The Semigroup generated by a Temple Class System with Large Data

Paolo Baiti and Alberto Bressan

S.I.S.S.A., Via Beirut 4, Trieste 34014, Italy

Abstract: We consider the Cauchy problem

$$u_t + \big[F(u)\big]_x = 0, \qquad u(0,x) = \bar{u}(x) \tag{$*$}$$

for a nonlinear $n \times n$ system of conservation laws with coinciding shock and rarefaction curves. Assuming the existence of a coordinates system made of Riemann invariants, we prove the existence of a weak solution of $(*)$ that depends in a lipschitz continuous way on the initial data, in the class of functions with arbitrarily large but bounded total variation.

1 - Introduction

We consider the Cauchy problem for the $n \times n$ system of conservation laws in one space dimension

$$u_t + \big[F(u)\big]_x = 0, \tag{1.1}$$

$$u(0,x) = \bar u(x), \tag{1.2}$$

where $F : \Omega \subset I\!R^n \mapsto I\!R^n$ is sufficiently smooth. Our basic assumptions will be

(H1) the system is strictly hyperbolic, with each characteristic field either linearly degenerate or genuinely nonlinear;

(H2) shock and rarefaction curves coincide;

(H3) as u ranges in Ω, there exists a system of coordinates $v = (v_1, \ldots, v_n)(u)$ consisting of Riemann invariants.

Consider a set $E \subset \Omega$ having the form

$$E = \Big\{ u \in \Omega; \quad v_i(u) \in [a_i,\ b_i] \ \ i = 1, \ldots, n \Big\}, \tag{1.3}$$

and assume that, as u varies in E, the characteristic speeds $\lambda_i(u)$, $i = 1, \ldots, n$ range over disjoint intervals, say $\big[\lambda_i^{min},\ \lambda_i^{max}\big]$.

Systems which satisfy the hypotheses (H1)–(H3) were studied in [12, 13]. With these assumptions it is well known that, for any initial data $\bar u \in \mathbf{BV}$ taking values in E, the Cauchy problem (1.1), (1.2) has a weak entropic solution defined for all $t \geq 0$, still taking values inside E [11]. For some related uniqueness results, see [5, 8]. The purpose of this paper is to prove the existence of solutions which depend on the initial data in a Lipschitz continuous way, w.r.t. the $\mathbf{L}^1$ norm. The Lipschitz constant depends only on the bound on the total variation.

More precisely, fix any point $\tilde u \in E$. By a translation in the Riemann coordinates, is not restrictive to assume that $(v_1, \ldots, v_n)(\tilde u) = (0, \ldots, 0)$. For $M > 0$, define the family of functions

$$\mathcal{D}_M \doteq \left\{ u : I\!R \mapsto E; \quad v(u) \in \mathbf{L}^1, \ \ \sum_i \mathrm{Tot.Var.}\big(v_i(u)\big) \leq M \right\}. \tag{1.4}$$

Our main result is the following.

Theorem 1. *For each $M > 0$ there exists a Lipschitz constant C_M and a continuous semigroup $S : [0, +\infty[\times \mathcal{D}_M \mapsto \mathcal{D}_M$ such that*

i) $S_0 \bar u = \bar u$, $S_t S_s \bar u = S_{t+s} \bar u$;

ii) $\|S_t\bar{u} - S_s\bar{v}\|_{\mathbf{L}^1} \le C_M\big(|t-s| + \|\bar{u}-\bar{v}\|_{\mathbf{L}^1}\big)$*;*

iii) each trajectory $t \mapsto S_t\bar{u}(\cdot)$ *is a weak solution of the Cauchy problem* (1.1)-(1.2)*;*

iv) if $\bar{u} \in \mathcal{D}_M$ *is a piecewise constant function, then, for* t *small,* $S_t\bar{u}$ *coincides with the function obtained by piecing together the solutions of the Riemann problems determined by the jumps of* $\bar{u}$*.*

From the results in [3] it follows that the above semigroup is unique and that its trajectories can be characterized as "Viscosity solutions" in terms of a family of local integral estimates. Moreover, every solution of the Cauchy problem (1.1), (1.2) obtained by the Glimm scheme [7] or by wave-front tracking [1, 6, 10] actually coincides with the semigroup trajectory $u(t,\cdot) = S_t\bar{u}$.

The existence of a semigroup for 2×2 systems with coinciding shock and rarefaction curves was first proved in [2], but only for a class of functions with small total variation.

Our approach is similar to that in [2]: we construct a sequence of uniformly Lipschitz approximate semigroups defined on certain domains of piecewise constant functions and obtain S in the limit. More precisely we prove that these semigroups are contractive w.r.t. a suitable weighted distance, uniformly equivalent to the standard $\mathbf{L}^1$ metric.

As in [2, 4], this weighted distance is defined as

$$\delta(u,u') \doteq \inf\left\{ \|\gamma\|_W : \gamma \text{ is a pseudopolygonal joining } u \text{ with } u' \right\}, \tag{1.5}$$

for a particular choice of the weighted length $\|\gamma\|_W$. By a *pseudopolygonal* we mean here a finite concatenation of *elementary paths*, of the form

$$\theta \mapsto u^\theta \doteq \sum_{\alpha=1}^{N} \omega_\alpha\, \chi_{]x^\theta_{\alpha-1},\, x^\theta_\alpha]}, \qquad x^\theta_\alpha = x_\alpha + \xi_\alpha\theta, \qquad \theta\in[a,b] \tag{1.6}$$

where χ_I is the characteristic function of the set I, $\omega_0,\dots,\omega_N \in I\!\!R^n$ are constant states and ξ_α is the shift rate of the jump at x_α. In (1.6) it is assumed that $x^\theta_1 < \ldots < x^\theta_N$ for $a<\theta<b$. If γ is an elementary path of the form (1.6), its $\mathbf{L}^1$-length is computed by

$$\|\gamma\|_{\mathbf{L}^1} = \int_a^b \sum_{\alpha=1}^{N} \big|\Delta u(x_\alpha)\big| \left|\frac{\partial x^\theta_\alpha}{\partial\theta}\right| d\theta = \sum_{\alpha=1}^{N} \big|\Delta u(x_\alpha)\big|\,|\xi_\alpha|\,(b-a), \tag{1.7}$$

where $\Delta u(x_\alpha) = \omega_{\alpha+1} - \omega_\alpha$. We will construct an equivalent metric of the form:

$$\|\gamma\|_W = \int_a^b \sum_{\alpha=1}^{N} \big|\Delta u(x_\alpha)\big| \left|\frac{\partial x^\theta_\alpha}{\partial\theta}\right| W_\alpha(u)\, d\theta \tag{1.8}$$

for suitable choices of the weights W_α. We recall that, in all previous works in this direction, the weights W_α always had the form

$$\begin{aligned} W_\alpha \doteq{} & 1 + C_1\cdot\big[\text{strength of all waves approaching the wave-front at } x_\alpha\big] + \\ & + C_2\cdot\big[\text{global wave interaction potential}\big] \end{aligned}$$

for some constants C_1, C_2. This choice, however, is successful only in the case of small total variation and cannot be used here. The main novelty of the present paper lies in the construction of the weights W_α, which is performed by backward induction, relative to the wave-front configuration of each approximate solution. The key step in the proof is the analysis in Section 4, which establishes an a-priori bound on these weights, depending only on the total variation. As a consequence, our weigthed distances remain uniformly equivalent to the usual $\mathbf{L}^1$ distance. This yields the continuity of the semigroup $S : \mathcal{D}_M \times [0, \infty[\mapsto \mathcal{D}_M$, with a Lipschitz constant depending only on the total variation of functions in $\mathcal{D}_M$.

