge-Kutta method of stage order $q = 3$ is needed. We shall analyse the Radau IIA method, which is given by its

Butcher-array

$$
\begin{array}{c|ccc}
\dfrac{4-\sqrt{6}}{10} & \dfrac{88-7\sqrt{6}}{360} & \dfrac{296-169\sqrt{6}}{1800} & \dfrac{-2+3\sqrt{6}}{255} \\
\dfrac{4+\sqrt{6}}{10} & \dfrac{296+169\sqrt{6}}{1800} & \dfrac{88+7\sqrt{6}}{360} & \dfrac{-2-3\sqrt{6}}{255} \\
1 & \dfrac{16-\sqrt{6}}{36} & \dfrac{16+\sqrt{6}}{36} & \dfrac{1}{9} \\
\hline
 & \dfrac{16-\sqrt{6}}{36} & \dfrac{16+\sqrt{6}}{36} & \dfrac{1}{9}
\end{array}
\tag{16}
$$

and which has stage order three, that is

$$
\begin{gathered}
\frac{\hat{u}_{k_l} - \hat{u}_{k-1}}{\Delta t} + \sum_{m=1}^{s} a_{l,m} P_{\Delta x}^{(3)}(\hat{u}_{k_m}, t_{k_m}) = O(\Delta t^q), \\
\frac{\hat{u}_{k} - \hat{u}_{k-1}}{\Delta t} + \sum_{l=1}^{s} b_l P_{\Delta x}^{(3)}(\hat{u}_{k_l}, t_{k_l}) = O(\Delta t^q), \\
k = 1, 2, \ldots, K, \quad l = 1, 2, \ldots, s,
\end{gathered}
$$

where $\hat{u}$ is the solution of (14).

This method has two further properties, which will allow us to prove stability of the overall scheme. It is algebraically stable and has a positive coercivity coefficient, $\alpha_0(A^{-1}) = 5/(4+\sqrt{6})$, where $A = \{a_{l,m}\}$ is the matrix of the Runge-Kutta coefficients, cf. [5].

Now we are in the position to state the main result

Theorem 3.1. *Consider a mixed system* (1), (2), *where* B_{11} *is strongly parabolic and* A_{22} *is strictly hyperbolic and regular. Let* $\tilde{u}$ *be the unique smooth solution of the initial-boundary value problem* (1)–(6).

For sufficiently small step sizes $\Delta t = \mu \Delta x$, *where* μ *is arbitrarily but fixed, the Radau IIA method* (15), (16), *applied to the space discrete problem* (13), (14), *has a unique solution* U *defined on* Ω_h, *which converges at second order to the analytical solution* $\tilde{u}$

$$
\|U - \tilde{u}|_h\|_{\infty,2}^{h} = \mathcal{O}(\Delta x^2), \quad \Delta x \to 0.
$$

The proof of this Theorem will be developed in the next two sections. Finally we collect the individual steps and summarise the proof in Section 6.

4. Approximation to the Numerical Solution, Boundary Layer Terms

According to the general concept in Section 2, we have to construct a high order consistent function u^{pi}, at which the scheme will be linearised. The stability of the linearised scheme will be discussed in the next section.

4.1. **Construction.** As the order of consistency of our space discretisation decays to one near the boundary, boundary layers will occure. The following ansatz takes this layers into account.

$$\begin{aligned} u^{pi}(t,x,\xi,\zeta) &= \tilde{u}(t,x) + \Delta x u^{(1)}(t,x) + \Delta x^2 u^{(2)}(t,x) \\ &\quad + \Delta x^2 l^{(1)}(t,\xi) + \Delta x^3 l^{(2)}(t,\xi) + \Delta x^2 r^{(1)}(t,\zeta) + \Delta x^3 r^{(2)}(t,\zeta) \\ &\qquad \xi = \frac{x}{\Delta x},\ \zeta = \frac{1-x}{\Delta x} \end{aligned} \tag{17}$$

The error terms are defined by equations that we get by

- substituting the ansatz (17) into the semi-discrete equations $u_t + P^{(3)}_{\Delta x}(u) = 0$, given by (14),
- expanding the terms which do not depend on ξ or ζ with respect to the step size,
- separating the terms depending on x, ξ and ζ,
- seting the coefficients of Δx and Δx^2 to zero.

The results are linear initial-boundary value problems that define $u^{(1)}$ and $u^{(2)}$, and finite difference equations that define $l^{(1)}$, $l^{(2)}$, $r^{(1)}$, and $r^{(2)}$ at the grid points.

By construction, u^{pi} satisfies the semi-discrete equations up to order three. As we shall see in Section 6, it is even third order consistent with the fully-discrete scheme.

The defining equations for $u^{(1)}$ are

$$u^{(1)}{}_t + A(\tilde{u})u^{(1)}{}_x + DA(\tilde{u})u^{(1)}\tilde{u}_x = B(\tilde{u})u^{(1)}{}_{xx} + DB(\tilde{u})u^{(1)}\tilde{u}_{xx} + DC(\tilde{u})u^{(1)}. \tag{18}$$

Additionally, we have homogeneous initial and boundary conditions, such that $u^{(1)}$ vanishes.

Next, we define the first boundary layer terms $l^{(1)}$ and $r^{(1)}$. To this end, we split up $l^{(i)}$ and $r^{(i)}$, $i = 1, 2$, into three parts, according to the components v, w^+, and w^- of u, i.e.

$$l^{(i)} = \left(l^{(i)^v}, l^{(i)^{w^+}}, l^{(i)^{w^-}} \right)^T, \quad r^{(i)} = \left(r^{(i)^v}, r^{(i)^{w^+}}, r^{(i)^{w^-}} \right)^T.$$

The defining difference equations are

$$\begin{aligned} l^{(1)^{w^+}}_{k,0} &= 0, \\ (\Delta x \mathcal{D}^{(3)}_- l^{(1)^{w^+}}_k)_j &= \begin{cases} \frac{5}{26}\tilde{w}^+_{xx}(t_k,x_j), & j = 1, \\ 0, & j = 2,3,\ldots,J. \end{cases} \end{aligned} \tag{19}$$

$$\begin{aligned} r^{(1)^{w^+}}_{k,0} &= 0, \\ (\Delta x \mathcal{D}^{(3)}_- r^{(1)^{w^+}}_k)_j &= \begin{cases} 0, & j = 1,2,\ldots,J-2, \\ -\frac{1}{26}\tilde{w}^+_{xx}(t_k,x_j), & j = J-1, \\ \frac{1}{10}\tilde{w}^+_{xx}(t_k,x_j) + \frac{1}{2}(\tilde{\Lambda}^+_j)^{-1}(\tilde{A}^+_{12})_j v_{xx}(t_k,x_j), & j = J. \end{cases} \end{aligned} \tag{20}$$

$$
(\Delta x \mathcal{D}_+^{(3)} l_k^{(1)^{w^-}})_j = \begin{cases} -\frac{1}{10}\widetilde{w}_{xx}^-(t_k,x_j) - \frac{1}{2}(\widetilde{\Lambda}_j^-)^{-1}(\widetilde{A}_{12}^-)_j v_{xx}(t_k,x_j), \\ \hfill j=0, \\ \frac{1}{26}\widetilde{w}_{xx}^-(t_k,x_j), \quad j=1, \\ 0, \quad j=2,3,\dots,J-1, \end{cases} \tag{21}
$$

$$
l_{k,J}^{(1)^{w^-}} = 0.
$$

$$
(\Delta x \mathcal{D}_+^{(3)} r_k^{(1)^{w^-}})_j = \begin{cases} 0, \quad j=0,1,\dots,J-2, \\ -\frac{5}{26}\widetilde{w}_{xx}^-(t_k,x_j), \quad j=J-1, \end{cases} \tag{22}
$$

$$
r_{k,J}^{(1)^{w^-}} = 0.
$$

The space derivatives of the parabolic component are discretised with second order accuracy including the boundary points. Thus, the v-parts of $r^{(1)}$ and $l^{(1)}$ are zero and we can prove the following lemma.

Lemma 4.1. *The first boundary layer terms $r^{(1)}$ and $l^{(1)}$ defined at the grid points by* (19)–(22) *are bounded independent of the grid parameter Δx.*

Proof. We prove the lemma for $r^{(1)}$ only. Since the v-components of $r^{(1)}$ are zero, we have to prove that the w^+- and the w^--components are bounded. These are defined by decoupled systems of difference equations. Thus, we can treat each component independent from the others. Let r denote any component of $r_k^{(1)^{w^+}}$ and $r_j = r(x_j)$. The vector r is defined through the following system of equations

$$
\begin{pmatrix}
1 & & & & & & & & & \\
-9 & 5 & 4 & & & & & & & \\
1 & -6 & 3 & 2 & & & & & & \\
 & 1 & -6 & 3 & 2 & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & & 1 & -6 & 3 & 2 & & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & & 1 & -6 & 3 & 2 & \\
 & & & & & & 2 & -12 & 5 & 5 \\
 & & & & & & & 2 & -9 & 7
\end{pmatrix}
\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_j \\ \vdots \\ r_{J-2} \\ r_{J-1} \\ r_J \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ y_{J-1} \\ y_J \end{pmatrix}. \tag{23}
$$

Here, y_{J-1} and y_J are the appropriate components of $-1/2\widetilde{w}_{xx}^+(t_k,x_{J-1})$ and $1/2\widetilde{w}_{xx}^+(t_k,x_J) + 5/2(\widetilde{\Lambda}_J^+)^{-1}(\widetilde{A}_{12}^+)_J v_{xx}(t_k,x_J)$ respectively.

