RIEMANN PROBLEMS WITH A KINK

Helge Holden, Nils Henrik Risebro

Abstract. We study the Riemann problem for isothermal flow of a gas in a thin pipe with a kink in it. This is modeled by a 2×2 system of conservation laws with Dirac measure sink term concentrated at the location of the bends in the pipe. We show that the Riemann problem for this system of equations always has a unique solution, given an extra condition relating the speeds on both sides of the kink. Furthermore, we study the related problem where the flow is perturbed by an continuous addition of momentum at distinct points. Under certain conditions we show that also this Riemann problem has a unique solution.

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0. Introduction. We consider the flow of an isothermal gas in a (infinitely) long thin pipe of constant cross section. If the walls of the pipe have no effect on the flow, and the pipe is straight, this can be modeled by the system of conservation laws ([5], p. 56)

$$\rho_t + (\rho v)_x = 0, \qquad (\rho v)_t + (\rho v^2 + \rho)_x = 0. \tag{0.1}$$

Here, $\rho(x,t)$ denotes the density of the gas, and $v(x,t)$ the velocity. The position along the pipe is described by the coordinate x, and t denotes the time variable. These equations describe the conservation of mass and momentum, respectively.

In this paper we discuss the situation where the pipe is not straight, but has a one or several kinks in it. In between these kinks the pipe is straight. Hence the pipe can be described by a polygonal curve, and we ignore gravity. As in the model without kinks, we let $\rho(x,t)$ denote the density of the gas, and $v(x,t)$ its velocity. We now let x be the arc-length parameter along the pipe, or rather the curve describing the pipe. Away from the kinks, conservation of mass and momentum is given by (0.1). It remains to determine the equations holding at the kinks.

Since the cross section of the pipe is assumed to be constant on each side of a kink, conservation of mass reads as before

$$\rho_t + (\rho v)_x = 0. \tag{0.2}$$

In general, we can not assume that ρ and v are continuous at the location of the kink. Since a kink is always located at the same x, which we for simplicity assume to be at $x = 0$, discontinuities at kinks must satisfy a Rankine-Hugoniot condition where the speed of the discontinuity is zero. Hence, from (0.2) we obtain

$$0 = 0\,(\rho_l - \rho_r) = (\rho v)_l - (\rho v)_r,$$

where we have used the notation $f_{l,r} = \lim_{x\to 0\mp} f(x)$. Therefore the product ρv is continuous across kinks.

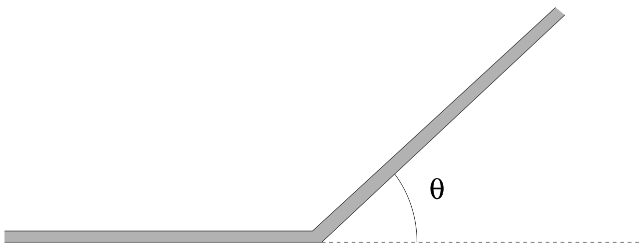

Fig. 1. The kink.

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To derive the momentum balance, we again consider a kink located at $x = 0$, the angle of the kink is given by θ, see Figure 1 above. Since the velocity of the gas is assumed to be parallel to the pipe (except at the kinks), we have

$$\mathbf{v}_l = v_l(1,0), \qquad \mathbf{v}_r = v_r(\cos\theta, \sin\theta).$$

Consequently, the change in momentum introduced by the kink is given by

$$\rho_r \mathbf{v}_r - \rho_l \mathbf{v}_l = \rho v(\cos\theta - 1, \sin\theta).$$

Therefore, the kink will act as a momentum sink, with a magnitude given by

$$|\rho_r \mathbf{v}_r - \rho_l \mathbf{v}_l| = \rho v \sqrt{2(1-\cos\theta)}. \tag{0.3}$$

To compensate for the complicated, and probably genuinely two dimensional, behavior at the kink we introduce a multiplicative empirical factor $f \in [0,1]$. This factor has dimension length(time)$^{-1}$, and is assumed to depend on the properties of the pipe and the gas. For simplicity, we will assume that the pipe is homogeneous, such that f is not dependent on location. Hence, we then arrive at the following model:

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho v)_t + \left(\rho v^2 + \rho\right)_x &= -f \sum_i k_i \delta_{x_i} \rho v, \end{aligned} \tag{0.4}$$

where δ_{x_i} is the Dirac measure located at the position of the ith kink; x_i, and k_i is given by the angle of this kink θ_i as

$$k_i = \sqrt{2\,(1-\cos\theta_i)}.$$

In order to examine one simple consequence of the model (0.4), we imagine a closed piecewise linear pipe approximating a circle of radius r. Assume that we have n equally spaced kinks, each of an angle $\theta = 2\pi/n$. We are interested in what happens for large n, then

$$k_i = \sqrt{2(1-\cos(2\pi/n))} \approx \frac{2\pi}{n}.$$

Also, the distance between adjacent kinks $\Delta x = x_i - x_{i-1} \approx r2\pi/n$. Therefore

$$\sum_i^n \rho v k_i \delta_{x_i} \approx \sum_i^n \rho v \frac{\Delta x}{r} \rightharpoonup \frac{\rho(x,t)v(x,t)}{r},$$

where $\rightharpoonup$ denotes weak convergence. Hence, for a smooth pipe, conservation of momentum is expressed by

$$(\rho v)_t + \left(\rho v^2 + \rho\right)_x = -\rho v f \kappa(x), \tag{0.5}$$

with $\kappa(x)$ denoting the curvature of the pipe at x and $f(x)$ is the local empirical bending factor at x.

Related to the model (0.4), is the model where one continuously *adds* momentum to the gas at distinct points x_i. A special case of that is the model one obtains by assuming $k_i \in [-2,0]$ on the right hand side of (0.4). This case is studied *mutatis mutandis* in Section 3.

The model (0.4) also arises as a model for the boundary behavior in an important two dimensional system of conservation laws. Consider the two dimensional version of (0.1)

$$\begin{aligned} \rho_t + (\rho v)_x + (\rho u)_y &= 0 \\ (\rho v)_t + \left(\rho v^2 + \rho\right)_x + (\rho v u)_y &= 0 \\ (\rho u)_t + (\rho v u)_x + \left(\rho u^2 + \rho\right)_y &= 0. \end{aligned} \tag{0.6}$$

Here, v and u denote the velocity in the x and y direction respectivly. Let $p(x)$ denote the function

$$p(x) = \begin{cases} 0 & \text{for } x < 0, \\ x\tan\theta & \text{for } x \geq 0, \end{cases}$$

and let Ω denote the set

$$\{(x, y) \mid y > p(x)\}.$$

Then consider (0.6) in Ω, together with the boundary condition that the velocity is parallell to $\partial\Omega$ at $\partial\Omega$. This system models the isothermal flow across a ramp in two dimensions. If one imposes the initial condition

$$(\rho, u, v)(x, 0) = \begin{cases} (\rho_l, u_l, v_l) & \text{for } x < 0, \\ (\rho_r, u_r, v_r) & \text{for } x \geq 0, \end{cases}$$

the solution along the boundary will be given by the solution of the Riemann problem considered here, (1.10). As an application, one could envisage using the solution computed in this paper as input at the boundary in a numerical scheme for solving (0.6).