2 - Construction of approximate solutions

By strict hyperbolicity, for every u the Jacobian matrix $A(u) = DF(u)$ has n real and distinct eigenvalues $\lambda_1(u) < \ldots < \lambda_n(u)$. For $u, u' \in E$, consider the averaged matrix

$$A(u, u') = \int_0^1 A\big(su' + (1-s)u\big)\, ds \tag{2.1}$$

and call $\lambda_1(u, u') < \ldots < \lambda_n(u, u')$ the corresponding eigenvalues. Throughout the following we shall use a fixed system of Riemann coordinates $v = (v_1, \ldots, v_n)$, and simply write $\lambda_i(v, v')$ in place of $\lambda_i\big(u(v),\ u(v')\big)$.

For each integer $\nu \geq 1$ we shall construct a semigroup S^ν of approximate solutions, defined on a set $\mathcal{D}_M^\nu \subset \mathcal{D}_M$ of piecewise constant functions. As $\nu \to \infty$, the Lipschitz constant of S^ν remains uniformly bounded, while the domains $\mathcal{D}_M^\nu$ become dense in $\mathcal{D}_M$. In the limit, a semigroup S will be obtained, satisfying all required properties.

Recalling (1.3), consider the box $E' \doteq [a_1, b_1] \times \cdots \times [a_n, b_n]$. Fix $\nu \geq 1$ and define the finite grid

$$\mathcal{G}^\nu \doteq 2^{-\nu} \mathbb{Z}^n \cap E'.$$

As domain of the approximating semigroup S^ν we choose

$$\mathcal{D}_M^\nu \doteq \left\{ v : \mathbb{R} \mapsto \mathcal{G}^\nu; \quad \sum_{i=1}^n \text{Tot.Var.}(v_i) \leq M \right\}. \tag{2.2}$$

Clearly, any function $v \in \mathcal{D}_M^\nu$ is piecewise constant with $\leq 2^\nu M$ jumps. In order to describe the flow of S^ν, it suffices to specify how each Riemann problem is solved.

Let $v^-, v^+ \in \mathcal{G}^\nu$ be the initial data for a standard Riemann problem. An approximate solution, within the class of functions taking values inside $\mathcal{G}^\nu$, is constructed as follows. Consider the intermediate states

$$\omega_0 = v^-, \qquad \ldots, \qquad \omega_i = (v_1^+, \ldots, v_i^+, v_{i+1}^-, \ldots, v_n^-), \qquad \ldots, \qquad \omega_n = v^+. \tag{2.3}$$

Set

$$\sigma_i \doteq v_i^+ - v_i^- \in 2^{-\nu}\mathbb{Z}. \tag{2.4}$$

We call σ_i the *size* of the i-th wave generated by the Riemann problem (v^-, v^+).

A shock (or a contact discontinuity) will be propagated as single wave-front, while a centered rarefaction wave will be partitioned along the nodes of the grid $\mathcal{G}^\nu$ and propagated as a rarefaction fan. In the following, p_i denotes the number of pieces in which the i-th wave is partitioned.

More precisely, if the i-th characteristic field is linearly degenerate, or if it is genuinely nonlinear and $\sigma_i < 0$, we then set $p_i = 1$ and define the shock speed $\lambda_{i,1} = \lambda_i(\omega_{i-1}, \omega_i)$. On the other hand, if the i-th characteristic field is genuinely nonlinear and $\sigma_i \geq 0$, we then set $p_i = 2^\nu \sigma_i$ and define the intermediate states $\omega_i^0 = \omega_{i-1}, \omega_i^1, \ldots, \omega_i^{p_i} = \omega_i$ to be precisely the points on the segment connecting ω_{i-1} with ω_i which also lie on the grid $\mathcal{G}^\nu$. In this case, we define the speeds of the corresponding rarefaction fronts as $\lambda_{i,h} = \lambda_i(\omega_i^{h-1}, \omega_i^h)$, for $h = 1, \ldots, p_i$.

The ν-approximate solution of the Riemann problem with data (v^-, v^+) can now be defined as

$$v(t,x) = \begin{cases} v^- & \text{if} \quad x < t\lambda_{1,1}, \\ \omega_i & \text{if} \quad t\lambda_{i-1,p_{i-1}} < x < t\lambda_{i,1}, \qquad i = 0, \ldots, n-1, \\ \omega_i^h & \text{if} \quad t\lambda_{i,h-1} < x < t\lambda_{i,h}, \qquad h = 0, \ldots, p_i - 1, \\ v^+ & \text{if} \quad t\lambda_{n,p_n} < x. \end{cases} \tag{2.5}$$

For every initial data $\bar{v} \in \mathcal{D}_M^\nu$, a ν-approximate solution $v = v(t,x)$ can now be constructed by a wave-front tracking method, as follows. At time $t = 0$ we solve the Riemann problems determined by the jumps in $\bar{v}$ according to (2.5). Patching together these local solutions, we obtain a piecewise constant function v defined up to the first time t_1 where two or more wave-fronts interact. At each point of interaction, the corresponding Riemann problems are again solved according to (2.5). The solution is then prolonged up to a time t_2 where the second set of interactions takes place, etc. This solution will be denoted as

$$v(t,\cdot) = S_t^\nu \bar{v}.$$

Observe that the total variation of $v(t,\cdot)$ (always measured w.r.t. the Riemann coordinates) coincides with the total strength of waves, and is non-increasing in time. Hence, $v(t,\cdot) \in \mathcal{D}_M^\nu$ for all $t \geq 0$. Moreover, the number of wave-fronts in $v(t,\cdot)$ is also non-increasing, at each interaction.

Thanks to the assumption that shock and rarefaction curves coincide, our choices of the wave speeds imply that all jumps in v satisfy the Rankine-Hugoniot conditions. Hence, every ν-approximate solution v is in fact a weak solution of (1.1). However, in the presence of genuinely nonlinear fields, the corresponding rarefaction fronts do not satisfy the usual entropy-admissibility conditions. Since these fronts have strength $2^{-\nu}$, as $\nu \to \infty$ we shall obtain a semigroup of entropy-admissible solutions $S : \mathcal{D}_M \times [0, \infty[\mapsto \mathcal{D}_M$, in the limit.

The main issue here is the Lipschitz continuity of the semigroup $S = \lim S^\nu$. This will be proved by providing a Lipschitz constant uniformly valid for all S^ν. For this purpose, given any two initial conditions $\bar v, \bar v' \in \mathcal{D}^\nu_M$, consider a pseudopolygonal $\gamma_0 : \theta \mapsto \bar v^\theta$, with $\gamma_0(0) = \bar v$, $\gamma_0(1) = \bar v'$. If $\bar v, \bar v'$ both have support inside some interval $[a, b]$, a simple example of such a path is

$$\bar v^\theta \doteq \bar v \cdot \chi_{[a,\lambda]} + \bar v' \cdot \chi_{]\lambda,b]} \qquad \lambda \doteq \theta b + (1-\theta)a. \tag{2.6}$$

Let $v^\theta(t,\cdot) = S^\nu_t \bar v^\theta$ be the corresponding solutions. Since the number of wave-fronts in these solutions is a-priori bounded and the locations of the interaction points in the t-x plane are determined by a linear system of equations, it is clear that, at any time $\tau > 0$, the corresponding path $\gamma_\tau : \theta \mapsto v^\theta(\tau,\cdot)$ is still a pseudopolygonal. Moreover, there exist finitely many parameter values $0 = \theta_0 < \theta_1 < \cdots < \theta_\nu = 1$ such that the wave-front configuration of v^θ remains the same as θ ranges on each of the open intervals $I_j \doteq]\theta_{j-1}, \theta_j[$. In this case, the lengths of the paths γ_0 and γ_τ are measured by an expression of the form

$$\|\gamma\|_{\mathbf{L}^1} = \sum_{j=1}^{\nu} \int_{\theta_{j-1}}^{\theta_j} \sum_\alpha \left|\Delta v^\theta(x^\theta_\alpha)\right| \left|\frac{\partial x^\theta_\alpha}{\partial\theta}\right| d\theta. \tag{2.7}$$

By carefully studying how the integrand in (2.7) varies in time, we will prove that $\|\gamma_\tau\|_{\mathbf{L}^1} \leq L\|\gamma_0\|_{\mathbf{L}^1}$, for some constant L depending only on the total variation. In turn, this will provide a uniform Lipschitz constant for the semigroups S^ν.