We get $r_0 = 0$ and define $b_j = r_j/r_{j-1}$, $j = 2,3,\dots,J$ and $c_j = r_j/r_1$, $j = 1,2,\dots,J$. Then, we have $b_2 = -5/4$, $b_3 = -39/10$ and for $j = 4,5,\dots,J-1$, b_j is defined recursively by

$$
b_j = -\frac{3}{2} + \frac{3}{b_{j-1}} - \frac{1}{2b_{j-1}b_{j-2}}
$$

and we get $b_4 = -185/78$, $b_5 = -1043/370$, $b_6 = -5505/2086$. From the recursion formula it is obvious that $b_j \in (-3,-2.5)$ if b_{j-1} and b_{j-2} are in this interval, which is true for b_5 and b_6. Therefore, we know that $|r_{j-1}| < |r_j|$ for $j = 1, 2, \dots, J-1$, and we thus have to prove that r_{J-1} and r_J are bounded.

From the last two equations in (23), we get

$$r_1 = \frac{7y_{J-1} - 5y_J}{80c_{J-1} - 94c_{J-2} + 14c_{J-3}}$$

and therefore

$$\begin{aligned} r_{J-1} &= c_{J-1}r_1 = \frac{c_{J-1}(7y_{J-1} - 5y_J)}{80c_{J-1} - 94c_{J-2} + 14c_{J-3}} \\ &= \frac{b_{J-1}b_{J-2}(7y_{J-1} - 5y_J)}{80b_{J-1}b_{J-2} - 94b_{J-2} + 14}. \end{aligned}$$

Now, we have

$$0.006 < \frac{b_{J-1}b_{j-2}}{80b_{J-1}b_{J-2} - 94b_{J-2} + 14} < 0.013,$$

which gives a bound for r_j, $j \le J-1$. For r_J, we get

$$\frac{y_J}{7} + \frac{(7y_{J-1} - 5y_J)(9b_{J-1}b_{J-2} - 2b_{J-2})}{7(80b_{J-1}b_{J-2} - 94b_{J-2} + 14)}$$

and we have

$$0.008 < \frac{9b_{J-1}b_{J-2} - 2b_{J-2}}{7(80b_{J-1}b_{J-2} - 94b_{J-2} + 14)} < 0.017,$$

from which it follows that $r_k^{(1)^{w^+}}$ is bounded independent of the grid size.

Now, we estimate the components of $r_k^{(1)^{w^-}}$. Let therefore q denote any component of $r_k^{(1)^{w^-}}$. The vector q is defined by

$$\begin{pmatrix} -7 & 9 & -2 & & & & & & & \\ -5 & -5 & 12 & -2 & & & & & & \\ & -2 & -3 & 6 & -1 & & & & & \\ & & -2 & -3 & 6 & -1 & & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & -2 & -3 & 6 & -1 & & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & -2 & -3 & 6 & -1 \\ & & & & & & & -4 & -5 & 9 \\ & & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_j \\ \vdots \\ q_{J-2} \\ q_{J-1} \\ q_J \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ z_{J-1} \\ 0 \end{pmatrix}, \tag{24}$$

with $z_{J-1} = -5/26\widetilde{w}_{xx}^-(t_k, x_{J-1})$. Obviously, the problem can be scaled such that $z_{J-1} = 1$.

To prove pointwise boundedness of q, we make use of the fact that we can always subtract a constant vector from the left hand side in (24) without changing the right hand side of the first $J-1$ equations. So, we define

$$s_j = q_j - q_0, \quad j = 0, 1, \dots, J, \tag{25}$$

and we have to solve the following reduced system

$$\begin{pmatrix} -7 & 9 & -2 & & & & & & & \\ -5 & -5 & 12 & -2 & & & & & & \\ & -2 & -3 & 6 & -1 & & & & & \\ & & -2 & -3 & 6 & -1 & & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & -2 & -3 & 6 & -1 & & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & -2 & -3 & 6 & -1 \\ & & & & & & & -4 & -5 & 9 \\ & & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_j \\ \vdots \\ s_{J-2} \\ s_{J-1} \\ s_J \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 1 \\ -q_0 \end{pmatrix}. \tag{26}$$

Now, it is obvious that we can proceed in the same way as for p. We define $d_j = s_j/s_{j-1},\ j = 2,3,\dots,J$ and $e_j = s_j/s_1,\ j = 1,2,\dots,J$, and we get $d_2 = 9/2,\ d_3 = 44/9$, and

$$d_j = 6 - \frac{3}{d_{j-1}} - \frac{2}{d_{j-1}d_{j-2}}, \quad j = 4,5,\dots,J.$$

We have $d_4 = 233/44,\ d_5 = 1248/233$, and from the recursion formula, we see that $d_j \in (5, 5.5)$ if $d_{j-1}, d_{j-2} \in (5, 5.5)$. Thus $|s_{j-1}| < |s_j|$ for $j = 1,2,\dots,J$, and we need to prove that s_J is uniformly bounded.

From the second last equation in (26), we get

$$s_J = e_J s_1 = \frac{e_J}{e_J - 5e_{J-1} - 4e_{J-2}} = \frac{d_J d_{J-1}}{9 d_J d_{J-1} - 5 d_{J-1} - 4}$$

and hence

$$0.102 \le s_J \le 0.157,$$

so that we have proved $|s|_\infty$ to be bounded independent of the grid size. With (25) it follows that the same is true for q. □

Remark 4.1. *The proof of Lemma 4.1 shows that, in the interior of the interval* $(0,1)$, *r is bounded by*

$$\exp\left(\log 1.25 \frac{1-x_j}{\Delta x}\right) + |r_1| \quad < \quad |r_j| \quad < \quad \exp\left(\log 3.9 \frac{1-x_j}{\Delta x}\right) + |r_1|,$$
$$j = 1,2,\dots,J-1.$$

The second smooth error term $u^{(2)}$ of the pilot function is defined by the following system of partial differential equations.

$$\begin{aligned} & {u^{(2)}}_t + A(\tilde u)({u^{(2)}}_x + 1/6\,(\tilde v_{xxx}, 0)) + DA(\tilde u)u^{(2)}\tilde u_x \\ & \quad + DA(\tilde u)l^{(1)}\tilde u_x + DA(\tilde u)r^{(1)}\tilde u_x \\ = \quad & B(\tilde u)({u^{(2)}}_{xx} + 1/12\,\tilde u_{xxxx}) + DB(\tilde u)u^{(2)}\tilde u_{xx} \\ & \quad + DB(\tilde u)l^{(1)}\tilde u_{xx} + DB(\tilde u)r^{(1)}\tilde u_{xx} \\ & \quad + DC(\tilde u)u^{(2)} + DC(\tilde u)l^{(1)} + DC(\tilde u)r^{(1)}, \end{aligned} \tag{27}$$

where $r^{(1)}$ and $l^{(1)}$ are extended to the whole interval by smoothly interpolating the grid functions defined above.