The model (0.5) is an example of a system of conservation laws with source, sometimes referred to as balance equations. The general form of such equations is

$$w_t + f(w)_x = g(x, w) \tag{0.7}$$

where $w = w(x, t)$ is in $\mathbb{R}^n$, and consequently f is a mapping $\mathbb{R}^n \to \mathbb{R}^n$ and g a mapping $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Balance equations of this form are used to model a variety of situations. One system of equations, which is somewhat related to the model presented here, is a quasi-one-dimensional model for gas flow in a variable area duct. This model reads

$$\begin{aligned} \rho_t + (\rho v)_x &= -\frac{a'(x)}{a(x)}\rho v \\ (\rho v)_t + \left(\rho v^2 + p(v)\right)_x &= -\frac{a'(x)}{a(x)}\rho v^2 \\ (\rho E)_t + (\rho E v + p(v)v)_x &= -\frac{a'(x)}{a(x)}\left(\rho E v + p(v)v\right), \end{aligned} \tag{0.8}$$

where ρ and v are as before, and $p(v)$ denotes the pressure and E the total energy of the gas. The cross sectional area of the duct is denoted by $a(x)$. This model (0.8) has been analyzed by Liu in [6, 7, 8, 9]. If the source term g is smooth and bounded, and if the eigenvalues of the Jacobian df are real and distinct, and different from zero, existence of a global (in t) solution was obtained in [6] by a generalization of Glimm's method, and uniqueness and stability was proved by Crasta and Piccoli in the 2×2 case, [1]. If, however, the eigenvalues may take the value zero, (0.7) is a so-called resonant hyperbolic system. This is the case for the system (0.8), as well as for the system discussed in this paper.

The wave structure for resonant hyperbolic systems may be surprisingly complicated, see Isaacson and Temple [4] as well as the above mentioned works by Liu.

If the cross sectional area of the duct in (0.8) is piecewise constant, the source term becomes a point source similar to the source in (0.4). The Riemann problem for (0.8) with a piecewise constant a was analyzed by Marchesin and Paes-Leme in [10].

In addition, we mention that systems of equations exhibiting nonlinear resonance also occur in models of two and three phase flow in porous media, see [2], and [3].

1. The Riemann problem. We consider the following system of equations

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho v)_t + \left(\rho v^2 + \rho\right)_x &= -\sum_i f_i k_i \delta_{x_i} \rho v, \\ \rho(x, 0) = \rho_0(x), &\qquad v(x, 0) = v_0(x) \end{aligned} \tag{1.1}$$

where the unknowns ρ and v are functions of x and t. The effects of the bends in the pipe are expressed by the term $k_i \delta_{x_i} \rho v$, where δ_{x_i} denotes the unit point mass located at x_i, and k_i is given by

$$k_i = k(\theta_i) = \sqrt{2\left(1 - \cos\left(\theta_i\right)\right)} \tag{1.2}$$

where θ_i is the angle of the bend located at x_i. Even in the absence of source terms, (1.1) generally develops discontinuities, so we seek weak solutions. By definition these satisfy

$$\begin{aligned} &\iint \rho\varphi_t + \rho v \varphi_x \, dxdt + \int \rho_0(x)\varphi(x,0)\, dx = 0 \\ &\iint \rho v \varphi_t + \left(\rho v^2 + \rho\right)\varphi_x \, dxdt + \int \rho_0(x) v_0(x)\varphi(x,0)\, dx = \\ &\qquad\qquad \int \sum_i f_i k_i \varphi\left(x_i,t\right)\rho\left(x_i,t\right) v\left(x_i,t\right)\, dt. \end{aligned} \tag{1.3}$$

If there are no source terms, (1.1) is a strictly hyperbolic conservation law, with eigenvalues

$$\lambda_1 = v - 1, \text{ and } \lambda_2 = v + 1. \tag{1.4}$$

The corresponding eigenvectors are

$$e_1 = (1, v-1), \text{ and } e_2 = (1, v+1). \tag{1.5}$$

When solving the Riemann problem we are interested in those states (ρ, v) that can be joined to a given state (ρ_0, v_0) by a simple wave, i.e., either a shock wave or a rarefaction wave. The shock waves satisfy the Lax entropy condition. For a definition of these concepts, see [11, 5]. Through each (ρ_0, v_0) there are two curves, $C_1(\rho_0, v_0)$ and $C_2(\rho_0, v_0)$, of such states. If (ρ, v) is on $C_1(\rho_0, v_0)$, then the initial value problem

$$\begin{gathered} \rho_t + (\rho v)_x = 0 \\ (\rho v)_t + \left(\rho v^2 + \rho\right)_x = 0, \\ \rho(x,0) = \begin{cases} \rho_0 & \text{for } x < 0, \\ \rho & \text{for } x > 0, \end{cases} \qquad v(x,0) = \begin{cases} v_0 & \text{for } x < 0, \\ v & \text{for } x > 0, \end{cases} \end{gathered} \tag{1.6}$$

is solved by a slow wave (also denoted as a one-wave). This wave is a (one-)shock wave if $\rho > \rho_0$ and a (one-)rarefaction wave if $\rho < \rho_0$. The curve C_1 is given by the following expression ([5], p. 71ff and p. 84f)

$$C_1 \; : \; v_1(\rho; \rho_0, v_0) = \begin{cases} v_0 - \ln\left(\frac{\rho}{\rho_0}\right) & \text{for } \rho < \rho_0 \\ v_0 - \frac{\rho - \rho_0}{\sqrt{\rho\rho_0}} & \text{for } \rho > \rho_0. \end{cases} \tag{1.7}$$

Similarly, if (ρ, v) is on $C_2(\rho_0, v_0)$, then the initial value problem (1.6) is solved by either a two-shock wave or a two-rarefaction wave. The curve C_2 is given by ([5], p. 71ff and p. 84f)

$$C_2 \; : \; v_2(\rho; \rho_0, v_0) = \begin{cases} v_0 + \frac{\rho - \rho_0}{\sqrt{\rho\rho_0}} & \text{for } \rho < \rho_0 \\ v_0 + \ln\left(\frac{\rho}{\rho_0}\right) & \text{for } \rho > \rho_0, \end{cases} \tag{1.8}$$

with two-shocks corresponding to $\rho < \rho_0$ and two-rarefaction waves corresponding to $\rho > \rho_0$. Whenever convenient we will denote the shock and rarefaction part of C_i by S_i and R_i, respectively. Furthermore, when we consider the *right state* R as fixed, we denote the corresponding wave curves by C_i^- etc. Note that C_1 is given by a decreasing convex function of $v_1(\rho)$, such that $\lim_{\rho\to 0+} v_1 = \infty$ and $\lim_{\rho\to\infty} v_1 = -\infty$, and that C_2 is given by an increasing concave function $v_2(\rho)$ such that $\lim_{\rho\to 0+} v_2 = -\infty$ and $\lim_{\rho\to\infty} v_2 = -\infty$. If the source terms in (1.1) are absent, this ensures the unique solution of the Riemann problem.

The speed of shock waves is given by the Rankine-Hugoniot condition, and may be computed from the first equation in (1.1)

$$s_j = \frac{\rho v_j - \rho_0 v_0}{\rho - \rho_0} = v_0 + (-1)^j \sqrt{\frac{\rho}{\rho_0}} = v_j + (-1)^j \sqrt{\frac{\rho_0}{\rho}}. \tag{1.9}$$

Note that both shock speeds and the speed of rarefaction waves can be both positive and negative. We will in our construction of the solution of the Riemann problem need certain points on the shock curves. If (ρ, v)

is a given state, we denote the point on the shock curve that can be connected with a shock of zero speed by $Z(\rho, v)$. From expressions (1.7)–(1.9) we find that

$$Z(\rho, v) = \left(\rho v^2, \frac{1}{v}\right)$$

which for one-shocks is defined for $v \geq 1$, while for two-shocks it is defined for $v \in [-1, 0\rangle$. Furthermore, we will need the intersection of $C_1(L)$ and the line $v = -1$, and we denote by $\widehat{L}$ the unique point in $C_1(L) \cap \{v = -1\}$. In addition, we let $\widetilde{L}$ denote the unique intersection of $C_1(L)$ and the line $v = v_c^-$, see (2.4).