3 - Interaction estimates

In this section we collect some basic estimates relating the speeds and shifts of wave-fronts before and after an interaction. In the following, we always consider wave-fronts of some ν-approximate solution constructed as in Section 2. Wave strengths will be measured as in (2.4), referring to the system of Riemann coordinates $v = (v_1, \ldots, v_n)$.

Lemma 1. *Assume that N wave-fronts belonging to different families $i_1 > \cdots > i_N$, of sizes $\sigma_1, \ldots, \sigma_N$, interact together at a single point. Call $\lambda^-_\alpha, \lambda^+_\alpha$ the speeds of the α-th wave respectively before and after the interaction. Then for all $\alpha = 1, \ldots, N$ these speeds satisfy:*

$$|\lambda^+_\alpha - \lambda^-_\alpha| \leq C_1 \sum_{\beta\neq\alpha} |\sigma_\beta|, \tag{3.1}$$

for a suitable constant C_1.

Lemma 2. *Assume that two interacting wave-fronts, both of the i-th genuinely nonlinear family, have sizes σ', σ'', respectively. Then, for some constant C_2, their speeds λ', λ'' satisfy:*

$$|\lambda' - \lambda''| \geq C_2\big(|\sigma'| + |\sigma''|\big). \tag{3.2}$$

Proof. Indeed, the two incoming waves may be both shocks, or else one is a shock (say, of size σ') and the other is a rarefaction (of size σ''). In this second case we must have $\sigma'' = 2^{-\nu}$, while $|\sigma'| \geq 2^{1-\nu}$. Indeed, if also $|\sigma'| = 2^{-\nu}$, the wave-fronts would have exactly the same speed, and could not interact. The estimate (3.2) is now a straightforward consequence of the genuine nonlinearity of the i-th family.

Lemma 3. *Assume that two waves of the (genuinely nonlinear) i-th family, of sizes σ', σ'', interact and produce an outgoing i-wave of size $\sigma^+ = \sigma' + \sigma''$. Call $\lambda', \lambda'', \lambda^+$ the speeds of the waves $\sigma', \sigma'', \sigma^+$, respectively. Then, for some constant C_3, one has the estimate*

$$\left|\lambda^+ - \frac{\lambda'\sigma' + \lambda''\sigma''}{\sigma' + \sigma''}\right| \leq C_3|\sigma'\sigma''|. \tag{3.3}$$

Proof. Define $\phi_i(v,\sigma) \doteq v + \sigma r_i$, where we set $r_i = e_i$ the i-th vector of the canonical basis, and $\lambda_i(v,\sigma) \doteq \lambda_i\big(v, \phi_i(v,\sigma)\big)$. Call v^l, v^m, v^r respectively the left, middle and right state before the interaction. Then $\lambda' = \lambda_i(v^l,\sigma')$, $\lambda'' = \lambda_i(v^m,\sigma'')$ and $\lambda^+ = \lambda_i(v^l,\sigma^+)$. Moreover we observe that $\sigma^+ < 0$. Writing $r_i \bullet \lambda_i$ for the directional derivative of λ_i in the direction of r_i, one has

$$\begin{aligned} \lambda_i(v,\sigma) &= \lambda_i(v) + \frac{\sigma}{2}(r_i \bullet \lambda_i)(v) + O(1)\cdot\sigma^2, \\ \frac{\partial}{\partial\sigma}\lambda_i(v,\sigma) &= \frac{1}{2}(r_i \bullet \lambda_i)(v) + O(1)\cdot\sigma. \end{aligned} \tag{3.4}$$

Observe that either

(i) the incoming waves are both shocks; or

(ii) the incoming waves are a shock and a rarefaction.

If (i) holds, then $\sigma', \sigma'' < 0$ and $|\sigma' + \sigma''| = |\sigma'| + |\sigma''|$. If (ii) holds, assume that $\sigma' < 0$, $\sigma'' > 0$, the other case being entirely similar. We then have $\sigma' = -h2^{-\nu}$ for some integer $h \geq 2$, while $\sigma'' = 2^{-\nu}$. Therefore

$$|\sigma' + \sigma''| \geq \frac{1}{3}\big(|\sigma'| + |\sigma''|\big). \tag{3.5}$$

Now consider two cases

CASE 1: $|\sigma'| \geq |\sigma''|$. Then by (3.4) we have

$$\begin{aligned} &\left|\lambda_i(v^l,\sigma' + \sigma'') - \frac{\sigma'\lambda_i(v^l,\sigma') + \sigma''\lambda_i(v^m,\sigma'')}{\sigma' + \sigma''}\right| \\ &\quad= \left|\lambda_i(v^l,\sigma') + \int_{\sigma'}^{\sigma'+\sigma''} \frac{\partial}{\partial\sigma}\lambda_i(v^l,\sigma)\,d\sigma - \frac{\sigma'\lambda_i(v^l,\sigma') + \sigma''\lambda_i(v^m,\sigma'')}{\sigma' + \sigma''}\right| \\ &\quad= \left|\int_{\sigma'}^{\sigma'+\sigma''} \frac{\partial}{\partial\sigma}\lambda_i(v^l,\sigma)\,d\sigma + \frac{\sigma''}{\sigma' + \sigma''}\big(\lambda_i(v^l,\sigma') - \lambda_i(v^m,\sigma'')\big)\right| \\ &\quad= \left|\frac{\sigma''}{2}(r_i \bullet \lambda_i)(v^l) + O(1)\cdot\big(\sigma''(\sigma' + \sigma'')\big) + \frac{\sigma''}{\sigma' + \sigma''}\big(\lambda_i(v^l,\sigma') - \lambda_i(v^m,\sigma'')\big)\right| \end{aligned}$$

$$= \left| \frac{\sigma''}{2}(r_i \bullet \lambda_i)(v^l) + \frac{\sigma''}{\sigma' + \sigma''}\big(\lambda_i(v^l, \sigma') - \lambda_i(v^m, \sigma'')\big) \right| + O(1) \cdot |\sigma'\sigma''|. \tag{3.6}$$

Moreover

$$\begin{aligned} \lambda_i(v^m, \sigma'') &= \lambda_i\big(\phi_i(v^l, \sigma'), \sigma''\big) \\ &= \lambda_i\big(\phi_i(v^l, \sigma')\big) + \frac{\sigma''}{2}(r_i \bullet \lambda_i)\big(\phi_i(v^l, \sigma')\big) + O(1) \cdot |\sigma''|^2 \\ &= \lambda_i(v^l) + \sigma'(r_i \bullet \lambda_i)(v^l) + \frac{\sigma''}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot \big(|\sigma'|^2 + |\sigma'\sigma''| + |\sigma''|^2\big) \\ &= \lambda_i(v^l) + \frac{2\sigma' + \sigma''}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot \big(|\sigma'|^2 + |\sigma''|^2\big). \end{aligned} \tag{3.7}$$

Estimates (3.4)-(3.7) yield

$$\begin{aligned} &\left| \lambda_i(v^l, \sigma' + \sigma'') - \frac{\sigma'\lambda_i(v^l, \sigma') + \sigma''\lambda_i(v^m, \sigma'')}{\sigma' + \sigma''} \right| \\ &\quad = \left| \frac{\sigma''}{2}(r_i \bullet \lambda_i)(v^l) + \frac{\sigma''}{\sigma' + \sigma''}\left(- \frac{\sigma' + \sigma''}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot \big(|\sigma'|^2 + |\sigma''|^2\big)\right) \right| \\ &\quad = O(1) \cdot |\sigma''| \frac{|\sigma'|^2 + |\sigma''|^2}{|\sigma' + \sigma''|} = O(1) \cdot |\sigma''| \big(|\sigma'| + |\sigma''|\big) = O(1) \cdot |\sigma'\sigma''|, \end{aligned} \tag{3.8}$$

hence (3.3) holds.