Remark 4.2. *The second error term $u^{(2)}$ depends on Δx as $r^{(1)}$ and $l^{(1)}$ do, however for any fixed $\Delta x > 0$, it is clear that $u^{(2)}$ is a smooth function. Since $r^{(1)}$ and $l^{(1)}$, now regarded as functions in x, are uniformly bounded for $\Delta x \to 0$, it follows that $u^{(2)}$ is bounded independently of Δx, and the pilot function is properly defined so far.*

It remains to define the second boundary layer terms $l^{(2)}$ and $r^{(2)}$. This is done through difference equations, which are similar to those for $l^{(1)}$ and $r^{(1)}$.

$$
\begin{aligned}
&l_{k,0}^{(2)^{w^+}} = 0,\\
&(\Delta x \mathcal{D}_-^{(3)} l_k^{(2)^{w^+}})_j = \begin{cases} -\frac{1}{6}\widetilde{w}^+_{xxx}(t_k, x_j), & j = 1,\\ 0, & j = 2, 3, \ldots, J. \end{cases}
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&r_{k,0}^{(2)^{w^+}} = 0,\\
&(\Delta x \mathcal{D}_-^{(3)} r_k^{(2)^{w^+}})_j = \begin{cases} 0, & j = 1, 2, \ldots, J-2,\\ -\frac{1}{78}\widetilde{w}^+_{xxx}(t_k, x_j), & j = J-1,\\ \frac{7}{30}\widetilde{w}^+_{xxx}(t_k, x_j), & j = J. \end{cases}
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&(\Delta x \mathcal{D}_+^{(3)} l_k^{(2)^{w^-}})_j = \begin{cases} \frac{7}{30}\widetilde{w}^-_{xxx}(t_k, x_j), & j = 0,\\ -\frac{1}{78}\widetilde{w}^-_{xxx}(t_k, x_j), & j = 1,\\ 0, & j = 2, 3, \ldots, J-1, \end{cases}\\
&l_{k,J}^{(2)^{w^-}} = 0.
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&(\Delta x \mathcal{D}_+^{(3)} r_k^{(2)^{w^-}})_j = \begin{cases} 0, & j = 0, 1, \ldots, J-2,\\ -\frac{1}{6}\widetilde{w}^-_{xxx}(t_k, x_j), & j = J-1, \end{cases}\\
&r_{k,J}^{(2)^{w^-}} = 0.
\end{aligned}
\tag{31}
$$

The remaining components of $r^{(2)}$ and $l^{(2)}$ are zero. The proof of the following lemma follows the one of Lemma 4.1.

Lemma 4.2. *The second boundary layer terms, $r^{(2)}$ and $l^{(2)}$, defined at the grid points by* (28)–(31) *are bounded independent of the grid parameter Δx.*

4.2. **Visualisation.** Here, we illustrate the boundary layers by solving two model problems numerically, each with different schemes. The schemes are chosen in such a way that they differ in the first boundary layer terms, which can be expressed by the difference of the two numerical solutions. This difference will be plotted below. First, we investigate the scalar advection equation, for which we can compute the first boundary layer terms explicitely. The second model problem is an initial-boundary value problem for the one-dimensional Navier-Stokes equations.

Since the correction terms of the pilot function are defined by linear equations, it is obviously enough to look at linear problems.

4.2.1. *Linear Advection Equation.* We discretise the initial-boundary value problem

$$\begin{aligned} u_t + u_x &= 0, \\ u(t,0) &= 0, \quad t \geq 0 \\ u(0,x) &= \sin^3(\pi x), \quad x \in [0,1]. \end{aligned}$$

Clearly, this initial boundary value problem has the solution

$$\tilde{u}(t,x) = \begin{cases} 0, & 0 \leq x \leq t, \\ \sin^3(\pi(x-t)), & t < x \leq 1. \end{cases}$$

In time direction, we apply the implicite Euler method and for the space discretisation, we use the operator $\mathcal{D}_-^{(3)}$. We are interested in the first boundary layer term $r^{(1)}$ on the right boundary. For this reason, we compare the solution of this scheme with the one of a scheme with boundary conditions that are second order accurate. To this end, we replace the discretisation for $j = J-1, J$ by

$$\begin{aligned} \frac{u_{k,J-1} - u_{k-1,J-1}}{\Delta t} + \frac{2u_{k,J} + 3u_{k,J-1} - 6u_{k,J-2} + u_{k,J-3}}{6\Delta x} &= 0, \\ \frac{u_{k,J} - u_{k-1,J}}{\Delta t} + \frac{3u_{k,J} - 4u_{k,J-1} + u_{k,J-2}}{2\Delta x} &= 0. \end{aligned} \tag{32}$$

Let U_1 denote the solution of the first scheme with the first order boundary conditions, i.e. the method with layer. By U_2, we denote the solution of the scheme with the artificial boundary conditions (32), i.e. without layer.

The boundary layer, which is caused by the first order conditions in $\mathcal{D}_-^{(3)}$, can be approximated by the difference $R = \Delta x^2(U_1 - U_2)$. On the other hand, $r^{(1)}(t_k)$ can be computed by formula (23) with

$$\begin{aligned} y_{J-1} &= -\frac{1}{2}\left(6\sin(x_1)\cos^2(x_1) - 3\sin^3(x_1)\right), \quad x_1 = 1 - \Delta x - t_k, \\ y_J &= \frac{1}{2}\left(6\sin(x_2)\cos^2(x_2) - 3\sin^3(x_2)\right), \quad x_2 = 1 - t_k. \end{aligned}$$

In Figures 1 and 2, we compare the analytical expression for $r^{(1)}$, defined by (23) and interpolated with the standard cubic spline, with the numerical approximation R, both at time $t = 0.1$ and $t = 0.5$. The computations are carried out with $\mu = \Delta t/\Delta x = 1$ and $\Delta x = 10^{-3}$. We observe that the numerical approximation R—printed as discrete values—represents the analytical expression quite well.

4.2.2. *Navier-Stokes Equations.* Next, we investigate a linear system of partial differential equations. Let us consider the following initial-boundary value problem

$$\begin{pmatrix} v \\ \rho \end{pmatrix}_t + \begin{pmatrix} 1 & 1.4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v \\ \rho \end{pmatrix}_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ \rho \end{pmatrix}_{xx}$$

$$\left.\begin{aligned} v(0,x) &= 1 + \alpha\sin^3(\pi x) \\ \rho(0,x) &= 1 - \alpha\sin^3(\pi x) \end{aligned}\right\} \quad x \in [0,1]$$

$$\left.\begin{aligned} v(t,0) &= v(t,1) = 1 \\ \rho(t,0) &= 1 \end{aligned}\right\} \quad t \geq 0.$$

This is a linearised version of the Navier-Stokes equations.

Again, we apply the implicit Euler method for the time discretisation. The space variable is discretised with the centered two point stencil for the velocity v. Since we have physical boundary conditions on either side of the interval, we do not need to define artificial boundary conditions for v.

The ρ component is discretised with the two methods from Section 4.2.1 and we compare the results of the two schemes.

Figures 3 and 6 display the solution at time $t = 0.1$ and $t = 0.5$, respectively. The v-component of the second order error terms is displayed in Figures 4 and 7. As expected, there is no boundary layer in v. For the hyperbolic component, ρ, the superposition of the second order error terms is displayed in Figures 5 and 8. Due to the low order boundary condition, we clearly observe the boundary layer.

As in Section 4.2.1, the computations are carried out with $\mu = \Delta t/\Delta x = 1$ and $\Delta x = 10^{-3}$. The parameter α is set to 0.1.

5. Stability and Convergence

Having defined the third order consistent pilot function, it remains to show stability of the scheme linearised at u^{pi}. In a first step an energy estimate for linearised, strictly hyperbolic problems, discretised with the third order approximations $\mathcal{D}_-^{(3)}$ and $\mathcal{D}_+^{(3)}$ will be proved. In combination with the estimates for the parabolic component in [3], an estimate for the mixed system follows. Finally, in Section 5.4 we shall see that the overall scheme is linearly stable in the sense of (9). Let us begin with collecting some technical preliminaries.

5.1. **Some Useful Tools.**

Definition 5.1. *Let $A = \operatorname{diag}(A_0, A_1, \ldots, A_J)$, $A_j \in \mathcal{R}^{d_1\times d_2}$, $j = 0, 1, \ldots, J$. The commutator $[\mathcal{D}, A]$ of some difference operator and A is defined by*

$$[\mathcal{D}, A] = \mathcal{D}\otimes I_{d_1}\cdot A - A\cdot\mathcal{D}\otimes I_{d_2}.$$

Lemma 5.1. *Let H be positive definite. For any difference approximation $\mathcal{D}$ on $\Omega_{\Delta x}$ and for the discretisation A of any smooth matrix function $\mathcal{A}\in\mathcal{C}^1([0,1],\mathcal{R}^{d_1\times d_2})$, it holds that $\|[\mathcal{D}, A]\|_H$ can be bounded independently of Δx.*

Proof. As usual, the matrix A is given by

$$A = \operatorname{diag}(A_0, A_1, \ldots, A_J).$$

We have

$$[\mathcal{D}, A]_{l,m} = \frac{1}{\Delta x} d_{l,m}(A_m - A_l), \quad l, m = 0, 1, \ldots, J$$

where $d_{l,m} = 0$ if $|l - m| > s$ and s is the bandwidth of $\mathcal{D}$. Since $A(x)$ is differentiable, it holds that

$$|A_m - A_l| = \left|\int_{x_l}^{x_m} DA(\xi)d\xi\right| \le \Delta x|l - m| \sup_{x\in[0,1]}\|DA(x)\|.$$

Therefore, we have

$$|[\mathcal{D}, A]_{l,m}| = |d_{l,m}(l - m)| \sup_{x\in[0,1]}|DA(x)|,$$

which is independent of the gridsize Δx. □

Lemma 5.2. *Let $\mathcal{D}$ be any consistent finite difference approximation to d/dx and let $H = \operatorname{diag}(h_0, h_1, \dots, h_J)$ be a diagonal norm matrix, then there exist constants c' and c'' such that the following estimates hold for all functions u*

(i) $\|\mathcal{D}u\|_H^{(1,J)} \le c'\|D_-u\|^{(1,J)}$,
(ii) $\|\mathcal{D}u\|_H^{(0,J-1)} \le c''\|D_+u\|^{(0,J-1)}$.