The Riemann problem for (1.1) is the initial value problem

$$\begin{gathered} \rho_t + (\rho v)_x = 0 \\ (\rho v)_t + \left(\rho v^2 + \rho\right)_x = -k\delta_0 \rho v, \\ \rho(x,0) = \begin{cases} \rho_L & \text{for } x < 0, \\ \rho_R & \text{for } x > 0, \end{cases} \qquad v(x,0) = \begin{cases} v_L & \text{for } x < 0, \\ v_R & \text{for } x > 0, \end{cases} \end{gathered} \tag{1.10}$$

where we have absorbed the empirical factor f in the geometric factor k. We study two distinct but related cases; k positive (where $\theta = \arccos(1 - k^2/2)$ denotes the angle of the kink), and k negative. In both cases, $|k| \leq 2$. We seek self similar solutions to (1.10), that is $\rho = \rho\,(x/t)$ and $v = v\,(x/t)$. Away from the point $x = 0$, we have can use the curves C_1 and C_2 to connect states (ρ, v). We label such connections C-waves. At the point $x = 0$, we will in general have a discontinuity. This discontinuity will satisfy the Rankine-Hugoniot conditions

$$[\![\rho v]\!] = 0, \qquad [\![\rho v^2 + \rho]\!] = -k\rho v, \tag{1.11}$$

where by $[\![\phi]\!]$ we denote the jump in ϕ, i.e., $[\![\phi]\!] = \phi_r - \phi_l$. Given (ρ_l, v_l), the last equation in (1.11) can be solved for v_r giving

$$v_r := g_\pm\,(v_l) := \frac{1}{2v_l}\left(\alpha\,(v_l) \pm \sqrt{\alpha^2\,(v_l) - 4v_l^2}\right), \tag{1.12}$$

where

$$\alpha(v) = v^2 - kv + 1. \tag{1.13}$$

We can now use the first equation in (1.11) to calculate ρ_r,

$$\rho_r := \rho_l f_\pm\,(v_l) := \rho_l \frac{v_l}{g_\pm\,(v_l)}. \tag{1.14}$$

For any point (ρ_l, v_l) we therefore have two candidates for (ρ_r, v_r). In order to choose between these we impose the extra condition that

$$\frac{dv_r}{dv_l} \geq 0, \tag{1.15}$$

as an increase in the velocity of the gas coming into the kink, should result in an increase in the outgoing velocity. We will use the same criterion for k negative. The choice (1.15) selects a branch of $g_\pm$ and hence of $f_\pm$. The remaining discussion will depend on the properties of this function determined by the sign of k.

2. The case k positive. We will need detailed properties of the function $g_\pm$ which satisfies

$$vg^2 - \alpha(v)g + v = 0, \quad g = g_\pm. \tag{2.1}$$

and this implies that the solution satisfies

$$v = -g(-g(v)). \tag{2.2}$$

The two solutions fulfill

$$g_{\pm}(v) = g_{\pm}(1/v), \quad g_{+}(v)g_{-}(v) = 1. \tag{2.3}$$

Define

$$v_c^{+} = -g_{+}(-1), \quad v_c^{-} = -g_{-}(-1). \tag{2.4}$$

We have that $v_c^{-} \leq 1 \leq v_c^{+}$, equality holding only if $k = 0$. The symmetry (2.2) implies that $g_{\pm}(v_c^{\pm}) = 1$. Furthermore,

$$\begin{aligned} g'_{+}(v) \geq 0 \quad &\text{for } v \leq -1 \text{ and for } v \geq v_c^{+}, \\ g'_{-}(v) \geq 0 \quad &\text{for } v \in \left[-1, v_c^{-}\right]. \end{aligned}$$

In Figure 2 below we show $g_{\pm}$ and $f_{\pm}$ for $k = 0.5$.

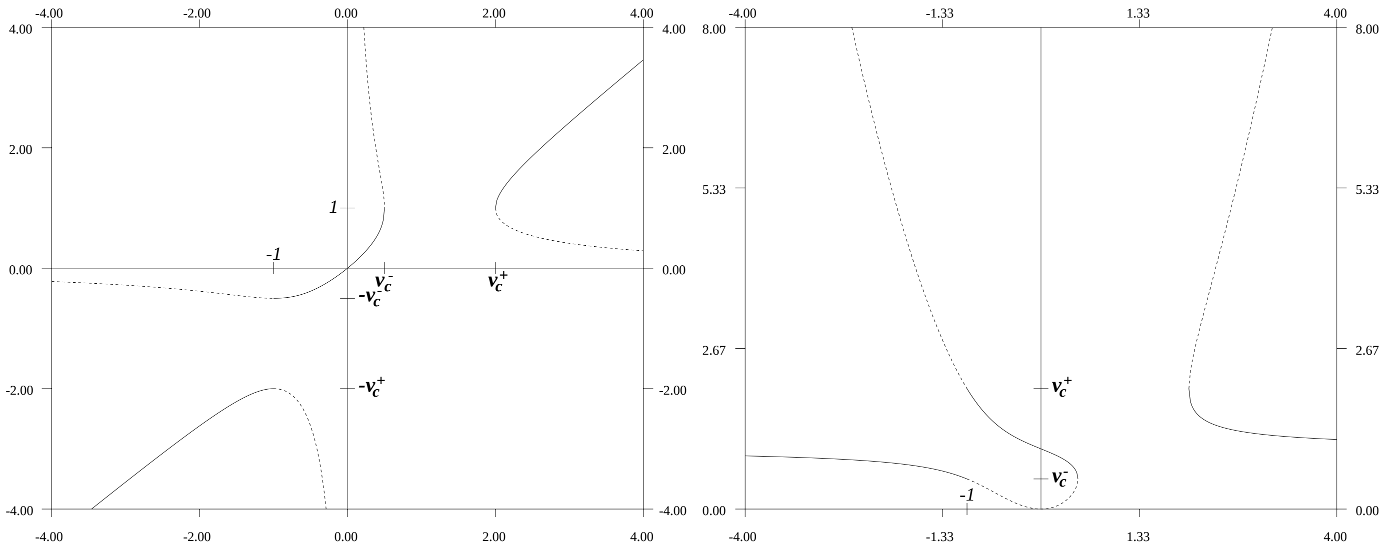

Fig. 2. The functions $g_{\pm}$ (left) and $f_{\pm}$ (right) for $k = 0.5$.

This means that for $v_l \in \langle -\infty, -1] \cup [v_c^{+}, \infty\rangle$, we choose the pluss sign in (1.12) and (1.14), while for $v \in [-1, v_c^{-}]$, we choose the minus sign. There are no solutions in the region $\langle v_c^{-}, v_c^{+}\rangle$. In the figure this choice is indicated by solid lines. Let therefore the mapping K be given by

$$K(\rho, v) = \begin{cases} (\rho f_{+}(v), g_{+}(v)) & \text{for } v \leq -1 \text{ and for } v \geq v_c^{+}, \\ (\rho f_{-}(v), g_{-}(v)) & \text{for } v \in [-1, v_c^{-}]. \end{cases} \tag{2.5}$$

In the following we use the term "K-wave" meaning the mapping K. An important property of the K mapping is that it commutes with the stationary shocks.

Lemma 2.1. *The K mapping commutes with Z, i.e., $Z(K(\rho, v)) = K(Z(\rho, v))$ whenever the two mappings are defined.*

Proof. Let (ρ, v) be a point with $v \geq v_c^{+}$. We then compute

$$\begin{aligned} Z\left(K\left(\rho, v\right)\right) &= Z\left(\rho v g_{+}\left(v\right), g_{-}\left(v\right)\right) \\ &= \left(\rho v g_{+}\left(v\right) g_{-}^{2}\left(v\right), \frac{1}{g_{-}\left(v\right)}\right) \\ &= \left(\rho v g_{-}\left(v\right), g_{+}\left(v\right)\right). \end{aligned}$$

and

$$\begin{aligned} K(Z(\rho,v)) &= K\left(\rho v^2, \frac{1}{v}\right) \\ &= \left(\rho v^2 \frac{1}{v} g_-\left(\frac{1}{v}\right), g_+\left(\frac{1}{v}\right)\right) \\ &= (\rho v g_-(v), g_+(v)) \\ &= Z(K(\rho,v)). \end{aligned}$$

The case for fast waves is similar. □

So when solving the Riemann problem we have three waves at our disposal; C_1-waves, C_2-waves and K-waves. The K-waves always have zero speed, and the C-waves can have both positive and negative speed. The solution has to contain a K-wave to bring us from one part of the pipe to the other. This means that *a priori* the solution may consist of up to five different waves: $C_1C_2KC_1C_2$. However, the following lemma limits this to four.