CASE 2: $|\sigma|' < |\sigma''|$. In this case

$$\begin{aligned} &\left| \lambda_i(v^l, \sigma' + \sigma'') - \frac{\sigma'\lambda_i(v^l, \sigma') + \sigma''\lambda_i(v^m, \sigma'')}{\sigma' + \sigma''} \right| \\ &\quad = \left| \lambda_i(v^m, \sigma'') - \int_0^{\sigma'} \left[\frac{d}{ds}\lambda_i\big(\phi_i(v^l, s), \sigma' + \sigma'' - s\big) \right] ds - \frac{\sigma'\lambda_i(v^l, \sigma') + \sigma''\lambda_i(v^m, \sigma'')}{\sigma' + \sigma''} \right| \\ &\quad = \left| - \int_0^{\sigma'} \left[\frac{d}{ds}\lambda_i\big(\phi_i(v^l, s), \sigma' + \sigma'' - s\big) \right] ds + \frac{\sigma'}{\sigma' + \sigma''}\big(\lambda_i(v^m, \sigma'') - \lambda_i(v^l, \sigma')\big) \right|. \end{aligned} \tag{3.9}$$

Now, as in [4], we have

$$\frac{d}{ds}\lambda_i\big(\phi_i(v^l, s), \sigma' + \sigma'' - s\big) = \frac{1}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot \big(|\sigma'| + |\sigma''|\big). \tag{3.10}$$

Moreover,

$$\begin{aligned} &\lambda_i(v^m, \sigma'') - \lambda_i(v^l, \sigma') \\ &\quad = \lambda_i(v^m) + \frac{\sigma''}{2}(r_i \bullet \lambda_i)(v^m) + O(1) \cdot |\sigma''|^2 - \left[\lambda_i(v^l) + \frac{\sigma'}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot |\sigma'|^2\right] \\ &\quad = \lambda_i(v^l) + \frac{2\sigma' + \sigma''}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot \big(|\sigma'|^2 + |\sigma''|^2\big) + \\ &\qquad - \left[\lambda_i(v^l) + \frac{\sigma'}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot |\sigma''|^2\right] \\ &\quad = \frac{\sigma' + \sigma''}{2}(r_i \bullet \lambda_i)(v^l) + O(1) \cdot \big(|\sigma'|^2 + |\sigma''|^2\big). \end{aligned} \tag{3.11}$$

Estimates (3.9), (3.10) and (3.11) yield

$$\begin{aligned}
&\left|\lambda_i(v^l,\sigma' + \sigma'') - \frac{\sigma'\lambda_i(v^l,\sigma') + \sigma''\lambda_i(v^m,\sigma'')}{\sigma' + \sigma''}\right| \\
&\quad = \left| - \int_0^{\sigma'} \left[\frac{1}{2}(r_i \bullet \lambda_i)(v^l) + O(1)\cdot|\sigma''|\right] ds + \frac{\sigma'}{\sigma' + \sigma''}\left(\frac{\sigma' + \sigma''}{2}(r_i \bullet \lambda_i)(v^l) + O(1)\cdot|\sigma''|^2\right)\right| \\
&\quad = O(1)\cdot|\sigma'\sigma''|,
\end{aligned}$$

hence (3.3) again holds. This completes the proof of Lemma 3.

At any given interaction, we now study the relations between the shifts of the incoming and of the outgoing wave-fronts. Consider a one-parameter family of piecewise constant solutions v^θ. Assume that each v^θ contains N incoming wave-fronts, say located along the lines

$$x = x^0 + \Lambda_i(t - t^0) + \xi_i\theta \qquad i = 1, \ldots, N, \tag{3.12}$$

with $\Lambda_1 > \cdots > \Lambda_N$, interacting all together at a single point P^θ. Introducing the vector

$$\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \doteq \frac{\partial P^\theta}{\partial\theta}, \tag{3.13}$$

one easily checks that the shifts ξ_i satisfy the relations

$$\xi_i = \mathbf{v}_2 - \Lambda_i\mathbf{v}_1 \qquad i = 1, \ldots, N. \tag{3.14}$$

An outgoing wave with speed Λ^+ will have a shift ξ^+ computed by

$$\xi^+ = \mathbf{v}_2 - \Lambda^+\mathbf{v}_1 = \frac{(\Lambda^+ - \Lambda_i)\xi_j - (\Lambda^+ - \Lambda_j)\xi_i}{(\Lambda_j - \Lambda_i)}, \tag{3.15}$$

for every distinct indices $i, j \in \{1, \ldots, N\}$. The next two lemmas provide a bound on the shifts of the outgoing waves, in the case where the incoming fronts belong all to different families, or all to the same characteristic family, respectively.

Lemma 4. *Assume that N wave-fronts of different families $i_1 > \cdots > i_N$ interact at a single point. Let the incoming fronts have sizes $\sigma_1, \ldots, \sigma_N$, speeds $\lambda_1^-, \ldots, \lambda_N^-$ and shifts $\xi_1^-, \ldots, \xi_N^-$, respectively. Then, N outgoing wave-fronts will emerge from the interaction, of the same sizes as the incoming ones, but with different speeds $\lambda_1^+, \ldots, \lambda_N^+$ and shifts $\xi_1^+, \ldots, \xi_N^+$. For a suitable constant C_4, the shifts of the outgoing fronts satisfy*

$$|\xi_i^+| \le \left(1 + C_4\sum_{k\neq i}|\sigma_k|\right)|\xi_i^-| + \left(C_4\sum_{k\neq i}|\sigma_k||\xi_k^-|\right). \tag{3.16}$$

Proof. Since all the incoming waves are of different families, for some constant C we have

$$|\lambda_k^- - \lambda_i^-| \geq C. \tag{3.17}$$

From (3.15), (3.1), (3.14) and (3.17) it follows

$$\begin{aligned} |\xi_i^+ - \xi_i^-| = |\lambda_i^+ - \lambda_i^-| \left| \frac{\xi_j^- - \xi_i^-}{\lambda_j^- - \lambda_i^-} \right| &\leq C_3 \left(\sum_{k \neq i} |\sigma_k| \right) \left| \frac{\xi_j^- - \xi_i^-}{\lambda_j^- - \lambda_i^-} \right| \\ = C_3 \sum_{k \neq i} \left(|\sigma_k| \left| \frac{\xi_k^- - \xi_i^-}{\lambda_k^- - \lambda_i^-} \right| \right) &\leq C_4 \sum_{k \neq i} |\sigma_k| \left(|\xi_k^-| + |\xi_i^-| \right). \end{aligned} \tag{3.18}$$

This clearly implies (3.16).