Proof. Let L and R denote the lower and upper bandwidth of $\mathcal{D}$ respectively and define $L' = L'(j) = \max(0, j-L)$ and $R' = R'(j) = \min(J, j+R)$. To simplify notation, we set $\Delta x = 0$, then

$$(\mathcal{D}u)_j = \sum_{m=L'}^{R'} d_{j,m}(E^{m-j}u)_j.$$

Due to consistency of $\mathcal{D}$, it holds that

$$\sum_{m=L'}^{R'} d_{j,m} = 0,$$

therefore, there exist numbers $\beta_{j,m},\ L'+1 \le m \le R'$ with

$$\sum_{m=L'}^{R'} d_{j,m}E^{m-j} = \sum_{m=L'+1}^{R'} \beta_{j,m}(E^{m-j}D_-).$$

With $\beta_j = \sum\limits_{m=L'+1}^{R'} |\beta_{j,m}|$ we get by elementary computation

$$\begin{aligned}
|(\mathcal{D}u)_j|^2 &= |\sum_{m=L'}^{R'} d_{j,m}(E^{m-j}u)_j|^2 = |\sum_{m=L'+1}^{R'} \beta_{j,m}(E^{m-j}D_-u)_j|^2 \\
&\le \beta_j^2 \max_{L'+1\le m\le R'} |(E^{m-j}D_-u)_j|^2 \le \beta_j^2 \sum_{m=L'+1}^{R'} |(E^{m-j}D_-u)_j|^2.
\end{aligned}$$

Hence with $\beta = \max\limits_{1\le j\le J}(h_j\beta_j^2)$ we have

$$\begin{aligned}
(\|\mathcal{D}u\|_H^{(1,J)})^2 &= \sum_{j=1}^{J} h_j|(\mathcal{D}u)_j|^2 \le \sum_{j=1}^{J} h_j\beta_j^2 \left(\sum_{m=L'+1}^{R'} |(E^{m-j}D_-u)_j|^2\right) \\
&\le \beta \sum_{m=-L}^{R} \left(\sum_{j=\max(1-m,1)}^{\min(J-m,J)} |(E^m D_-u)_j|^2\right) \\
&\le \beta(R+L+1)\sum_{j=1}^{J} |(D_-u)_j|^2 \\
&= \beta(R+L+1)(\|D_-u\|^{(1,J)})^2,
\end{aligned}$$

from which (i) follows. (ii) can be proved analogously. □

Definition 5.2. *A pair of difference approximations $(\mathcal{D}_-,\ \mathcal{D}_+)$ is called dissipative with respect to a norm matrix H, if the difference $H(\mathcal{D}_- - \mathcal{D}_+)$ is positive semidefinite and if at least one of its eigenvalues is positive.*

Lemma 5.3. *The pair of third order approximations $(\mathcal{D}_-^{(3)}, \mathcal{D}_+^{(3)})$ with its related normmatrix H is dissipative.*

Proof. We have

$$\Delta x H(\mathcal{D}_-^{(3)} - \mathcal{D}_+^{(3)}) = \frac{1}{6}\begin{pmatrix}
1 & -2 & 1 & & & & & & & \\
-2 & 5 & -4 & 1 & & & & & & \\
1 & -4 & 6 & -4 & 1 & & & & & \\
 & 1 & -4 & 6 & -4 & 1 & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & 1 & -4 & 6 & -4 & 1 & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & 1 & -4 & 6 & -4 & 1 \\
 & & & & & & 1 & -4 & 5 & -2 \\
 & & & & & & & 1 & -2 & 1
\end{pmatrix}$$

The Cholesky factor of this operator is given by

$$\Delta x R = \frac{\sqrt{6}}{6}\begin{pmatrix}
1 & -2 & 1 & & & & & \\
 & 1 & -2 & 1 & & & & \\
 & & \ddots & \ddots & \ddots & & & \\
 & & & 1 & -2 & 1 & & \\
 & & & & \ddots & \ddots & \ddots & \\
 & & & & & 1 & -2 & 1 \\
 & & & & & & 0 & 0 \\
 & & & & & & & 0
\end{pmatrix}$$

which proves the lemma. □

5.2. **Strictly Hyperbolic Problems.** The discretisation of a strictly hyperbolic system at internal points reads:

$$(w_t)_j + (P_{\Delta x}^{(3)}(t)w)_j = 0, \quad j = 1, 2, \dots J-1, \tag{33}$$

with

$$\begin{aligned}
(P_{\Delta x}^{(3)}(t)w)_j &= \Lambda^+(t, x_j, \Delta x)(\mathcal{D}_-^{(3)}w)_j + \Lambda^-(t, x_j, \Delta x)(\mathcal{D}_+^{(3)}w)_j \\
&\quad - C(t, x_j, \Delta x)w_j.
\end{aligned}$$

The unknown vector w is a function of the time variable t. Furthermore, we have initial and inflow boundary data

$$\begin{aligned}
w_j(0) = 0, \quad j = 0, 1, \dots J, \\
w_0^+(t) = 0, \quad w_J^-(t) = 0, \quad t \in [0, T].
\end{aligned} \tag{34}$$

At the outflow boundaries, the conditions are

$$\begin{aligned}
(w_t^-)_0 &= -\widetilde{\Lambda}^-(t, x_0, \Delta x)(\mathcal{D}_+^{(3)}w^-)_0 + \widetilde{C}^-(t, x_0, \Delta x)w_0, \\
(w_t^+)_J &= -\widetilde{\Lambda}^+(t, x_J, \Delta x)(\mathcal{D}_-^{(3)}w^+)_J + \widetilde{C}^+(t, x_J, \Delta x)w_J.
\end{aligned} \tag{35}$$

Now, the time derivative of $\|w\|_H^2$ is

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|_H^2 &= (w, w_t)_H \\
&= -(w, P_{\Delta x}^{(3)}w)_H^{(1,J-1)} + \Delta x < w_0, (w_t)_0 >_H + \Delta x < w_J, (w_t)_J >_H .
\end{aligned}$$

Using the homogeneous boundary conditions, we get

$$\begin{aligned}(w,w_t)_H &= -(w,\Lambda^+\mathcal{D}_-^{(3)}w)_H^{(1,J-1)} - (w,\Lambda^-\mathcal{D}_+^{(3)}w)_H^{(1,J-1)} + (w,Cw)_H^{(1,J-1)}\\ &\quad - < w_0^-,\widetilde{\Lambda}_0^-\mathcal{D}_+^{(3)}w_0^- >_H + < w_0^-,\widetilde{C}_0^- w_0 >_H\\ &\quad - < w_J^+,\widetilde{\Lambda}_J^+\mathcal{D}_-^{(3)}w_J^+ >_H + < w_J^+,\widetilde{C}_J^+ w_J >_H\\ &= -(w^+,\widetilde{\Lambda}^+\mathcal{D}_-^{(3)}w^+)_H^{(1,J)} - (w^-,\widetilde{\Lambda}^-\mathcal{D}_+^{(3)}w^-)_H^{(0,J-1)}\\ &\quad + (w^+,\widetilde{C}^+w)_H^{(1,J)} + (w^-,\widetilde{C}^-w)_H^{(0,J-1)}\\ &= -(w^+,\widetilde{\Lambda}^+\mathcal{D}_-^{(3)}w^+)_H - (w^-,\widetilde{\Lambda}^-\mathcal{D}_+^{(3)}w^-)_H + (w,Cw)_H\end{aligned}$$

and energy boundedness of the semidiscrete problem can be proved as follows.