Lemma 2.2. *The wave configuration* $\ldots C_2KC_1 \ldots$ *is impossible.*

Proof. The smallest speed of the right edge of any C_2 wave with a right state (ρ_l, v_l) is $v_l + 1$, and the largest speed of the left edge of a C_1 wave with a left state (ρ_r, v_r) is $v_l - 1$. Therefore we have the inequalities

$$v_l + 1 \leq 0 \leq v_r - 1$$

which means that $v_l \leq -1$ and $v_r \geq 1$. Since ρv is constant across K-waves, there can be no K-wave connecting (ρ_l, v_l) and (ρ_r, v_r). □

Consequently, we are left with the four possible wave configurations to solve the Riemann problem:

$$C_1KC_1C_2, \quad C_1KC_2, \quad C_1C_2KC_2, \text{ and } KC_1C_2.$$

When solving the Riemann problem, we use the strategy that for each fixed left state $L = (\rho_L, v_L)$ we construct waves with increasing speed until we reach the right state R. In this way we partition the (ρ, v) plane into regions such that for each $R = (\rho_R, v_R)$ in a region, the wave structure of the solution is constant. Hence, we may have a $C_1C_2KC_2$ region, a C_1KC_2 region etc. States to the left of the K-wave are denoted by l and states to the right are denoted by r.

It turns out that there are two cases; $v_L > v_c^+$ and $v_L \leq v_c^+$.

Case 1: $v_L > v_c^+$. Consider first a right state R near L. The waves in a neighborhood of L all have positive speed, and hence we have to start with a K-wave. Let $r_1 = (\rho_{r_1}, v_{r_1}) = K(L)$. Then $v_{r_1} \in [1, v_L]$. We now construct wave curves $C_1(r_1)$ and $C_2(r_1)$. For $r_2 \in R_1(r_1)$ wave speeds are all positive, and we may continue with a fast wave $C_2(r_2)$ to reach a right state R, viz. $R \in C_2(r_2)$. If, however, $r_2 \in C_1(r_1)$, the shock speed is only positive down to the point $Z(r_1)$, which by Lemma 2.1 equals $K(Z(L))$. Thus for each $r_2 = (\rho_{r_2}, v_{r_2})$ with $v_{r_2} \in [1/v_{r_1}, v_{r_1}]$, we may continue with a fast wave to a right state $R \in C_2(r_2)$. In this way we fill the region denoted by KC_1C_2 above the curve $C_2(K(Z(L)))$. To prove uniqueness, consider a right state R in this region. The Riemann problem with left state $K(L)$ and right state R has a unique solution by standard techniques, and hence the only alternative would be to start from the right with a K-wave to the state $K^{-1}(R)$, which, however, is in the region with only waves with positive speed, and hence is impossible.

Consider now a point l_1 on $S_1(L)$ between $Z(L)$ and $\widehat{L}$. L connects to l_1 with a slow shock with negative speed. For each such state l_1 we may continue with a K-wave to a state $r_1 = K(l_1)$. At the state $\widehat{L}$ the K-map ceases to be continuous, and the wave structure will be different. Fast rarefaction waves from r_2 will all have positive speed and may be used in the construction to reach a right state $R \in R_2(r_2)$. For fast shock waves emanating from r_1 with positive v_{r_1}, the shock speed remains positive (the Z map is not defined), and hence R may be any point on $S_2(r_1)$. If, however, r_1 is between $v = 0$ and $v = -v_c^-$, the shocks on $C_2(r_1)$ have positive speed down to $Z(r_1)$, and only states R above this point can be reached with this wave structure. Let c denote the part of $S_1(L)$ between $v = 0$ and $v = -v_c^-$, and let $\kappa = K(c)$, and finally $\zeta = Z(K(c))$. We then find that the solution reads C_1KC_2 in the region bounded from above by $C_2(K(Z(L)))$ and bounded from below by ζ and $C_2(K(\widehat{L}))$, the two latter curves starting from $Z(K(\widehat{L}))$. To prove uniqueness we first define $\tilde{c}$ as the part of $C_1(L)$ between $Z(L)$ and $\widehat{L}$, and subsequently $\tilde{\kappa} = Z(\tilde{c})$. First we have to prove that the curve $\tilde{\kappa}$ is transversal to C_2 curves starting from $\tilde{\kappa}$. This is the content of Lemma A.1 in the Appendix. Furthermore, let R be above ζ. The only alternative to the given solution would be to connect the states $K^{-1}(R)$ and R instead of using a fast shock. As K^{-1} is monotone in the v variable, $K^{-1}(R)$ will be above $K^{-1}(\zeta) = Z(c)$. But then the state $K^{-1}(R)$ can only be reached with shocks with positive speed, making it impossible to end with a K-wave.

Consider now a right state R in the region denoted C_1C_2K, i.e., below the curves ζ and $v = -v_c^+$. Let $l_2 = K^{-1}(R)$. Then $v_{l_2} \le -1$. The curves $C_2^-(l_2)$ and $S_1(L)$ intersect uniquely at a point l_1. If l_2 (yes, l_2) is below $S_1(L)$, the states l_1 and l_2 will connect using a rarefaction wave with negative speed as $v_{l_2} \le -1$. If, on the other hand, l_2 is below $S_1(L)$, we will use a fast shock wave to connect l_1 with l_2. As R is below ζ and the map K^{-1} is monotone in the v variable, also $l_2 = K^{-1}(R)$ will be below $K^{-1}(\zeta) = Z(c)$ using Lemma 2.1, and hence the shock will have negative speed as required. In this way we solve the Riemann problem in the region C_1C_2K.

Finally, the region $C_1C_2KC_2$ is bounded from above by the curve $C_2(K(\widehat{L}))$ and by the line $v = -v_c^+$ from below, both curves emanating from $Z(K(\widehat{L}))$. Let R be in this region. Assume R is such that $v_R \ge -v_c^-$. Then we let $r = (\rho_r, -v_c^-)$ be the unique point such that $R \in R_2(r)$, and subsequently $l_2 = K^{-1}(r) = (\rho_{l_2}, -1)$. There is a unique state l_1 on $S_1(L)$ such that $l_2 \in R_2(l_1)$. Thus the solution consists of the following waves: The left state L is connected to at state l_1 using a slow shock wave (with negative speed), followed by a fast rarefaction wave (with nonpositive speed) up to the state l_2 on the line $v = -1$. This state is connected with a K wave to the state r. Finally, we use a fast wave to connect to R. If R has v_R less than $-v_c^-$, we start by using a fast shock with positive speed from $r = (\rho_r, -v_c^-)$ down to R, i.e., $R \in S_2(r)$. The remaining part of the solution is the same. For the uniqueness question, consider first the case with $v_R > -v_c^-$. Any waves connecting to $K^{-1}(R)$ would have positive speed, which would make it impossible to connect $K^{-1}(R)$ and R with a stationary K-wave. If v_R is between $-v_c^+$ and $-v_c^-$, we can not apply $K^{-1}(R)$, and all fast waves have negative speed in that region, making it impossible to construct a different solution.