Lemma 5. *Assume that N wave-fronts of the i-th (genuinely nonlinear) family interact all together at a single point. Let these incoming fronts have sizes $\sigma_1, \ldots, \sigma_N$ and shifts $\xi_1, \ldots, \xi_N$, respectively. From the interaction, a single wave-front of the i-th family will then emerge, with size $\sigma^+ = \sum_k \sigma_k \leq 0$ and shift ξ^+, satisfying*

$$|\sigma^+ \xi^+| \leq \sum_{k=1}^{N} \left(|\sigma_k \xi_k| \prod_{\substack{j \neq k \\ j=1,\ldots,N}} \left(1 + C_5 |\sigma_j|\right) \right), \tag{3.19}$$

for some constant C_5.

Proof. Assume first $N = 2$. Call λ_1 and λ_2 the speeds of the interacting waves σ_1, σ_2; call λ^+ the speed of the outgoing wave. From (3.15) it follows that

$$\begin{aligned} |\sigma^+ \xi^+| &= |\sigma_1 + \sigma_2| \left| \frac{(\lambda^+ - \lambda_1)\xi_2 - (\lambda^+ - \lambda_2)\xi_1}{\lambda_2 - \lambda_1} \right| \\ &\leq |\sigma_1 + \sigma_2| \left(\left| \frac{(\lambda^+ - \lambda_1)\xi_2}{\lambda_2 - \lambda_1} \right| + \left| \frac{(\lambda^+ - \lambda_2)\xi_1}{\lambda_2 - \lambda_1} \right| \right). \end{aligned} \tag{3.20}$$

Using (3.2) and (3.3) we obtain

$$\begin{aligned} |\sigma_1 + \sigma_2| \left| \frac{\lambda^+ - \lambda_1}{\lambda_2 - \lambda_1} \right| |\xi_2| &\leq \left(\left| \lambda^+ - \frac{\sigma_1 \lambda_1 + \sigma_2 \lambda_2}{\sigma_1 + \sigma_2} \right| + \left| \frac{\sigma_1 \lambda_1 + \sigma_2 \lambda_2}{\sigma_1 + \sigma_2} - \lambda_1 \right| \right) \left| \frac{\sigma_1 + \sigma_2}{\lambda_2 - \lambda_1} \right| |\xi_2| \\ &\leq \left(C_4 |\sigma_1 \sigma_2| + \left| \frac{\sigma_2}{\sigma_1 + \sigma_2} \right| |\lambda_2 - \lambda_1| \right) \left| \frac{\sigma_1 + \sigma_2}{\lambda_2 - \lambda_1} \right| |\xi_2| \\ &\leq \left(1 + C_5 |\sigma_1|\right) |\sigma_2 \xi_2| \end{aligned} \tag{3.21}$$

and similarly

$$|\sigma_1 + \sigma_2| \left| \frac{\lambda^+ - \lambda_2}{\lambda_2 - \lambda_1} \right| |\xi_1| \leq \left(1 + C_5 |\sigma_2|\right) |\sigma_1 \xi_1|. \tag{3.22}$$

Inserting (3.21) and (3.22) in (3.20) we deduce

$$|\sigma^+\xi^+| \le \big(1 + C_5|\sigma_1|\big)|\sigma_2\xi_2| + \big(1 + C_5|\sigma_2|\big)|\sigma_1\xi_1|. \tag{3.23}$$

Hence (3.19) holds when $N = 2$.

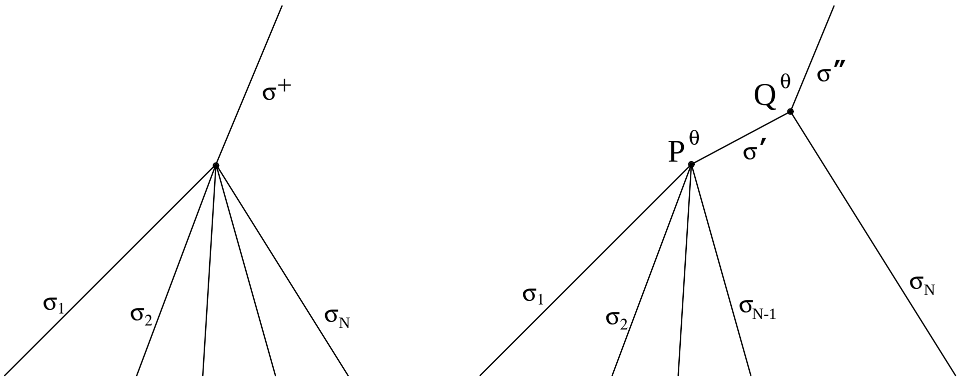

Figure 1

Next, assume that (3.19) is true for every set of $(N-1)$ interacting waves. We will show that the same holds in the case of N wave-fronts. Consider a slightly perturbed configuration, where the first $(N-1)$ wave-fronts interact together at a point P^θ, and then the outgoing front interacts with the last one at Q^θ (fig. 1). Denote by σ', ξ' and σ'', ξ'' the sizes and shifts of the waves emerging from P^θ and Q^θ, respectively. We now have

$$\sigma' = \sum_{k=1}^{N-1} \sigma_k, \qquad \sigma'' = \sigma' + \sigma_N = \sigma^+.$$

Moreover one easily checks that $\partial P^\theta/\partial\theta = \partial Q^\theta/\partial\theta$, hence $\xi'' = \xi^+$. The inductive hypothesis now implies

$$|\sigma'\xi'| \le \sum_{k=1}^{N-1} \left(|\sigma_k\xi_k| \prod_{\substack{j=1,\dots,N-1 \\ j\neq k}} \big(1 + C_5|\sigma_j|\big)\right),$$

hence from (3.23) we get

$$\begin{aligned}
|\sigma^+\xi^+| &= |\sigma''\xi''| \le \big(1 + C_5|\sigma'|\big)|\sigma_N\xi_N| + \big(1 + C_5|\sigma_N|\big)|\sigma'\xi'| \\
&\le \left(1 + C_5\left|\sum_{k=1}^{N-1}\sigma_k\right|\right)|\sigma_N\xi_N| + \big(1 + C_5|\sigma_N|\big)\sum_{k=1}^{N-1}\left(|\sigma_k\xi_k| \prod_{\substack{j=1,\dots,N-1 \\ j\neq k}} \big(1 + C_5|\sigma_j|\big)\right) \\
&\le |\sigma_N\xi_N| \prod_{\substack{j=1,\dots,N \\ j\neq N}} \big(1 + C_5|\sigma_j|\big) + \sum_{k=1}^{N-1}\left(|\sigma_k\xi_k| \prod_{\substack{j=1,\dots,N \\ j\neq k}} \big(1 + C_5|\sigma_j|\big)\right) \\
&= \sum_{k=1}^{N}\left(|\sigma_k\xi_k| \prod_{\substack{j=1,\dots,N \\ j\neq k}} \big(1 + C_5|\sigma_j|\big)\right).
\end{aligned}$$

By induction on N, the lemma is proved.

Remark 1. The most general case of N_1 fronts of the first family, N_2 of the second family, ... and N_n fronts of the n-th family, interacting all together at a single point P can be reduced to the two previous cases, as shown in fig. 2. We first let all wave-fronts of the same i-th family interact at a point P_i, generating a single outgoing i-wave. We then let the wave-fronts emerging from the points P_i interact together at a single point Q. This second interaction satisfies the assumptions of Lemma 4.