Proposition 5.1. *The semidiscrete problem given by* (33)–(35) *is energy bounded, i.e. there is a constant c such that*

$$\frac{d}{dt}\|w\|_H \le c\|w\|_H.$$

Proof. First, we estimate the term $-(w^+,\widetilde{\Lambda}^+\mathcal{D}_-^{(3)}w^+)_H$, we split $\mathcal{D}_-^{(3)}$ into its almost antisymmetric and its symmetric part

$$-2(w^+,\widetilde{\Lambda}^+\mathcal{D}_-^{(3)}w^+)_H = -(w^+,\widetilde{\Lambda}^+(\mathcal{D}_-^{(3)}+\mathcal{D}_+^{(3)})w^+)_H - (w^+,\widetilde{\Lambda}^+(\mathcal{D}_-^{(3)}-\mathcal{D}_+^{(3)})w^+)_H.$$

From Lemma 5.3 we know that $(\mathcal{D}_-^{(3)}-\mathcal{D}_+^{(3)})$ is positive semidefinite and therefore we have

$$-2(w^+,\widetilde{\Lambda}^+\mathcal{D}_-^{(3)}w^+)_H \le -(w^+,\widetilde{\Lambda}^+(\mathcal{D}_-^{(3)}+\mathcal{D}_+^{(3)})w^+)_H. \tag{36}$$

Using summation by parts (11) and the inflow condition on the left boundary it follows

$$\begin{aligned}&-(w^+,\widetilde{\Lambda}^+(\mathcal{D}_-^{(3)}+\mathcal{D}_+^{(3)})w^+)_H\\ &= ((\mathcal{D}_-^{(3)}+\mathcal{D}_+^{(3)})w^+,\widetilde{\Lambda}^+w^+)_H + (w^+,[\mathcal{D}_-^{(3)}+\mathcal{D}_+^{(3)},\widetilde{\Lambda}^+]w^+)_H - 2 < w_J^+,\widetilde{\Lambda}_J^+w_J^+ > .\end{aligned}$$

By Lemma 5.1 and the fact that $\widetilde{\Lambda}_J^+$ is positive definite, we get

$$-(w^+,\widetilde{\Lambda}^+(\mathcal{D}_-^{(3)}+\mathcal{D}_+^{(3)})w^+)_H \le ((\mathcal{D}_-^{(3)}+\mathcal{D}_+^{(3)})w^+,\widetilde{\Lambda}^+w^+)_H + c_1\|w^+\|_H^2.$$

Because H is diagonal, it is obvious that $\widetilde{\Lambda}^+$ is self-adjoint with respect to $(\cdot,\cdot)_H$, and it follows that

$$-(w^+,\widetilde{\Lambda}^+(\mathcal{D}_-^{(3)}+\mathcal{D}_+^{(3)})w^+)_H \le \frac{1}{2}c_1\|w^+\|_H^2.$$

The term $-(w^-,\widetilde{\Lambda}^-\mathcal{D}_+^{(3)}w^-)_H$ is estimated similarly and to $(w,Cw)_H$ we can apply Schwarz's inequality, which concludes the proof. □

As we have seen in the proof of Proposition 5.1, it is necessary to apply the *backward* difference $\mathcal{D}_-^{(3)}$ to w^+ and the *forward* operator $\mathcal{D}_+^{(3)}$ to w^-. Doing so, the pair $(\mathcal{D}_-^{(3)},\mathcal{D}_+^{(3)})$ becomes dissipative in the sense of Definition 5.2. Otherwise, inequality (36) would not hold and the scheme could no longer be stable, since this would contradict a result by Jeltsch and Smit, cf. [6].

In the case where forward and backward approximations are identical, $\mathcal{D}_- = \mathcal{D}_+ = D_0$ for example, the matrix $H(\mathcal{D}_- - \mathcal{D}_+)$ vanishes and the scheme is non-dissipative.

5.3. **Mixed Systems.** The semidiscretisation for mixed systems is given by

$$(u_t)_j + (P^{(3)}_{\Delta x}(t)u)_j = 0, \quad j = 1, 2, \dots J-1, \tag{37}$$

where $P^{(3)}_{\Delta x}$ is defined as

$$\begin{aligned}(P^{(3)}_{\Delta x}(t)u)_j &= A_j(\mathcal{D}^{(2)}_0 v, \mathcal{D}^{(3)}_- w^+, \mathcal{D}^{(3)}_+ w^-)_j \\ &\quad - B_j(D_+D_-u)_j - C_j u_j.\end{aligned}$$

At the boundaries we have the inflow conditions (34) for the hyperbolic part and Dirichlet conditions for the parabolic part

$$\begin{aligned} v_j(0) &= 0, & j &= 0, 1, \dots J, \\ v_0(t) &= v_{k,J} = 0, & t &\in [0, T]. \end{aligned} \tag{38}$$

The outflow conditions are essentially given by (35), but we have to add the coupling terms

$$\begin{aligned}(w^-_t)_0 &= -(\tilde{A}^-_{21})_0(\mathcal{D}^{(3)}_+ v)_0 - \tilde{\Lambda}^-(t, x_0, \Delta x)(\mathcal{D}^{(3)}_+ w^-)_0 + \tilde{C}^-(t, x_0, \Delta x)u_0, \\ (w^+_t)_J &= -(\tilde{A}^+_{21})_J(\mathcal{D}^{(3)}_+ v)_J - \tilde{\Lambda}^+(t, x_J, \Delta x)(\mathcal{D}^{(3)}_- w^+)_J + \tilde{C}^+(t, x_J, \Delta x)u_J\end{aligned} \tag{39}$$

Let H be the norm–matrix defined in (12) and let $\mathcal{H}$ be defined by

$$\mathcal{H} = \mathrm{diag}(I_m, h_1 I_{n-m}, I_m, h_2 I_{n-m}, \dots, I_m, h_J I_{n-m}).$$

We can now prove an energy estimate for the semidiscrete mixed system.

Proposition 5.2. *The semidiscrete problem given by* (37)–(39) *is energy bounded, i.e. there is a constant c such that*

$$\frac{d}{dt}\|u\|_{\mathcal{H}} \le c\|u\|_{\mathcal{H}}.$$

Proof. Differentiating $\|u\|_{\mathcal{H}}$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{\mathcal{H}} &= (u, u_t)_{\mathcal{H}} \\ &= -(u, P^{(3)}_{\Delta x}u)^{(1,J-1)}_{\mathcal{H}} + \Delta x < u_0, (u_t)_0 >_{\mathcal{H}} + \Delta x < u_J, (u_t)_J >_{\mathcal{H}} .\end{aligned}$$

Making use of the boundary conditions, we get

$$\begin{aligned}(u, u_t)_{\mathcal{H}} &= \underbrace{\frac{1}{2}(v, B_{11}D_+D_-v)^{(1,J-1)}}_{I} \underbrace{+\frac{1}{2}(v, B_{11}D_-D_+v)^{(1,J-1)}}_{II} \\ &\quad \underbrace{-(v, A_{11}D_0 v)^{(1,J-1)}}_{III} \underbrace{-\,(v, \tilde{A}^+_{12}\mathcal{D}^{(3)}_- w^+)^{(1,J-1)}}_{IV} \\ &\quad \underbrace{-\,(v, \tilde{A}^-_{12}\mathcal{D}^{(3)}_+ w^-)^{(1,J-1)}}_{V} \underbrace{-(w^-, \tilde{A}^-_{21}\mathcal{D}^{(3)}_+ v)^{(0,J-1)}_H}_{VI} \\ &\quad \underbrace{-(w^+, \tilde{A}^+_{21}\mathcal{D}^{(3)}_- v)^{(1,J)}_H}_{VII} \underbrace{-(w^-, \tilde{\Lambda}^-\mathcal{D}^{(3)}_+ w^-)^{(0,J-1)}_H}_{VIII} \\ &\quad \underbrace{-\,(w^+, \tilde{\Lambda}^+\mathcal{D}^{(3)}_- w^+)^{(1,J)}_H}_{IX} \underbrace{+(u, Cu)_{\mathcal{H}}}_{X}.\end{aligned}$$

For the terms I–III, we have the following estimates, which are proved in [3], Section 4

$$\begin{aligned} I &\leq -\beta(\|D_-v\|^{(1,J)})^2 + c_1\|v\|^{(1,J-1)}\|D_+v\|^{(1,J-1)},\\ II &\leq -\beta(\|D_+v\|^{(0,J-1)})^2 + c_2\|v\|^{(1,J-1)}\|D_-v\|^{(1,J-1)},\\ III &\leq c_3\|v\|^{(1,J-1)}(\|D_+v\|^{(1,J-1)} + \|D_-v\|^{(1,J-1)}). \end{aligned}$$

For $VIII$ and IX, we have the estimates from Proposition 5.1 and for IV, we have

$$\begin{aligned} IV &= (v, \tilde{A}_{12}^+\mathcal{D}_-^{(3)}w^+) = (v, HH^{-1}\tilde{A}_{12}^+\mathcal{D}_-^{(3)}w^+) = (v, (H^{-1}\tilde{A}_{12}^+)\mathcal{D}_-^{(3)}w^+)_H\\ &= (v, \mathcal{D}_-^{(3)}(H^{-1}\tilde{A}_{12}^+)w^+)_H - (v, [\mathcal{D}_-^{(3)}, H^{-1}\tilde{A}_{12}^+]w^+)_H. \end{aligned}$$

With Lemma 5.1, the summation by parts formula (10), the boundary conditions, and by Schwarz' inequality, we get

$$IV \leq c_4\|\mathcal{D}_+^{(3)}v\|_H\|w^+\|_H + c_5\|v\|_H\|w^+\|_H$$

and by Lemma 5.2

$$IV \leq c_6(\|D_-v\|^{(1,J)} + \|D_+v\|^{(0,J-1)})\|w^+\|_H + c_7\|v\|_H\|w^+\|_H.$$