Case 2: $v_L \le v_c^+$. If $v_L \le v_c^+$, we can connect to the state $l_1 = \widetilde{L}$ on the intersection $C_1(L) \cap \{v = v_c^-\}$ with a wave of non-positive speed. (If $v_L < v_c^-$ the wave is a rarefaction wave, and if $v_L > v_c^-$ it will be a shock wave.) l_1 can be connected to the state $r_1 = K(l_1)$ with a K-wave. The slow rarefaction wave starting from r_1 will have non-negative speed, and hence we may use fast waves from any point $r_2 \in R_1(r_1)$ to reach a right state $R \in C_2(r_2)$. In this way we fill the region above the curve $C_2(K(\widetilde{L}))$ with a solution of the form $C_1KC_1C_2$. The remaining part of the construction is similar to that of case 1.

The curves separating the various regions are illustrated in Figures 3 (case 1) and 4 (case 2). An illustration of solution curves in all cases are given in Figure 5, the left column for case 1, and the right column for case 2.

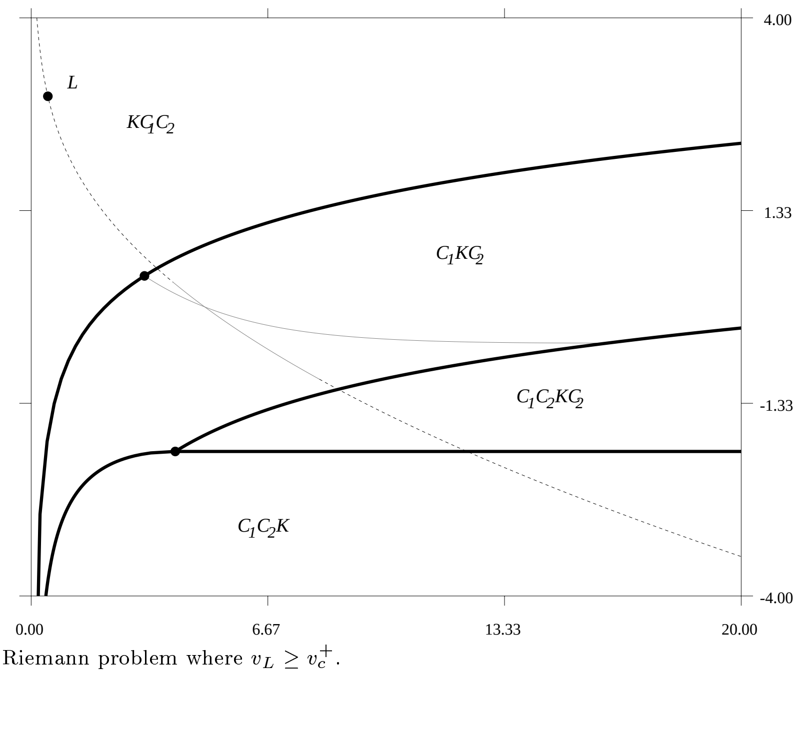

Fig. 3. The solution to the Riemann problem where $v_L \ge v_c^+$.

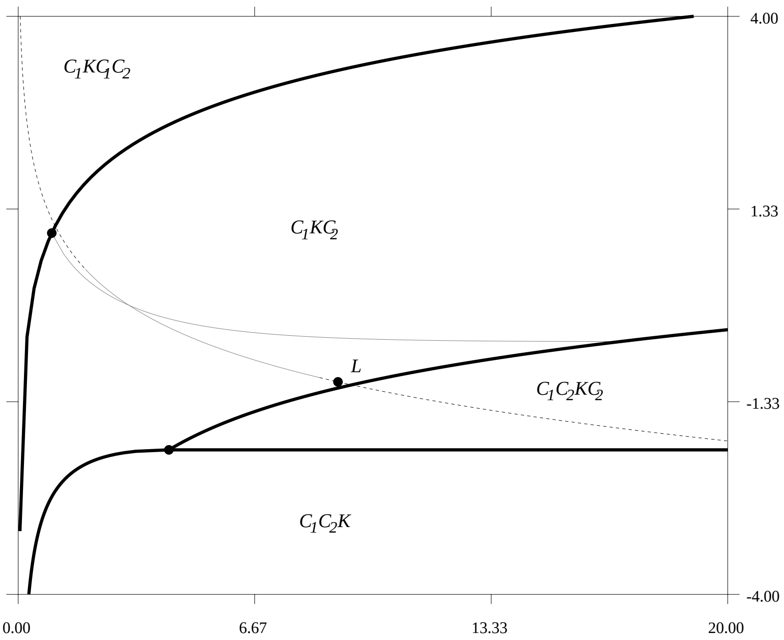

Fig. 4. The solution to the Riemann problem where $v_L \leq v_c^+$.

Hence we have proved the following result.

Theorem 2.3. *Let $0 \leq k < 2$. Then the Riemann problem*

$$
\begin{gathered}
\rho_t + (\rho v)_x = 0 \\
(\rho v)_t + \left(\rho v^2 + \rho\right)_x = -k\delta_0 \rho v, \\
\rho(x,0) = \begin{cases} \rho_L & \text{for } x < 0, \\ \rho_R & \text{for } x > 0, \end{cases} \qquad v(x,0) = \begin{cases} v_L & \text{for } x < 0, \\ v_R & \text{for } x > 0, \end{cases}
\end{gathered} \tag{2.6}
$$

has a unique solution in the class of combinations of Lax shocks, rarefaction waves and K-waves for any left state (ρ_L, v_L) and right state (ρ_R, v_R) with positive densities ρ_L and ρ_R given by the above construction.