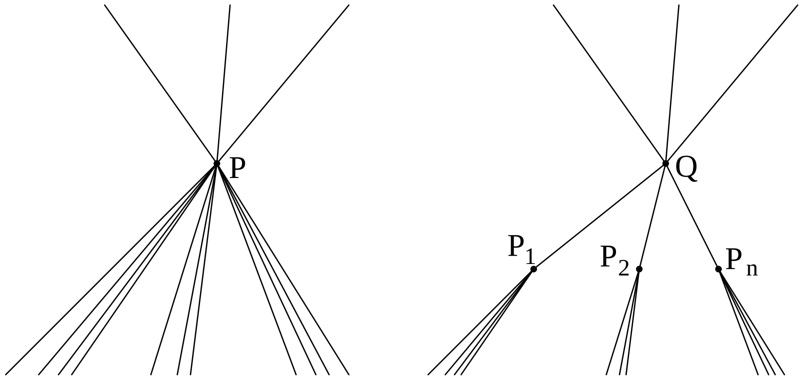

Figure 2

It is clear that the sizes and shifts of the wave-fronts emerging from Q in this perturbed configuration are exactly the same as those of the fronts emerging from P in the original configuration. Throughout the following, it is thus not restrictive to assume that each wave-front interaction is either of the type described in Lemma 4 (all waves of distinct families), or of the type described in Lemma 5 (all waves of the same family).

4 - Weighted path lengths

Let $v = v(t,x)$ be a ν-approximate solution, defined as in Section 2. By construction, v is thus piecewise constant in the t-x plane, with finitely many wave-fronts, say of size σ_α, located on the segments

$$J_\alpha = \Big\{(t,x);\quad t \in [t_\alpha,\ t'_\alpha[\,,\ x = \lambda_\alpha t + c_\alpha\Big\}. \tag{4.1}$$

Since the total number of interaction points is finite, we can choose a time T so large that no interaction occurs for $t \in [T, \infty[$. To each wave-front σ_α we now assign a weight W_α, so that the following two properties hold.

(i) At time $t = T$, all fronts have weight $W_\alpha = 1$.

(ii) Let P be a point of interaction. Call σ_α, Λ_α, W_α, $(\alpha = 1, \dots, N)$ respectively the sizes, speeds and weights of the incoming fronts, and σ'_β, Λ'_β, W'_β $(\beta = 1, \dots, N')$ the sizes, speeds and weights of the outgoing fronts. Then, for any vector $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$, the shifts

$$\xi_\alpha \doteq \mathbf{v}_2 - \Lambda_\alpha \mathbf{v}_1, \qquad\qquad \xi'_\beta \doteq \mathbf{v}_2 - \Lambda'_\beta \mathbf{v}_1 \tag{4.2}$$

satisfy

$$\sum_{\beta=1}^{N'} \left|\sigma'_\beta \xi'_\beta\right| W'_\beta \leq \sum_{\alpha=1}^{N} \left|\sigma_\alpha \xi_\alpha\right| W_\alpha. \tag{4.3}$$

In order to satisfy **(i)**-**(ii)**, we define the weights W_α by backward induction. To all wave-fronts of v at time $t = T$ we assign weight 1. Next, let P be an interaction point, and assume that weights W'_β have already been assigned to all of the outgoing fronts. Suitable weights W_α will be assigned to the incoming fronts as follows.

Choose a constant C larger than the constants C_4, C_5 in Lemmas 4 and 5.

CASE 1: All incoming wave-fronts belong to the same family. A possible situation is that of a shock of size $2^{1-\nu}$ surrounded by two rarefaction fronts both of size $2^{-\nu}$. In this case there is complete cancellation and no outgoing front. We then set $W_\alpha = 1$ for all three incoming fronts.

In all other situations, the interaction produces a single outgoing front, so that $N' = 1$. We then define

$$W_\alpha \doteq W'_1 \cdot \prod_{\beta \neq \alpha} \big(1 + C|\sigma_\beta|\big). \tag{4.4}$$

CASE 2: The incoming fronts all belong to different families. Then $N = N'$. Denoting by W_α, W'_α the weights of the i_α-waves respectively before and after the interaction, we define

$$W_\alpha \doteq \left(1 + C \sum_{\beta \neq \alpha} |\sigma_\beta|\right) W'_\alpha + C \sum_{\beta \neq \alpha} |\sigma_\beta|\, W'_\beta. \tag{4.5}$$

The general case, of N_1 incoming fronts of the first family, N_2 of the second family, etc., can always be reduced to a superposition of the above two cases, as in Remark 1.

Recalling (3.16) and (3.19), it is now easy to check that the properties **(i)**, **(ii)** hold. In turn, (4.3) implies that, for every pseudopolygonal $\gamma_0 : \theta \mapsto \bar{v}^\theta$ and every $t \geq 0$, the pseudopolygonal $\gamma_t : \theta \mapsto S_t^\nu\big(\gamma_0(\theta)\big)$ satisfies

$$\|\gamma_t\|_W \leq \|\gamma_0\|_W. \tag{4.6}$$

Lemma 6. *For any ν-approximate solution v, let the weights W_α be assigned as in (4.4), (4.5). Then all these weights are bounded by a constant L_M, depending only on the bound M on the total variation, and neither on ν nor on v.*

Proof. Consider a wave-front $\bar{\sigma}$, say of the i-th characteristic family, defined on some time interval $[\tau', \tau[$. Some additional notation must be introduced. Let the polygonal line $x = x_i(t)$ be the continuation of the front $\bar{\sigma}$ for all $t > \bar{t}$ (fig. 3). Call $\mathcal{I}(\bar{\sigma})$ the set of all waves which impinge on x_i after time τ. Let $\mathcal{J}(\bar{\sigma})$ be the subset of $\mathcal{I}(\bar{\sigma})$ consisting of waves of families $j \neq i$. If σ_α is (the size of) a wave-front located on a segment J_α such as (4.1), we denote by $W(\sigma_\alpha)$ its weight. Moreover, we write $W^+(\sigma_\alpha)$ for the weight of the front σ'_α of the same family as σ_α, originating at the terminal point of the J_α. Observe that we always have $W^+(\sigma_\alpha) < W(\sigma_\alpha)$.

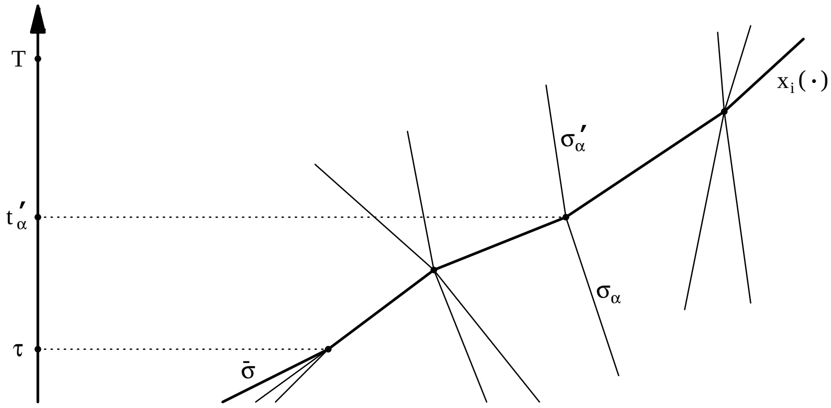

Figure 3

Using (4.4) and (4.5), by backward induction along the line $x_i(\cdot)$ we deduce

$$\begin{aligned} W(\bar{\sigma}) &\leq \exp\left\{ C \sum_{\sigma \in \mathcal{I}(\bar{\sigma})} |\sigma| \right\} \cdot \left(1 + C \sum_{\sigma \in \mathcal{J}(\bar{\sigma})} W^+(\sigma)\,|\sigma| \right) \\ &\leq e^{CM} \left(1 + C \sum_{\sigma \in \mathcal{J}(\bar{\sigma})} W^+(\sigma)\,|\sigma| \right), \end{aligned} \tag{4.7}$$