Analogously, we have

$$V \leq c_8(\|D_-v\|^{(1,J)} + \|D_+v\|^{(0,J-1)})\|w^+\|_H + c_9\|v\|_H\|w^-\|_H.$$

Finally, it follows from the equivalence of the norms $\|\cdot\|_H$ and $\|\cdot\|_2$ that

$$IV + V \leq c_{10}\|w\|_H(\|D_-v\|^{(1,J)} + \|D_+v\|^{(0,J-1)} + \|v\|^{(1,J-1)}).$$

With Lemma 5.2, we get for VI and VII

$$\begin{aligned} VI &\leq c_{11}\|w^-\|_H^{(0,J-1)}\|\mathcal{D}_+^{(3)}v\|_H^{(0,J-1)} \leq c_{12}\|w^-\|_H^{(0,J-1)}\|D_+v\|^{(0,J-1)}\\ VII &\leq c_{13}\|w^+\|_H^{(1,J)}\|\mathcal{D}_-^{(3)}v\|_H^{(1,J)} \leq c_{14}\|w^+\|_H^{(1,J)}\|D_-v\|^{(1,J)} \end{aligned}$$

and finally by Schwarz's inequality

$$X \leq c_{15}\|u\|_{\mathcal{H}}^2.$$

The proof is concluded by applying the algebraic inequality

$$ab \leq \frac{\sigma^2a^2}{2} + \frac{b^2}{2\sigma^2}, \quad a, b, \sigma \in \mathcal{R}\backslash\{0\}.$$ □

5.4. **Stable Time Integration.** Now, we have to prove that implicit Runge-Kutta methods are stable integrators for energybounded linear semidiscrete problems. First, we show that there exists a unique solution of the time integrator and then we can prove that the scheme is stable in the sense of (9). The general theory of Runge-Kutta methods, as we apply it here, can be found, for example, in [5].

Consider the linear semidiscrete problem

$$u_t + P_{\Delta x}^{(3)}(t)u = 0, \quad u(0) = 0. \tag{40}$$

As a direct consequence of Proposition 5.2, we get the following corollary.

Corollary 5.1 (One-sided Lipschitz condition). *The linear space discretisation operator $P_{\Delta x}^{(3)}(t)$ fulfills the one-sided Lipschitz condition*

$$-(P_{\Delta x}^{(3)}(t)u, u)_{\mathcal{H}} \leq Lip\|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{F}(\Omega_{\Delta x}, \mathcal{R}^n). \tag{41}$$

Proof. From Proposition 5.2, we get that

$$-(P^{(3)}_{\Delta x}(t)u, u)_{\mathcal{H}} = (u_t, u)_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}\|u\|^2_{\mathcal{H}} \leq Lip\|u\|^2_{\mathcal{H}}. \qquad \square$$

The system (40) is solved by the linear Runge-Kutta method

$$\begin{gathered} u_0 = z|_{\Delta x}, \\ \frac{u_{k_l} - u_{k-1}}{\Delta t} + \sum_{m=1}^{s} a_{l,m} P^{(3)}_{\Delta x}(t_{k_m}) u_{k_m} = 0, \\ \frac{u_k - u_{k-1}}{\Delta t} + \sum_{l=1}^{s} b_l P^{(3)}_{\Delta x}(t_{k_l}) u_{k_l} = 0, \\ k = 1, 2, \ldots, K, \end{gathered} \tag{42}$$

for which we have the following existence and uniqueness result, cf. [5], Theorem IV.14.2 and Theorem IV.14.4.

Theorem 5.1. *Consider the system of ordinary differential equations* (40) *which satisfies the one-sided Lipschitz conditions* (41). *If the Runge-Kutta matrix A of the method* (42) *is invertible and*

$$\Delta t Lip < \alpha_0(A^{-1}), \tag{43}$$

then there exists a unique solution of (42).

We remark that the restriction of the time step that is given by (43) is independent of the spatial grid parameter Δx.

Corollary 5.2. *The time-step can be chosen small enough $\Delta t \leq \alpha_0(A^{-1})/Lip$, such that the Radau IIA method* (16), *applied to* (42), *possesses a unique solution.*

Proof. For the Radau IIA method it holds that $\alpha_0(A^{-1}) = 5/(4+\sqrt{6}) > 0$, cf. [5], Theorem IV.14.5. Thus, Δt can always be chosen small enough to fulfill (43). $\square$

Now, we can prove that an algebraically stable Runge-Kutta method with positive coercivity coefficient is stable in the sense of (9).

Proposition 5.3. *An algebraically stable Runge-Kutta method with invertible coefficient matrix A is a stable time integrator for the energy bounded space discretisation* (42), *if $\Delta t\, Lip < \alpha_0(A^{-1})$.*

Proof. From Theorem 5.1, we know that there is a solution operator R_k such that the solution on the k-th time level can be written as

$$u_k = R_k u_{k-1}.$$

It is proved in [5], Proposition IV.15.2 that R_k is bounded as

$$\|R_k\|_{\mathcal{H}} \leq 1 + c_1\Delta t, \quad \text{for } \Delta t\, Lip < \alpha_0(A^{-1}),$$

from which it follows that the overall scheme is stable. $\square$

Corollary 5.3. *The Radau IIA method is a stable time integrator for* (40).

6. Proof of the Main Results

Proof of Theorem 3.1. The method (13)–(16) is continuously differentiable. Its derivative is Lipschitz continuous with a Lipschitz constant $Lip_{\Delta x} = O(\Delta x^{-2.5})$, $\Delta x \to 0$. Furthermore, the scheme linearised at u^{pi} is stable in the sence of (9), cf. Corollary 5.3. In Section 4, we constructed a pilot function, which is of third order consistent with the semi-discrete system (14). However, it is not obvious that u^{pi} is third order consistent with the full-discrete scheme as the theory for stiff ordinary differential equations requires a one-sided Lipschitz condition for the nonlinear operator $P^{(3)}_{\Delta x}$, cf. [5]. We have only proved a one-sided Lipschitz condition for the linearised problem and can therefore proceed as follows, cf. [2], Lemma 8.2.

From linearised stability and consistency of the first equation in (42) and by the theory of López-Marcos and Sanz-Serna [8], we get consistency of the nonlinear Runge-Kutta method, i.e.

$$\frac{u_k - u_{k-1}}{\Delta t} + \sum_{l=1}^{s} b_l P^{(3)}_{\Delta x}(u_{k_l}) = u_t + P^{(3)}_{\Delta x}(u) + O(\Delta t^r)$$

where $r = 3$ is the stage-order of the Radau IIA method and $u = u(t)$ is any smooth function. Especially for the pilot function, it follows

$$\frac{u^{pi}_k - u^{pi}_{k-1}}{\Delta t} + \sum_{l=1}^{s} b_l P^{(3)}_{\Delta x}(u^{pi}_{k_l}) = u^{pi}_t + P^{(3)}_{\Delta x}(u^{pi}) + O(\Delta t^3) = O(\Delta x^3),$$

therefore

$$\|\Phi_h(u^{pi}|_h)\|^h_{\infty,2} = O(\Delta x^3)$$

and hence, a unique numerical solution U exists with

$$\|U - u^{pi}|_h\|^h_{\infty,2} = O(\Delta x^3).$$

The pilot function converges by construction at second order to the analytical solution and the sought convergence result follows

$$\|U - \tilde{u}|_h\|^h_{\infty,2} \le \|U - u^{pi}|_h\|^h_{\infty,2} + \|u^{pi}|_h - \tilde{u}|_h\|^h_{\infty,2} = O(\Delta x^2), \quad \Delta x \to 0. \qquad \square$$

7. Conclusion

An argument to prove convergence of higher order schemes for nonlinear initial boundary value problems of mixed type was introduced. We have demonstrated, that the technique is applicable to a scheme with the following features:

The convective terms are discretised by a noncentered dissipative formula with special treatment of the boundaries, such that a summation by parts formula holds. The difference formula is third order consistent in the interiour but only first order at the boundary. The viscous terms are discretised by standard second order differences. For time integration a Radau IIA Runge–Kutta method is applied.

The main result is that the overall scheme converges at second order to the true solution of the initial boundary value problem. Some comments on this result are appropriate:

The fact that we get second order accuracy for the non-compact scheme, even though the boundary conditions in the hyperbolic part are only of first order, is in accordance with the classical theory. For ordinary differential equations, this phenomenon is well-known, and for linear initial-boundary value problems to hyperbolic systems, such results can be found in [4].

To obtain a convergence order more than two, a higher order formula for the viscous terms would be neccesary. Then artificial boundary conditions for the viscous component must be constructed and additional boundary layers will be introduced.

Nevertheless, independent of the space discretisation, one can not expect to observe the full classical convergence order of the implicit Runge-Kutta method. The effect of order reduction is well known in the literature [5]. In the worst case the convergence order can drop down to the stage order of the RK-method. The order reduction may be overcome by using multistep methods for time integration, but then additional initial layers will be introduced.