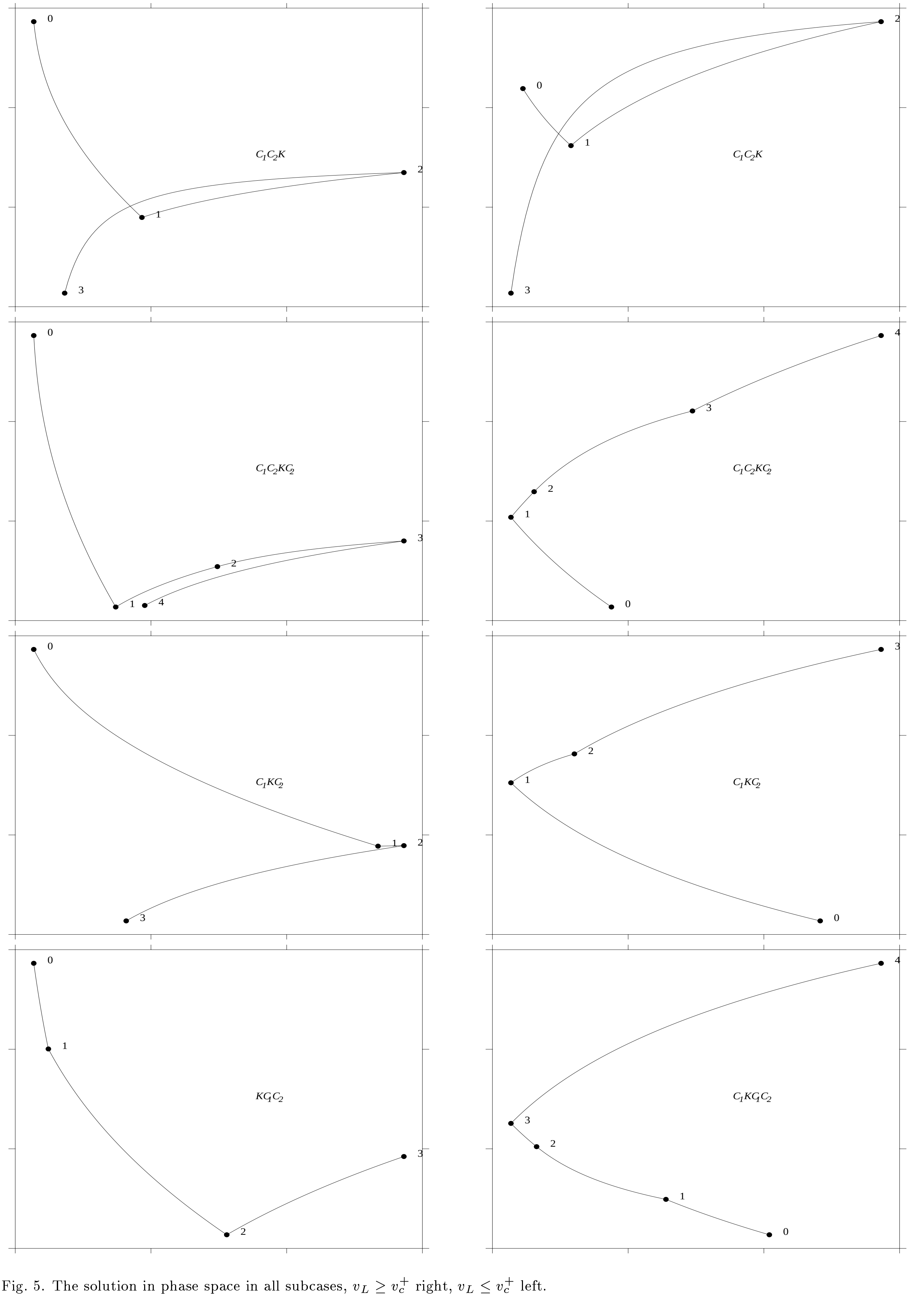

Fig. 5. The solution in phase space in all subcases, $v_L \geq v_c^+$ right, $v_L \leq v_c^+$ left.

3. The case k negative. The general structure of the argument is identical to that used in the case with k positive. However, the properties of the function $g_\pm$ and $f_\pm$ are different in the two cases, and a separate discussion is required. To keep the presentation short, we only give the details where they are different from those of the previous section.

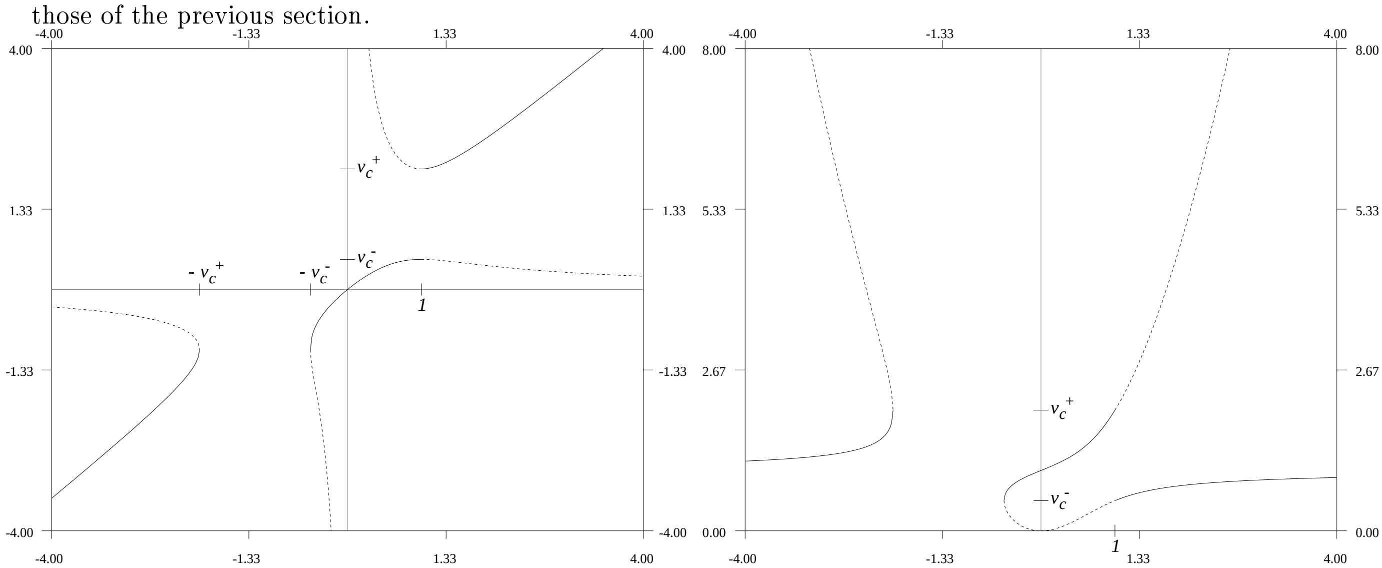

Fig. 6. The functions $g_\pm$ (left) and $f_\pm$ (right) for $k = -.5$.

Define

$$v_c^\pm = g_\pm(1), \quad v_c^- \leq 1 \leq v_c^+.$$

The symmetry properties of $g_\pm$ imply that $g_\pm(-v_c^\pm) = -1$. Furthermore, we have that

$$\frac{dg_+(v)}{dv} \geq 0 \text{ for } v \in \langle -\infty, -v_c^+] \cup [1, \infty\rangle$$

and

$$\frac{dg_-(v)}{dv} \geq 0 \text{ for } v \in [-v_c^+, 1]$$

and hence the K-wave reads

$$K(\rho, v) = \begin{cases} (\rho f_+(v), g_+(v)) & \text{for } v \leq -v_c^+ \text{ and for } v \geq 1, \\ (\rho f_-(v), g_-(v)) & \text{for } v \in [-v_c^-, 1]. \end{cases} \tag{3.1}$$

The Z map (defined as before) and the K-map still commute, i.e., Lemma 2.1 remains valid. As designated points on the slow wave curve, it is convenient to define $\widehat{L}$ as the unique intersection of $C_1(L)$ and $\{v = -v_c^-\}$, and $\widetilde{L}$ as the unique intersection between $C_1(L)$ and $\{v = 1\}$.

The general strategy for the construction of the solution of the Riemann problem is independent of the sign of k. Lemma 2.2 still reduces the number of possible wave combinations to four. As in the previous case, the analysis requires two distinct subcases; $v_L > 1$ and $v_L \leq 1$.

Case 1: $v_L > 1$. Waves in the neighborhood of L all have positive speed and we start with a K-wave to the state $r_1 = K(L)$. From the state r_1 we construct fast wave curve $C_1(r_1)$ and $C_2(r_1)$, which are all permissible except states on the slow shock below $Z(r_1) = K(Z(L))$. This results in a wave structure of the solution given by KC_1C_2 above the curve $C_2(K(Z(L)))$.

Let now l_1 be a state on $S_1(L)$ between $Z(L)$ and $\widehat{L}$. The left state L connects to l_1 with a shock with negative speed. We now employ the K map to a state $r_1 = K(l_1)$. Fast rarefaction waves emanating from r_1 all have positive speed and can be used if the right state $R \in R_2(r_1)$. If v_{r_1} is positive, fast shocks on $S_2(r_1)$ will always have positive speed, while if $v_{r_1} \geq -1$ but negative, then the fast shocks will have positive speed only down to $Z(r_1)$. Let as before c denote the part of $S_1(L)$ between $v = 0$ and $v = -v_c^-$, and $\kappa = K(c)$ and $\zeta = Z(\kappa)$. Then we obtain a solution with structure C_1KC_2 in the region bounded from above by $C_2(K(Z(L))$ and below by ζ and $C_2(K(\widehat{L}))$ (starting from $Z(K(\widehat{L}))$). The question of uniqueness is more difficult in this case as we see that the curve $\tilde{\kappa} = K(\tilde{c})$ where $\tilde{c}$ is the part of $C_1(L)$ between $Z(L)$ and $\widehat{L}$, is in fact tangent to the curve $C_2(K(\widehat{L}))$. A proof that $\tilde{\kappa}$ is transversal to C_2 curves originating from $\tilde{\kappa}$ (except at $K(\widehat{L})$) is given in the Appendix under the assumption that $k > 2(1-\sqrt{2})$.

The construction of a solution with right state R in the region below ζ and $v = -1$ is similar to that in the case k positive.