because the total amount of waves is $\leq M$. Assume now that $W(\bar{\sigma}) > e^{3CM}$, and that $\bar{\sigma}$ is the last front with this property, i.e. $W(\sigma) \leq e^{3CM}$ for all wave-fronts at times $t > \bar{\tau}$. Using the bound (4.7) itself, in order to estimate each term $W^+(\sigma)$ on the right hand side of (4.7), we obtain

$$\begin{aligned} \sum_{\sigma \in \mathcal{J}(\bar{\sigma})} W^+(\sigma)\,|\sigma| &\leq \sum_{\sigma \in \mathcal{J}(\bar{\sigma})} W(\sigma)\,|\sigma| \\ &\leq e^{CM} \left(\sum_{\sigma \in \mathcal{J}(\bar{\sigma})} |\sigma| \right) + Ce^{CM} \left(\sum_{\sigma \in \mathcal{J}(\bar{\sigma})} \sum_{\sigma' \in \mathcal{J}(\sigma)} W(\sigma')\,|\sigma\sigma'| \right) \\ &\leq Me^{CM} + Ce^{4CM} \sum_{\sigma \in \mathcal{J}(\bar{\sigma})} \sum_{\sigma' \in \mathcal{J}(\sigma)} |\sigma\sigma'| \\ &\leq Me^{CM} + Ce^{4CM} 2\big(Q(\tau-) - Q(T)\big), \end{aligned} \tag{4.8}$$

where $Q(\tau-)$ is the interaction potential of $v(t,\cdot)$ immediately before τ. By assumption, in (4.7) we have $W(\bar{\sigma}) > e^{3CM}$, $W^+(\sigma) \leq e^{3CM}$.

Therefore

$$\sum_{\sigma\in\mathcal{J}(\bar\sigma)} W^+(\sigma)\,|\sigma| \geq \frac{e^{2CM}-1}{C}. \tag{4.9}$$

Combining (4.8) and (4.9) we deduce

$$Q(\tau-)-Q(T) \geq K_M \doteq \frac{e^{2CM}-CMe^{CM}-1}{2C^2e^{4CM}} > 0. \tag{4.10}$$

In other words, if our inductive procedure assigns a weight $W > e^{3CM}$ to some wave-front at time $\tau-$, then over the interval $[\tau, T]$ the interaction potential Q must decrease at least K_M, a constant depending only on M.

An entirely similar computation shows that, if all the weights at a fixed time t_k are $\leq e^{3kCM}$ and if there exists a wave with weight $> e^{3(k+1)CM}$ at some time $t_{k+1} < t_k$, then the wave interaction potential Q must decrease by an amount $\geq K_M$ over the time interval $[t_{k+1}, t_k]$.

We now partition the interval $[0, T]$ into a finite number of subintervals $I_k \doteq [t_{k+1}, t_k]$, $(k = 1, \ldots, p)$, with $0 = t_{p+1} < \ldots < t_1 = T$, such that the following holds. Denoting $\Delta_k Q \doteq Q(t_{k+1}) - Q(t_k)$, one has

- either I_k contains only one interaction time and $\Delta_k Q \geq K_M$ (this is the case for example when two big shocks interact, or several small shocks interact at a single point),
- or $\Delta_k Q < K_M$ and I_k is maximal with this property. In other words, if $J_k = [\tau, t_k]$ is any interval containing I_k and containing also one additional interaction time not included in I_k, then $Q(\tau) - Q(t_k) \geq K_M$.

This choice of the intervals implies that, for every k, $\Delta_k Q + \Delta_{k+1} Q \geq K_M$. Since $Q(t)$ is non-increasing and $Q(0) - Q(T) \leq Q(0) \leq M^2$, this implies that the total number of these intervals is $p \leq 2M^2/K_M$. Observe that this upper bound for p is independent of $v \in \mathcal{D}_M^\nu$ and of ν.

By the previous analysis, if I_k is an interval satisfying $\Delta_k Q < K_M$ and if B_{k-1} is a bound for all the weights in the interval I_{k-1}, then $W(\sigma) \leq B_{k-1}e^{3CM}$ for all the waves σ for $t \in I_k$.

If, instead, $\Delta_k Q \geq K_M$, then I_k contains only an interaction time, say $\tau_k \in \,]t_{k+1}, t_k[\,$. Clearly at time τ_k more than one interaction can occur. If σ is an incoming wave and $W(\sigma)$ is its weight, then (4.4) and (4.5) both imply that

$$W(\sigma) \leq 2B_{k-1}e^{CM} < B_{k-1}e^{3CM}.$$

So also in this case $W(\sigma) \leq B_{k-1}e^{3CM}$ for all the waves and $t \in I_k$.

A simple inductive argument now yields

$$1 \leq W(\sigma) \leq L_M \doteq \big[\exp(3CM)\big]^{(2M^2/K_M)}. \tag{4.11}$$

This completes the proof of the lemma.

Using Lemma 6, we can now prove that the semigroups S^ν are globally Lipschitz with a uniform Lipschitz constant. For $\bar v, \bar v' \in \mathcal{D}^\nu_M$ define the distance

$$d^M_\nu(\bar v, \bar v') \doteq \inf \left\{ ||\gamma||_W \ : \gamma \text{ is a pseudopolygonal with values in } \mathcal{D}^\nu_M, \text{ joining } \bar v \text{ with } \bar v' \right\}. \quad (4.12)$$

By (4.6), this distance is contractive w.r.t the semigroup S^ν. From (1.7), (1.8), and (4.11) it follows

$$||\gamma||_{\mathbf{L}^1} \leq ||\gamma||_W \leq L_M ||\gamma||_{\mathbf{L}^1} \quad (4.13)$$

for every pseudopolygonal γ taking values inside $\mathcal{D}^\nu_M$. Let now $\bar v, \bar v' \in \mathcal{D}^\nu_M$ and consider the path $\gamma_0 : \theta \mapsto \bar v^\theta$ defined as in (2.6). Since $\bar v^\theta \in \mathcal{D}^\nu_{2M}$ for all θ, from (4.13) it follows

$$||\bar v - \bar v'||_{\mathbf{L}^1} \leq d^{2M}_\nu(\bar v, \bar v') \leq ||\bar v^\theta||_W \leq L_{2M} ||\bar v^\theta||_{\mathbf{L}^1} = L_{2M} ||\bar v - \bar v'||_{\mathbf{L}^1}. \quad (4.14)$$

Hence, for $\nu \geq 1$, the metrics d^{2M}_ν restricted to $\mathcal{D}^\nu_M$ are all uniformly equivalent to the usual $\mathbf{L}^1$ distance. Finally, the contractivity of the semigroup S^ν w.r.t. the metric d^{2M}_ν implies

$$||S^\nu_t \bar v - S^\nu_t \bar v'||_{\mathbf{L}^1} \leq L_{2M} ||\bar v - \bar v'||_{\mathbf{L}^1}. \quad (4.15)$$

5 - Proof of Theorem 1

In the previous sections we constructed a sequence of uniformly Lipschitz semigroups $S^\nu : \mathcal{D}^\nu_M \times [0, \infty[\mapsto \mathcal{D}^\nu_M$. Letting $\nu \to \infty$, we now show that the semigroups S^ν converge to a limit semigroup S, satisfying all required properties.