The stage order three of the Radau IIA method was neccesary in order to construct a third order pilot function.

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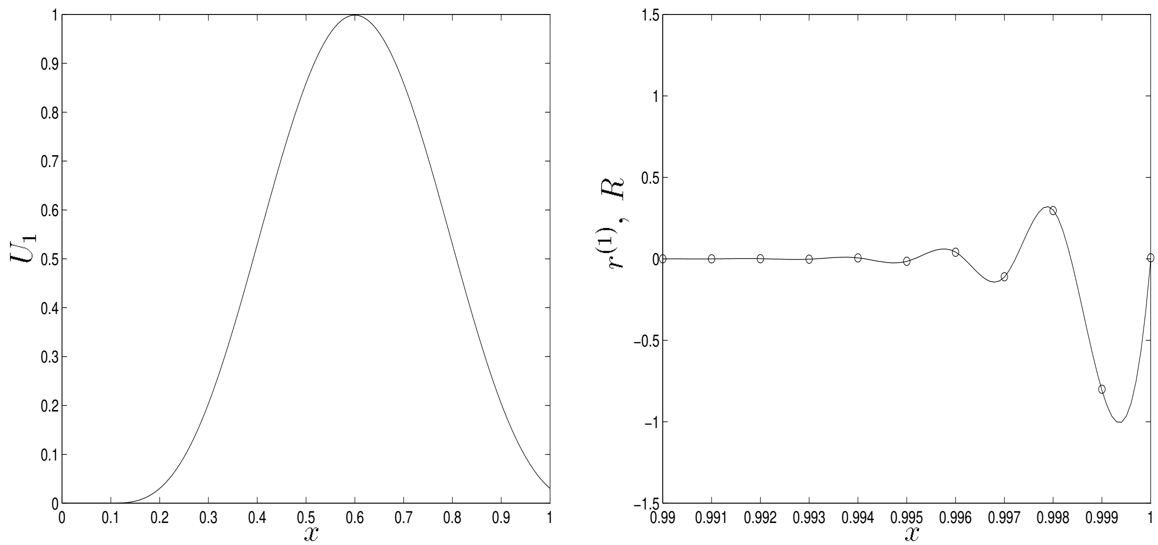

FIGURE 1. The solution U_1 and the boundary layers $r^{(1)}$ and R at time $t = 0.1$.

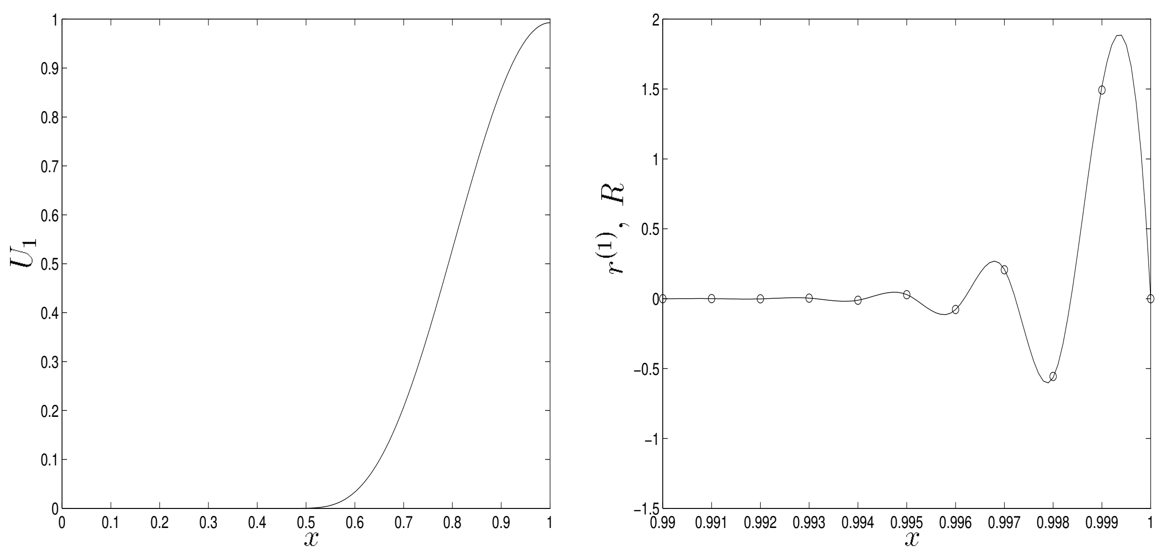

FIGURE 2. The solution U_1 and the boundary layers $r^{(1)}$ and R at time $t = 0.5$.

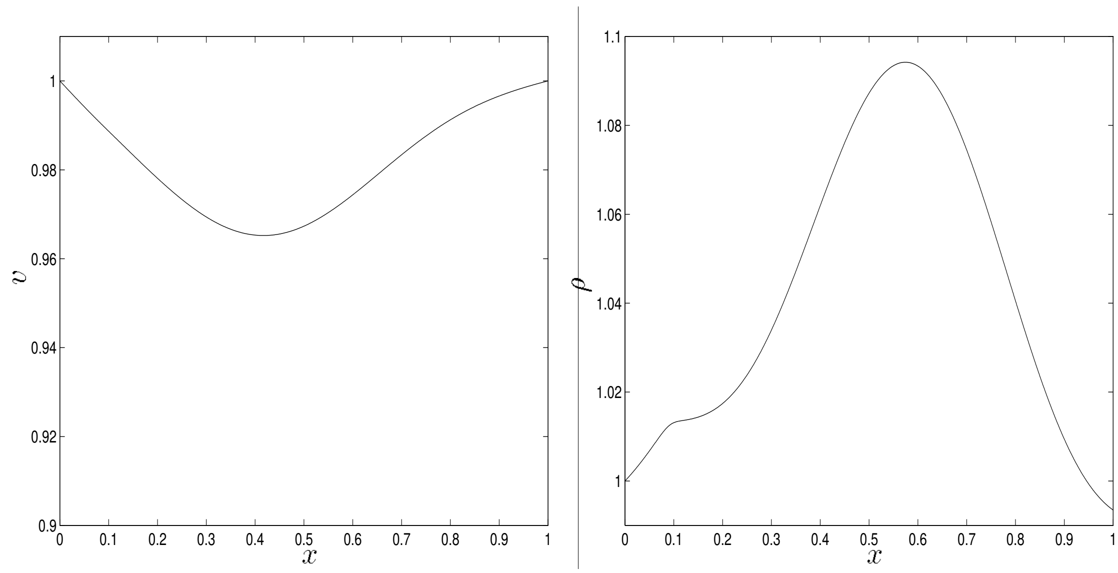

FIGURE 3. The v- and the ρ-component of the solution at $t = 0.1$.

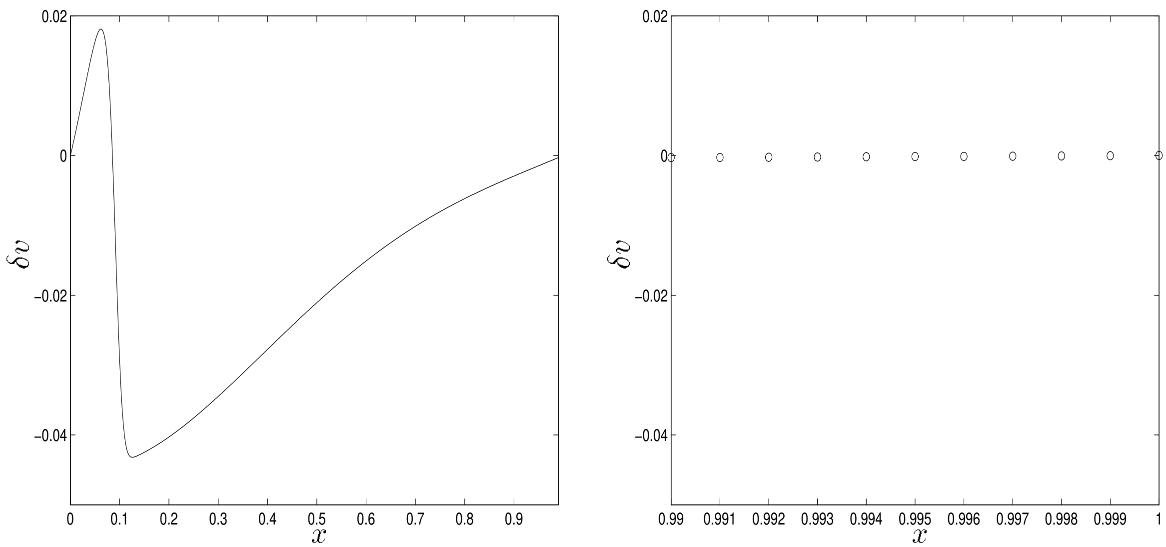

FIGURE 4. At $t = 0.1$: v-component of the second order error terms. No right boundary layer exists (right picture).