Finally, let R be in the region bounded from above by $R_2(K(\widehat{L}))$ and from below by $v = -1$. Then there is a unique state r on $v = -1$ such that $R \in R_2(r)$, and hence using only positive speeds. We have that r can be connected with K-wave from a state l_2 with $v_{l_2} = -v_c^+$. For each such state there is a unique state l_1 on $C_1(L)$ such that $l_2 \in C_2(l_1)$. This concludes the discussion of case 1.

Case 2: $v_L \leq 1$. Because $v_L \leq 1$ we can use a one-rarefaction wave with non-positive speed to reach the state $\widetilde{L}$ on the intersection of $C_1(L)$ and $v = 1$. When $v = 1$ we have two options for the map K, and denote $K^{\pm}(\rho, 1) = (\rho v_c^{\mp}, v_c^{\pm})$. (For the curve $\tilde{\kappa}$ we used the map K^-.) From the state $\widetilde{L}$ we use the K^+ map to the state $r_1 = K^+(\widetilde{L})$. We can now employ both slow and fast waves using $C_1(r_1)$ and $C_2(r_1)$. However, as in case with k positive, we can only use slow shocks down to the point $Z(K^+(\widetilde{L}))$ which is equal to $K^-(\widetilde{L})$. Hence we obtain a solution structure of the form $C_1KC_1C_2$ in the region above $C_2(K^-(\widetilde{L}))$.

The remaining part of the construction equals that of case 1.

We have proved the following theorem.

Theorem 3.1. *Let $0 \geq k > 2(1-\sqrt{2})$. Then the Riemann problem*

$$\begin{gathered}\rho_t + (\rho v)_x = 0 \\ (\rho v)_t + \left(\rho v^2 + \rho\right)_x = -k\delta_0 \rho v, \\ \rho(x,0) = \begin{cases} \rho_L & \text{for } x < 0, \\ \rho_R & \text{for } x > 0, \end{cases} \qquad v(x,0) = \begin{cases} v_L & \text{for } x < 0, \\ v_R & \text{for } x > 0, \end{cases}\end{gathered} \tag{3.2}$$

has a unique solution in the class of combinations of Lax shocks, rarefaction waves and K-waves for any left state (ρ_L, v_L) and right state (ρ_R, v_R) with positive densities ρ_L and ρ_R given by the above construction.

See Figures 7 and 8 below for an illustration of the various regions in cases 1 and 2, respectively.

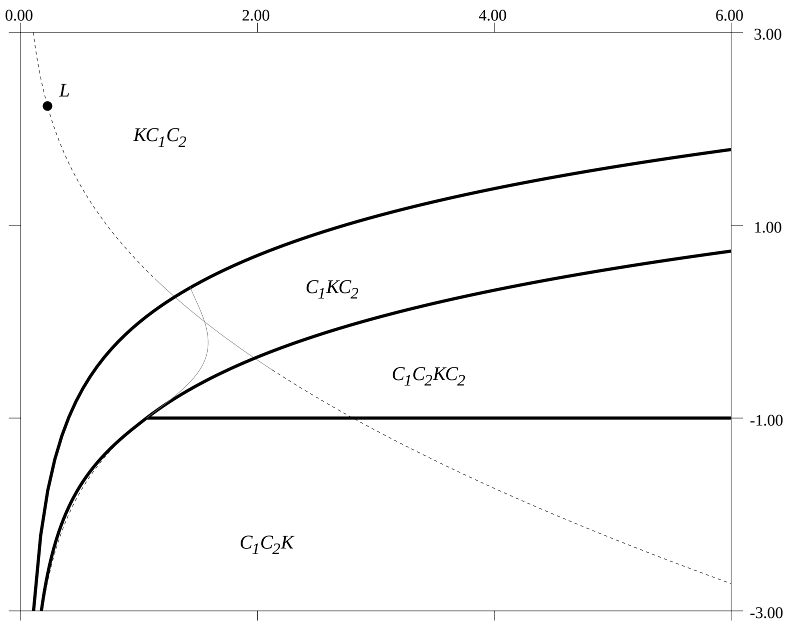

Fig. 7. The solution of the Riemann problem for $v_L > 1$.

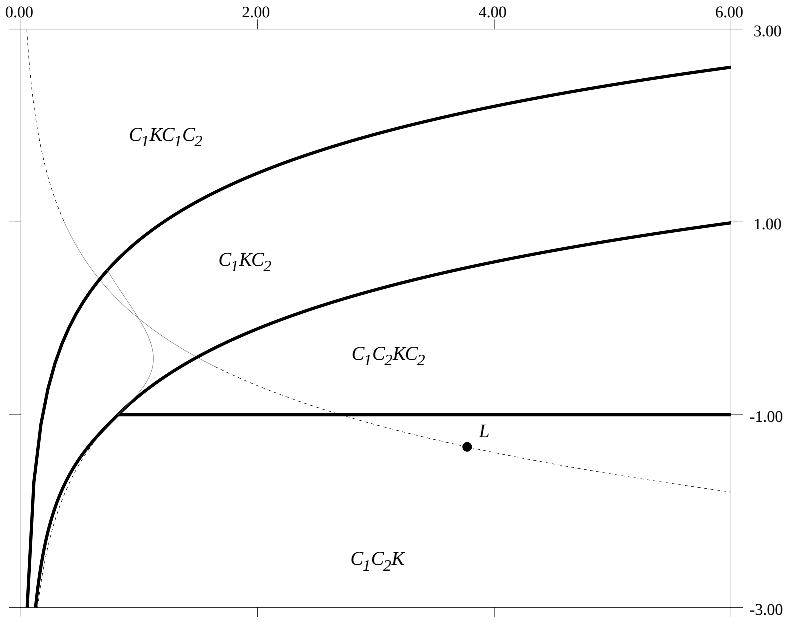

Fig. 8. The solution of the Riemann problem for $v_L < 1$.

4. Appendix. In this appendix we prove that the curve $\tilde{\kappa}$ and C_2 curves starting from $\tilde{\kappa}$ are transversal. We first observe that for the C_2 curves originating on $\tilde{\kappa}$ we have that $d\tilde{\rho}/d\tilde{v} = \tilde{\rho}$, and hence it suffices to show that $d\tilde{\rho}/d\tilde{v}$ for the curve $\tilde{\kappa}$ is not equal to $\tilde{\rho}$. In the case k positive we show that indeed $d\tilde{\rho}/d\tilde{v}$ is negative, while in the case k is negative, the estimates have to be sharper as the curves are infact tangent at the end point.

We start with the case when k is positive.

Lemma A.1. *Assume that* $0 \le k < 2$. *Let* $(\tilde{\rho}, \tilde{v})$ *be an arbitrary point on* $\tilde{\kappa}$. *Then*

$$\frac{d\tilde{\rho}}{d\tilde{v}} \le 0. \tag{A.1}$$

Proof. By the chain rule

$$\frac{d\tilde{\rho}}{d\tilde{v}} = \frac{\tilde{\rho}}{g'_-(v)} \left(\frac{\rho'}{\rho} + \frac{1}{v} - \frac{g'_-}{g_-} \right), \tag{A.2}$$

where we let the C_1 curve through (ρ_L, v_L) be parameterized by $(\rho(v), v)$. We have that $\rho'/\rho \le 0$, and the lemma will follow if

$$\frac{1}{v} - \frac{g'_-}{g_-} = \frac{g_- - g'_- v}{v g_- g'_-} \le 0. \tag{A.3}$$

The last inequality holds since g'_- satisfies the differential equation

$$g'_- = \frac{1 - v^2}{v\sqrt{\alpha^2 - 4v^2}} g_- \tag{A.4}$$

which implies that $g_- - g'_- v \le 0$ as long as $k \le 2$. $\square$

We now discuss to the more complicated question of transversality of the $\tilde{\kappa}$ curve in the case with k negative.