Recalling (1.4), define

$$\mathcal{D}'_M \doteq \left\{ v : I\!R \mapsto E' ; \quad v \in \mathbf{L}^1, \ \sum_i \text{Tot.Var.}(v_i) \leq M \right\} = \left\{ v(u);\ u \in \mathcal{D}_M \right\}.$$

Take any $\bar v \in \mathcal{D}'_M$. Since the union $\bigcup_{\nu \geq 1} \mathcal{D}^\nu_M$ is dense in $\mathcal{D}'_M$, there exists a sequence of functions $v^\nu \in \mathcal{D}^\nu_M$ such that $v^\nu \to \bar v$ in $\mathbf{L}^1$ as $\nu \to \infty$. We claim that the assignment

$$S_t \bar v \doteq \mathbf{L}^1\text{-} \lim_{\nu \to \infty} S^\nu_t v^\nu \quad (5.1)$$

uniquely defines a uniformly Lipschitz continuous semigroup on $\mathcal{D}'_M$. First, we show that the sequence $S^\nu_t v^\nu$ is a Cauchy sequence in $\mathbf{L}^1$. Let us recall Lemma 4 in [3]:

Lemma. *Let $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ be a globally Lipschitz semigroup with Lipschitz constant L. Let $v : [0, T] \mapsto \mathcal{D}$ be a continuous map whose values are piecewise constant in the (t, x)-plane, with jumps occurring along finitely many polygonal lines, say $\{x_\alpha(t)\}_{\alpha=1,\dots,N}$. Then*

$$||v(T) - S_T v(0)||_{\mathbf{L}^1} \leq L \int_0^T \limsup_{h\to 0+} \frac{\left\|v(t+h) - S_h v(t)\right\|_{\mathbf{L}^1}}{h}\, dt. \tag{5.2}$$

For any $\mu > \nu$, we now apply the estimate (5.2) with $S = S^\mu$, $v(t, \cdot) = S_t^\nu v^\nu$ and obtain

$$\begin{aligned}\|S_t^\nu v^\nu - S_t^\mu v^\mu\|_{\mathbf{L}^1} &\leq \|S_t^\mu v^\nu - S_t^\mu v^\mu\|_{\mathbf{L}^1} + \|S_t^\mu v^\nu - S_t^\nu v^\nu\|_{\mathbf{L}^1} \\ &\leq L_{2M}\|v^\nu - v^\mu\|_{\mathbf{L}^1} + L_{2M}\int_0^t \limsup_{h\to 0} \frac{1}{h}\left\|S_h^\mu S_\tau^\nu v^\nu - S_{\tau+h}^\nu v^\nu\right\|_{\mathbf{L}^1} d\tau\end{aligned} \tag{5.3}$$

At any time τ where no interaction occurs, call $\tilde{v} = S_\tau^\nu v^\nu$. We now estimate the difference $\left\|S_h^\mu \tilde{v} - S_h^\nu \tilde{v}\right\|_{\mathbf{L}^1}$. Let $x_1 < \cdots < x_q$ be the points where $\tilde{v}$ is discontinuous. Observe that, if the Riemann problem at x_α is solved by a shock wave or by a contact discontinuity, then by construction $S_h^\mu \tilde{v}(x) = S_h^\nu \tilde{v}(x)$ for x near x_α and h small enough.

Next, consider the case where the Riemann problem at x_α is solved by a rarefaction wave, say of the j-th family. Call $v_\alpha^\pm = \tilde{v}(\tau, x_\alpha(\tau)\pm)$. Then

- the ν-approximate solution of the Riemann problem is given by a unique j-wave connecting the states v_α^-, v_α^+ and moving with speed $\lambda^\alpha = \lambda_j(v_\alpha^-, v_\alpha^+)$. Our previous construction also implies $|v_\alpha^- - v_\alpha^+| = 2^{-\nu}$.
- the μ-approximate solution of the Riemann problem is given by a centered rarefaction fan, containing $2^{\mu-\nu}$ wave-fronts of the j-th family. More precisely, the jump (v_α^-, v_α^+) is decomposed into $2^{\mu-\nu}$ smaller jumps each of size $2^{-\mu}$, with the insertion of the intermediate states $v_\alpha^\ell = v_\alpha^- + \ell 2^{-\mu}\mathbf{e}_j$, $\ell = 0, 1, \dots, 2^{\mu-\nu}$. Here $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis in $\mathbb{R}^n$. Call $\lambda_\alpha^\ell = \lambda_j(v_\alpha^{\ell-1}, v_\alpha^\ell)$ the speeds of these wave-fronts.

Indicating by $\mathcal{R}$ be the set of points x_α corresponding to the rarefaction waves, for ρ and h sufficiently small we now have

$$\begin{aligned}\left\|S_h^\mu \tilde{v} - S_h^\nu \tilde{v}\right\|_{\mathbf{L}^1} &= \sum_{\alpha\in\mathcal{R}} \int_{x_\alpha-\rho}^{x_\alpha+\rho} \left|S_h^\mu \tilde{v}(x) - S_h^\nu \tilde{v}(x)\right| dx \\ &\leq \sum_{\alpha\in\mathcal{R}} \sum_{\ell=1}^{2^{\mu-\nu}-1} \int_{h\lambda_\alpha^\ell}^{h\lambda_\alpha^{\ell+1}} \left|v_\alpha^\ell - S_h^\nu \tilde{v}(x)\right| dx \\ &\leq h \sum_{\alpha\in\mathcal{R}} \sum_{\ell=1}^{2^{\mu-\nu}} \max\left\{|v_\alpha^\ell - v_\alpha^-|, |v_\alpha^\ell - v_\alpha^+|\right\} \left|\lambda_\alpha^{\ell+1} - \lambda_\alpha^\ell\right| \\ &\leq h \sum_{\alpha\in\mathcal{R}} C|v_\alpha^+ - v_\alpha^-|^2 \\ &\leq hC \cdot 2^{-\nu} M,\end{aligned} \tag{5.4}$$

for some constant C. Together, (5.3) and (5.4) yield

$$\|S_t^\nu v^\nu - S_t^\mu v^\mu\|_{\mathbf{L}^1} \leq L_{2M}\|v^\nu - v^\mu\|_{\mathbf{L}^1} + L_{2M} M C 2^{-\nu} t. \tag{5.5}$$

As $\nu, \mu \to \infty$, the right hand side of (5.5) clearly tends to zero. Hence the limit in (5.1) exists and does not depend on the choice of sequence v^ν. In particular, the map $S : \mathcal{D}'_M \times [0, +\infty[\mapsto \mathcal{D}'_M$ is well-defined.

Returning to the original coordinates u, it is now clear that the properties i), ii) and iv) hold, possibly with a different Lipschitz constant C_M. Since each trajectory $u(t, \cdot) = S_t \bar{u}$ is the limit of wave-front tracking approximations, a standard argument [1, 6, 10] shows that u is a weak solution of the Cauchy problem (1.1), (1.2).

Remark 2. Every integrable **BV** function $\bar{u} : I\!R \mapsto E$ lies in the domain $\mathcal{D}_M$ at (1.4), for some M sufficiently large. Hence, a unique viscosity solution of the corresponding Cauchy problem (1.1), (1.2) exists. Observe that a semigroup generated by (1.1) can be defined also on a domain $\mathcal{D}^* \supset \big(\cup \mathcal{D}_M\big)$ containing functions with unbounded variation, namely

$$\mathcal{D}^* \doteq \left\{ u : I\!R \mapsto E\,; \quad \begin{array}{l} u \text{ measurable, there exist sequences} \\ M_\nu,\ u_\nu \in \mathcal{D}_{M_\nu} \text{ such that } \lim_{\nu\to\infty} C_{M_\nu}\|u_\nu - \bar{u}\|_{\mathbf{L}^1} = 0 \end{array} \right\}. \tag{5.6}$$

Indeed, if $u \in \mathcal{D}^*$, we can define $S_t u \doteq \lim S_t u_\nu$, for any sequence $u_\nu \to u$ satisfying the conditions in (5.6). It is then easy to check that this semigroup is well defined and continuous. Of course, since $C_M \to \infty$ as $M \to \infty$, we do not expect this semigroup to satisfy any uniform Lipschitz estimate on $\mathcal{D}^*$.

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