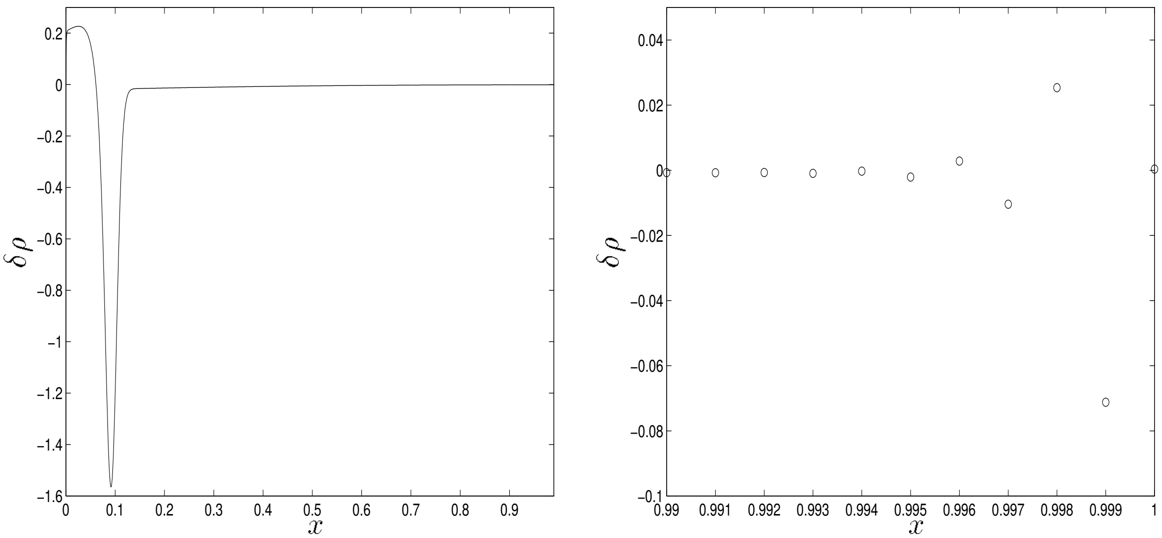

FIGURE 5. ρ-component: Superposition of the left boundary layer and second order error term, $\rho^{(2)}$ (left picture), and the right boundary layer, at $t = 0.1$.

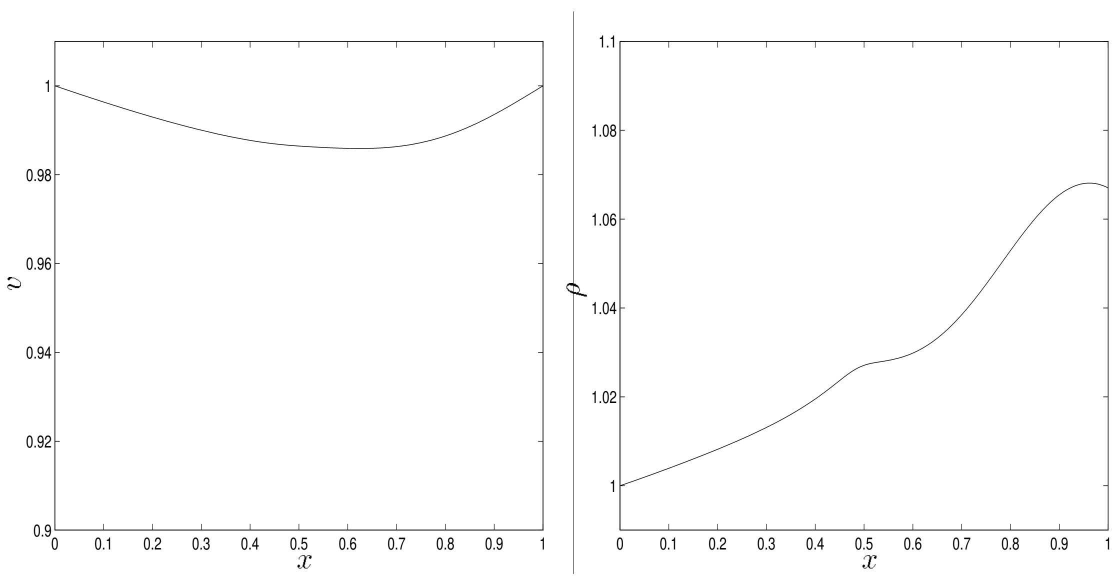

FIGURE 6. The v- and the ρ-component of the solution at $t = 0.5$.

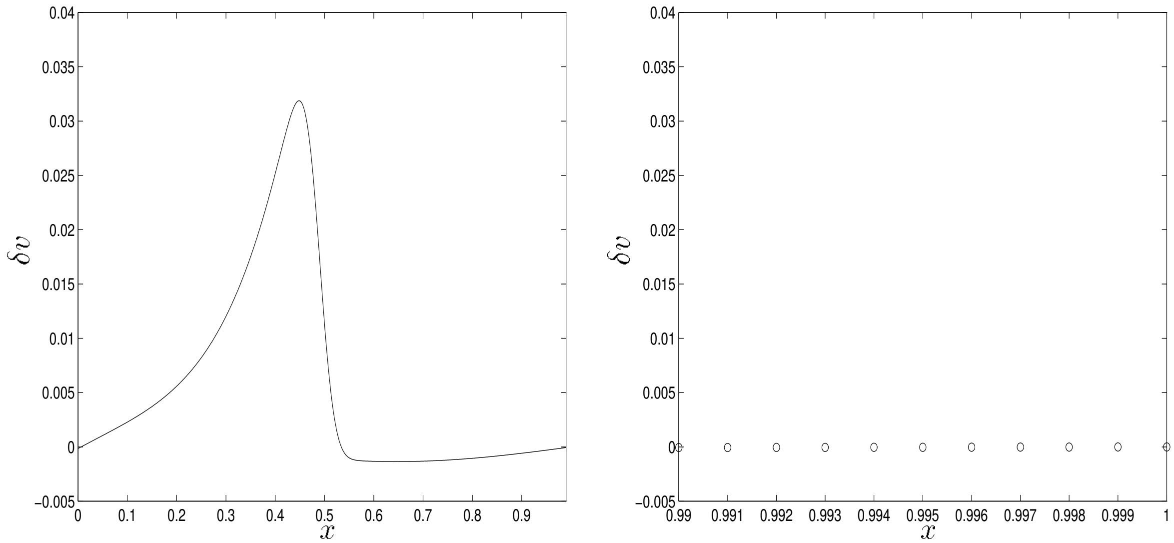

FIGURE 7. At $t = 0.5$: v-component of the second order error terms. No right boundary layer exists (right picture).

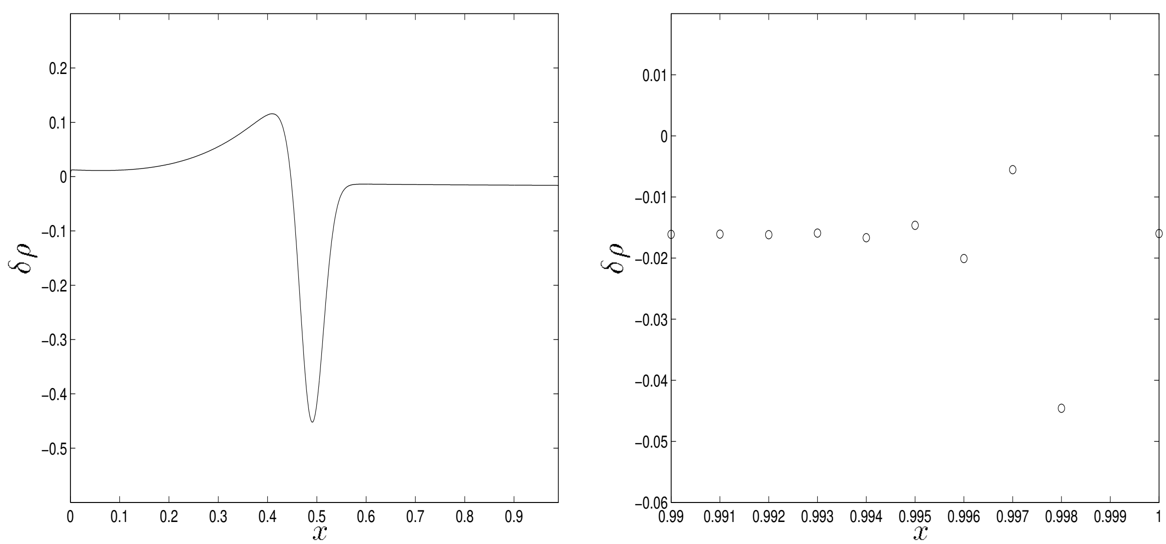

FIGURE 8. ρ-component: Superposition of the left boundary layer and second order error term, $\rho^{(2)}$ (left picture), and the right boundary layer, at $t = 0.5$.