Lemma A.2. *Assume* $2(1-\sqrt{2}) < k < 0$. *Let* $(\tilde{\rho}, \tilde{v})$ *an arbitrary point on* $\tilde{\kappa}$. *Then*

$$\frac{d\tilde{\rho}}{d\tilde{v}} < \tilde{\rho}. \tag{A.5}$$

except at the point $K(\widehat{L})$.

Proof. We parametrize the curve $\tilde{\kappa}$ using the parameter v running along the $C_1(L)$ curve, and hence write $\tilde{\rho} = \tilde{\rho}(\rho(v), v)$ and similarly for the other dependent variables.

We first compute

$$
\begin{aligned}
\frac{d\tilde\rho}{d\tilde v} &= \frac{\frac{d\tilde\rho}{dv}}{\frac{d\tilde v}{dv}} = \frac{\frac{(\rho' v+\rho)g-\rho v g'}{g^2}}{g'} \\
&= \frac{1}{g'}\left(\frac{\rho'\rho v}{g\rho}+\frac{\rho v}{gv}-\frac{\rho v g'}{gg}\right) \\
&= \frac{\tilde\rho}{g'}\left(\frac{\rho'}{\rho}+\frac{1}{v}-\frac{g'}{g}\right)
\end{aligned} \tag{A.6}
$$

What we aim to show is that

$$
\frac{1}{v}+\frac{\rho'}{\rho} < \frac{g'}{g}(g+1) \tag{A.7}
$$

except at the point $K(\widehat{L})$. From this it follows that $d\tilde\rho/d\tilde v < \tilde\rho$ away from the point $K(\widehat{L})$.

We have that

$$
\frac{\rho'}{\rho} = \begin{cases} -1 & \text{for } \rho \le \rho_L \\ -\frac{2\sqrt{\rho\rho_L}}{\rho+\rho_L} & \text{for } \rho > \rho_L. \end{cases}
$$

We first show that $\tilde\kappa$ is never tangent to the C_2 curves for $v>0$. Now we need the following lemma, the proof of which comes after this proof.

Lemma A.3. *Let $(\rho, v) = (\rho(v), v) \in C_1(L)$. Then*

$$
\frac{\rho'}{\rho} < \frac{-2v}{v^2+1} \tag{A.8}
$$

for $0<v<1$,

With (A.7) in mind we introduce the reparametrization β by

$$
\begin{aligned}
\beta(u) &= u - k + \frac{1}{u}, \\
u &= \frac{1}{2}\left(\gamma - \sqrt{\gamma^2-4}\right),
\end{aligned}
$$

where $\gamma = \beta + k \ge 2$. The parameter β is in the interval $[2-k, \infty\rangle$. We have that $d\beta/dv = 1 - 1/v^2 < 0$, in this interval. Therefore, (A.7) will follow if

$$
V(v) := \frac{\frac{1}{v}-\frac{2v}{v^2+1}}{1-\frac{1}{v^2}} = \frac{-v}{v^2+1} = -\frac{1}{\gamma}.
$$

is greater than

$$
H(v) := \frac{\frac{dg}{d\beta}}{g}(g+1) = -\frac{1}{2}\left(\sqrt{\frac{\beta+2}{\beta-2}}-1\right).
$$

Solving the equation $H=V$ for $\gamma = \tilde\gamma(k)$, we obtain

$$
\tilde\gamma(k) = \frac{2+k}{k+1}. \tag{A.9}
$$

Since $\gamma \ge 2$, and $\tilde\gamma(k) < 0$ for $k > -1$, we have that $H < V$ for all $k > -1$. This concludes the case $v > 0$.

Now for the (harder) case where $v<0$. In this case we use the crude estimate $\rho'/\rho < 0$, and wish to prove

$$
\frac{1}{v} < \frac{g'}{g}(g+1). \tag{A.10}
$$

Now we use the reparametrization

$$\beta = -v + k - \frac{1}{v},$$
$$v = -\frac{1}{2}\left(\gamma - \sqrt{\gamma^2 - 4}\right),$$

where $\gamma = \beta - k$. Now β is in the interval $[2, \infty\rangle$. We have that $d\beta/dv = 1/v^2 - 1 > 0$, consequently (A.10) will follow if we can show that

$$V(v) := \frac{\frac{1}{v}}{\frac{1}{v^2} - 1} = \frac{v}{1 - v^2} = \frac{-1}{\sqrt{\gamma^2 - 4}}$$

is less than

$$H(v) := \frac{\frac{dg}{d\beta}}{g}(g + 1) = \frac{1}{2}\left(\sqrt{\frac{\beta - 2}{\beta + 2}} - 1\right).$$

In this case we can solve the equation $V = H$ for $k = \tilde{k}(\beta)$, obtaining

$$\tilde{k}(\beta) = -\beta + \sqrt{\frac{1}{2}(\beta + 2)^2\left(\sqrt{\frac{\beta - 2}{\beta + 2}} + 1\right) - (\beta - 2)}.$$

For β in the interval $[2, \infty\rangle$, $\tilde{k}(\beta) \geq \tilde{k}(2) = 2(1 - \sqrt{2})$, see the figure below. Hence, for $k > 2(1 - \sqrt{2})$, $V(v) < H(v)$.

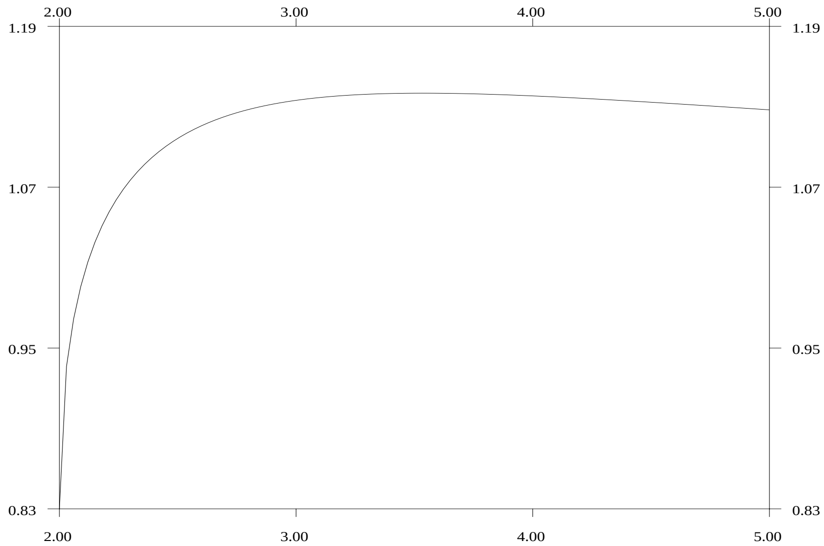

Fig. 8. The function $\tilde{k}(\beta)$.

Proof of Lemma A.3. If (ρ, v) is on the rarefaction part of the C_1 curve, then $\rho'/\rho = -1$, and the lemma certainly holds. Assume therefore that (ρ, v) is on the shock part of the C_1 curve. Then it is below $Z(L) = (\rho_L v_L^2, 1/v_L)$, and hence $v \leq 1/v_L$. In this case we have that

$$\frac{\rho'}{\rho} = \frac{-2\sigma}{\sigma^2 + 1}, \tag{A.21}$$

where $\sigma(v) = \sqrt{\rho_0/\rho(v)}$. We see that $\sigma' \leq 1/2$, and $\sigma(1/v_L) = 1/v_L$, showing that $\sigma(v) > v$ for $v \leq 1/v_L$. The function $\phi(t) = -2t/(t^2 + 1)$ is monotone decreasing for $|t| \leq 1$, and hence we infer that

$$\frac{\rho'}{\rho} < \frac{-2v}{v^2 + 1}.$$

□

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Department of Mathematical Sciences, Norwegian University of Science and Technology, N–7034 Trondheim, Norway (Holden)
E-mail address: holden@imf.unit.no

Department of Mathematics, University of Oslo, P.O. Box 1053, Blindern, N–0316 Oslo, Norway (Risebro)
E-mail address: nilshr@math.uio.no